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Continued Fractions

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ANALYTIC THEORY OF *Continued Fractions*

by

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D. VAN NOSTRAND COMPANY, INC.

PRINCETON, NEW JERSEY

TORONTO　　LONDON

NEW YORK

D. VAN NOSTRAND COMPANY, INC.
120 Alexander St., Princeton, New Jersey (*Principal office*)
257 Fourth Avenue, New York 10, New York

D. VAN NOSTRAND COMPANY, LTD.
358, Kensington High Street, London, W.14, England

D. VAN NOSTRAND COMPANY (Canada), LTD.
25 Hollinger Road, Toronto 16, Canada

08592a075

PRINTED IN THE UNITED STATES OF AMERICA

TO

LESTER R. FORD

PREFACE

In writing this book, I have tried to keep in mind the student of rather modest mathematical preparation, presupposing only a first course in function theory. Thus, I have included such things as a proof of Schwarz's inequality, theorems on uniformly bounded families of analytic functions, properties of Stieltjes integrals, and an introduction to the matrix calculus. I have presupposed a knowledge of the elementary properties of linear fractional transformations in the complex plane.

It has not been my intention to write a complete treatise on the subject of continued fractions, covering all the literature, but rather to present a unified theory correlating certain parts and applications of the subject within a larger analytic structure. I have not touched upon the arithmetic theory, and have, for the most part, refrained from developing formulas of a more general character than are actually used in the proofs. Neither have I made any attempt to compile a complete bibliography.

Certain parts of the book have been developed in courses. For instance, parts of Chapter X were used in a course in the theory of equations, and most of Part I was covered in a course in the theory of continued fractions. Some of the material of Chapters XII and XV was developed in seminar courses.

This approach to the theory of continued fractions is mainly the result of researches carried on during the past decade by my students and colleagues and myself. I wish to take this opportunity to thank all those who have had a part in this work, and who have made this book possible. I wish also to thank my wife, Mary Kate, for invaluable encouragement and for help in the preparation of the manuscript and correction of the proofs.

H. S. W.

January, 1948
The University of Texas

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INTRODUCTION

This book deals with the analytic theory of continued fractions, that is, with continued fractions in relation to analysis: the theory of equations, orthogonal polynomials, power series, infinite matrices and quadratic forms in infinitely many variables, definite integrals, the moment problem, analytic functions, and the summation of divergent series. In contrast with the analytic theory of continued fractions, there is an extensive arithmetic theory which is not touched upon here.

The celebrated memoir of T. J. Stieltjes [95],* *Recherches sur les fractions continues*, of 1894, may perhaps be regarded as marking the first major step in the creation of an analytic theory of continued fractions. Here is to be found the development of fundamental function theory and integral theory necessary for a complete treatment of an important class of continued fractions. For several years, Stieltjes had been interested in the problem of summation of divergent power series. His Thesis (1886), "Recherches sur quelques séries semi-convergentes" (*Oeuvres*, vol. 2, pp. 1–58), is a profound study of the remainders in several asymptotic series. In 1889–1890 he published a considerable number of examples of continued fraction expansions for series of this kind, all arising as formal power series expansions of definite integrals. The integrals are of the form

$$\int_0^\infty \frac{f(u)du}{z+u},$$

where $f(u) > 0$, and the continued fractions are of the form

$$\cfrac{1}{z+\cfrac{a_1}{1+\cfrac{a_2}{z+\cfrac{a_3}{1+\cdots}}}}, \tag{a}$$

* Numbers in brackets refer to the bibliography.

where the a_p are positive. The latter can be transformed into

$$\cfrac{1}{z + b_1 - \cfrac{p_1}{z + b_2 - \cfrac{p_2}{z + b_3 - \cdots}}}, \qquad \text{(b)}$$

where the b_k and p_k are positive functions of the a_k. [**93, 94.**]

In the memoir of 1894, Stieltjes developed a general theory of these continued fractions, covering questions of convergence and connection with definite integrals and divergent power series. In order to complete the theory, he had to extend the customary notion of integral, and to develop a general "convergence continuation theorem" for sequences of analytic functions.

In 1903, E. B. Van Vleck [**109**] undertook to extend the Stieltjes theory to continued fractions of the form (b) in which the p_k are arbitrary positive numbers and the b_k arbitrary real numbers. He was able to connect in certain cases these continued fractions with definite integrals of the type found by Stieltjes, but with the range of integration taken over the entire real axis. A complete extension of the Stieltjes theory to these continued fractions was first obtained by Hamburger [**26**] in 1920, following the pattern laid down by Stieltjes. In the interim, Hilbert and his pupils developed their famous theory of infinite matrices and quadratic forms in infinitely many variables, in which the ideas of Stieltjes are in the background. In 1914 Hellinger and Toeplitz [**32**] laid the groundwork for a matrix theory of the continued fraction (b) ($p_k > 0$, b_k real), and in 1922 Hellinger [**31**] obtained a complete theory from this point of view. Several other mathematicians reached the same goal by different methods at about the same time (Carleman [**6**], R. Nevanlinna [**62**], M. Riesz [**79**]).

Another kind of investigation had been going on in the meantime. Around 1900 Pringsheim [**73, 75**] and Van Vleck [**107, 108**] considered the question of convergence of continued fractions with complex elements

$$\cfrac{1}{1+\cfrac{c_1}{1+\cfrac{c_2}{1+\cdots}}} \tag{c}$$

and

$$\cfrac{1}{b_1+\cfrac{1}{b_2+\cdots}} \tag{d}$$

Pringsheim found that (c) converges if $|c_p| \leq (1 - g_{p-1})g_p$, where $0 < g_{p-1} < 1$, $p = 1, 2, 3, \cdots$, and Van Vleck arrived at the same conclusion but with $g_0 = 0$ and the requirement that the series

$$1 + \Sigma \frac{g_1 g_2 \cdots g_p}{(1 - g_1)(1 - g_2) \cdots (1 - g_p)}$$

be convergent. Both these results include an older theorem of Worpitzky [**143**] (1865), that (c) converges if $|c_p| \leq \frac{1}{4}$, $p = 1, 2, 3, \cdots$. Van Vleck found that (d) converges when $b_1 \neq 0$, $|\mathfrak{I}(b_p)| \leq k\mathfrak{R}(b_p)$, $k > 0$, $p = 1, 2, 3, \cdots$, if, and only if, the series $\Sigma |b_p|$ diverges. Van Vleck also found that if $c_p = a_p z$, $\lim_{p=\infty} a_p = a$, then (c) converges except for certain isolated values of z and except for values of z on the rectilinear cut from $-1/4a$ to ∞ in the direction of the vector from the origin to $-1/4a$. Szász [**98**] found that (c) converges if the c_p are in certain wedge-shaped domains extending beyond but not containing the circular domain found by Worpitzky. Szász [**99, 100**] also found that (c) converges if the series $\Sigma |c_p|$ converges and $\Sigma[|c_p| - \mathfrak{R}(c_p)] \leq 2$. This is an extension of an older theorem of von Koch [**116**]. These results and the proofs which were employed bear little relationship to one another or to the Stieltjes theory.

During the years 1940–1947, in which this book has been written, it has been our desire to develop a unified theory extending the various results indicated in the preceding sketch and tying them together within a larger analytic structure. First, we found that the inequalities $|c_p| \leq (1 - g_{p-1})g_p$ of Pringsheim and Van Vleck, which restrict the c_p to lie in the neighborhood of the origin, can be replaced by inequalities restricting the c_p to lie in domains bounded by certain parabolas with foci at the origin (Scott and Wall [**86**], Paydon and Wall [**68**]). Second, we developed the theory of positive definite continued fractions, which extends the Stieltjes theory to a class of continued fractions (b) with complex p_k and b_k, and also contains and extends the other results we have described, including the parabola theorems just mentioned (Hellinger and Wall [**35**], Wall and Wetzel [**138, 139**], Dennis and Wall [**9**]).

We shall now describe in some detail the general plan of the book.

The aforementioned "larger analytic structure" is obtained here by regarding the continued fraction as generated by an infinite sequence of linear fractional transformations in a single variable, and also as arising from a single linear transformation in infinitely many variables. There is often an interplay of these two ideas. From the first point of view the continued fraction theory becomes a part of the theory of Möbius transformations; whereas from the second point of view it becomes a part of the Hilbert theory of infinite matrices and quadratic forms in infinitely many variables.

Let us begin with the first point of view, and regard the continued fraction

$$\cfrac{1}{b_1 + z_1 - \cfrac{a_1^2}{b_2 + z_2 - \cfrac{a_2^2}{b_3 + z_3 - \cdots}}} \tag{e}$$

as being generated by the sequence of transformations

$$t = t_0(w) = \frac{1}{w}, \quad t = t_p(w) = b_p + z_p - \frac{a_p^2}{w}, \quad p = 1, 2, 3, \cdots,$$

of the variable w into the variable t. The symbolic product of the first $n + 1$ of these transformations is the transformation $t = t_0 t_1 \cdots t_n(w)$. The image of $w = \infty$ under this product transformation is the nth **approximant** of the continued fraction, and is a rational function of $z_1, z_2, z_3, \cdots$, with coefficients depending upon the constants a_p and b_p:

$$t_0 t_1 \cdots t_n(\infty) = \frac{A_n(z)}{B_n(z)}.$$

If the denominator $B_n(z) \neq 0$ for at most a finite number of values of n, and if $\lim_{n=\infty} [A_n(z)/B_n(z)] = L$, exists and is finite, then the continued fraction is said to **converge** to the **value** L. Otherwise the continued fraction **diverges.** Thus the value of a continued fraction is the limit of the images of a fixed point under a certain sequence of linear fractional transformations.

In order to investigate the continued fraction, we proceed as follows. First, we determine a sequence of half-planes $\{\pi_k\}$, such that $t_0(\pi_0) = K_0$, a finite circular region, and such that $t_p(\pi_p) \subset \pi_{p-1}$, $p = 1, 2, 3, \cdots$. If we put $t_0 t_1 \cdots t_p(\pi_p) = K_p$, then we see at once that $K_0 \supset K_1 \supset K_2 \supset \cdots$. Since ∞ is in π_p, it follows that $t_0 t_1 \cdots t_p(\infty)$, the pth approximant of (e), is in K_p. Thus, we determine a nest of circles such that the pth approximant of the continued fraction is in the pth circle.

There are two possible cases. Either the circles K_p have one, and only one, point L in common ("limit-point case"), or else the circles K_p have a circular region in common ("limit-circle case"). In the first case, the radius r_p of K_p has the limit 0 for $p = \infty$, whereas in the second case $r_p \to r > 0$. In the first case the continued fraction converges to the value L; in the second case the question of convergence remains undecided. Criteria for determining which of the two cases holds may be found if we first obtain an explicit formula for the radius r_p of K_p.

The above-described program can be carried out under various hypotheses upon the coefficients a_p and b_p of (e). We have sought to define a class of continued fractions for which this can be done, which is sufficiently general to include all particular classes which have been studied in the literature. The class of *positive definite continued fractions*, for which the quadratic forms

$$\sum_{p=1}^{n} \Im(b_p + z_p)|x_p|^2 - \sum_{p=1}^{n-1} \Im(a_p)(x_p\bar{x}_{p+1} + \bar{x}_p x_{p+1}),$$

$$n = 1, 2, 3, \cdots,$$

are positive definite for $\mathfrak{J}(z_p) > 0$, $p = 1, 2, 3, \cdots$, comes close to fulfilling our requirement.

The condition of positive definiteness can be formulated in the following convenient form.

$$\mathfrak{J}(b_p) \geq 0, \quad |a_p^2| - \mathfrak{R}(a_p^2) \leq 2\mathfrak{J}(b_p)\mathfrak{J}(b_{p+1})(1 - g_{p-1})g_p,$$
$$0 \leq g_{p-1} \leq 1, \quad p = 1, 2, 3, \cdots.$$

The continued fraction is positive definite if, and only if, numbers $g_0, g_1, g_2, \cdots$ can be found satisfying the above inequalities. This has a simple geometrical interpretation: namely, for each p, a_p^2 has its value in a certain parabola. This parabola has its focus at the origin and its vertex upon the negative half of the real axis, and it depends upon the index p.

The class of positive definite continued fractions is first introduced in Chapter IV. This could have been done earlier, but we have followed, instead, the plan of first investigating by appropriately simple methods certain special positive definite continued fractions. After covering some preliminary ideas in Chapter I, and some necessary conditions for convergence and a treatment of periodic continued fractions in Chapter II, we take up, in Chapter III, the aforementioned special positive definite continued fractions. These can be taken in the form (c), with the positive definiteness condition

$$|c_p| - \mathfrak{R}(c_p) \leq 2(1 - g_{p-1})g_p, \quad 0 \leq g_{p-1} \leq 1, \quad p = 1, 2, 3, \cdots.$$

This holds with $g_p = \frac{1}{2}$ in case $|c_p| \leq \frac{1}{4}$ (Worpitzky); for $0 \leq g_{p-1} \leq 1$ if $|c_p| \leq (1 - g_{p-1})g_p$ (Pringsheim, Van Vleck); and for $g_p = \frac{1}{2}$ if $|c_p| - \mathfrak{R}(c_p) \leq \frac{1}{2}$ (Scott and Wall).

The treatment of these cases in Chapter III is based upon a system of "fundamental inequalities" [86]. Sequences of the form $\{(1 - g_{p-1})g_p\}$ play an important part in our theory. They are called **chain sequences,** and some of their properties are developed systematically in Chapter IV after the introduction of positive definite continued fractions.

In Chapter V we prove a **theorem of invariability.** Consider the system of linear equations

$$L_p(x) = -a_{p-1}x_{p-1} + (b_p + z_p)x_p - a_px_{p+1} = 0,$$
$$p = 1, 2, 3, \cdots, \quad (a_0 = 1),$$

in the variables $x_1, x_2, x_3, \cdots$. Let $X_p(z)$ and $Y_p(z)$ denote the solutions of this system satisfying the initial conditions $X_0 = -1$, $X_1 = 0$, $Y_0 = 0$, $Y_1 = 1$. Then $X_{p+1}(z)/Y_{p+1}(z)$ is the pth approximant of the continued fraction, i.e., $t_0 t_1 \cdots t_p(\infty)$. When the continued fraction is positive definite, a sufficient condition for the limit-point case to hold is that at least one of the series $\Sigma|X_p(z)|^2$, $\Sigma|Y_p(z)|^2$ be divergent. We show that for any continued fraction (e) with $a_p \neq 0$, this condition is independent of the particular values of the z_p in every domain $|z_p| < M$, $p = 1, 2, 3, \cdots$, where M is a finite constant. We do this by expressing the general solution of the system $L_p = 0$ for parameter values z_p in terms of solutions with parameter values z_p^*, by means of a **Volterra sum equation.** From this theorem of invariability, it follows that a sufficient condition for the limit-point case to hold is that at least one of the series $\Sigma|X_p(0)|^2$, $\Sigma|Y_p(0)|^2$ be divergent. Since this condition is easier to handle than the condition "$r_p \to 0$," we emphasize it in preference to the "limit-point case," and call it the **determinate case.** When both the above series converge, we say that the **indeterminate case** holds. This classification actually replaces the other less convenient classification throughout much of the sequel.

The indeterminate case is in some respects easier to handle than the determinate case. We show that if the indeterminate case holds for any continued fraction (positive definite or not) and if the continued fraction or its reciprocal converges for one set of values of the z_p in the domain $|z_p| < M$, then the continued fraction or its reciprocal converges for every set of values of the z_p in this domain.

If the determinate case holds for a positive definite continued fraction, then it converges for $|z_p| < M$, $\mathfrak{I}(z_p) \geq \delta > 0$, $p = 1, 2, 3, \cdots$.

If $z_p = \zeta$, $p = 1, 2, 3, \cdots$, the continued fraction is called a **J-fraction.** We employ the **convergence continuation theorem** of Stieltjes (§ 24) to show that when the J-fraction is positive definite and the determinate case holds, then the J-fraction is an analytic function of ζ for $\mathfrak{I}(\zeta) > 0$. If the J-fraction is convergent in the indeterminate case, then it represents a meromorphic function of ζ.

The last two sections of Chapter V, and the whole of Chapter VI and of Chapter VII, deal with particular convergence theorems derived from the general theory of positive definite continued fractions. We mention, in particular, the theorems on bounded J-fractions (§ 26), on real J-fractions (§ 27) and on the continued fraction of Stieltjes (§ 28), the general theorem on the convergence of Stieltjes type continued fractions (§ 29), and theorems of Van Vleck, Hamburger, and Mall, which appear as corollaries of this theorem (§ 30).

The theorems of Chapter VII may be regarded as refinements or extensions of the theorems of Chapter III, and all deal with continued fractions of the form (c). We mention the "cardioid theorem," the theorem concerning convergence of the continued fraction for all c_p in some neighborhood of any point not on the interval $(-\infty, -\frac{1}{4})$, the theorem of Van Vleck mentioned before concerning continued fractions (c) in which $c_p = a_p z$, $\lim_{p=\infty} a_p = a$, and an extension of the theorem of Szász, also mentioned before, concerning continued fractions (c) for which $\Sigma|c_p|$ is convergent. These theorems all come out of the theory of positive definite continued fractions.

The concluding chapter of Part I (Chapter VIII) deals with the problem of finding estimates for the values of a continued fraction (c) whose elements are restricted to lie in a certain region of the complex plane.

The reader will find some applications of the convergence theory to the continued fraction of Gauss in Chapter XVIII. Other examples are given in Chapter XIX. For convenience in reference, it has seemed best to put the examples together in these chapters.

Part II, *Function Theory*, deals mainly with applications of continued fractions to the theory of equations, the moment problem, analytic functions, and the summation of divergent series. We begin, in Chapter IX, with the problem of expanding a rational function into a continued fraction, emphasizing techniques applicable to numerical examples. In Chapter X we show how these expansions can be used in the location of roots of polynomials. We are able to obtain (a) polygonal regions containing all the roots of a polynomial, (b) simple criteria for determining the number of roots in a half-plane, (c) the values of the roots by successive approximations.

In Chapter XI we give methods for expanding a formal power series into a continued fraction, connecting the problem with orthogonal polynomials and with the reduction of a quadratic form to a sum of squares. There are theorems connecting the sum of the power series with the value of its continued fraction expansion.

In Chapter XII we regard the continued fraction as arising from a single linear transformation in the space of infinitely many variables, and thereby connect continued fractions with the Hilbert theory of infinite matrices. We have included here an introduction to the matrix calculus. The matrices which are actually used are the **J-matrices** (§ 59), whose reciprocals $(\rho_{pq}(z))$ have the property that the leading coefficient $\rho_{11}(z)$ is formally equal to the J-fraction. The main problem is to determine a class of reciprocals which bear an essential relationship to the J-fraction. It turns out that these reciprocals are the ones for which the values of $\rho_{11}(z)$ are common to the circles $K_p(z)$ which were connected with the J-fraction in Chapter IV. We then find for these functions $\rho_{11}(z)$, called **equivalent functions** of the J-fraction, the important asymptotic formula

$$\rho_{11}(z) = \frac{1}{z} + \frac{0(1)}{z\mathfrak{J}(z)}, \quad \mathfrak{J}(z) > 0.$$

In Chapter XIII we show, by means of this asymptotic formula, that any equivalent function of a positive definite J-fraction can be expressed as a definite Stieltjes integral

$$\rho_{11}(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z-u},$$

where $\phi(u)$ is a bounded nondecreasing function of u. In this connection we develop in detail a number of the essential properties of Stieltjes integrals. Chapters XIV, XV, XVI and XVII are then devoted to problems growing out of this general continued fraction-definite integral tie-up. The first of these four chapters deals with the case where the above integral extends over only a finite interval, i.e., $\phi(u)$ is constant for $u > M_2$ and for

$u < M_1$. For the sake of convenience we take $M_1 = 0$, $M_2 = 1$, and show that $f(z)$ can be expressed as a definite integral of the form

$$f(z) = \int_0^1 \frac{d\phi(u)}{z - u},$$

where $\phi(u)$ is bounded and nondecreasing, if, and only if, $f(z)$ is equal to a continued fraction of the form (a), in which $-a_1$, $-a_2$, $-a_3$, $\cdots$ is a chain sequence. In this connection we consider the moment problem for the interval (0, 1).

In Chapter XV we give continued fraction expansions for functions which are analytic and have positive real parts in the domain exterior to the rectilinear cut from -1 to $-\infty$. This is done by connecting certain ideas of Schur [84] with positive definite continued fractions.

Chapter XVI contains the main outline of the theory of Hausdorff summability and of its connection with the moment problem for the interval (0, 1). There are some applications here of the material of Chapter XV.

The moment problem for an infinite interval is treated in Chapter XVII. We employ a modification of the method of R. Nevanlinna [62] in order to determine all solutions of the problem in terms of the equivalent functions of a J-fraction.

Chapters XVIII and XIX contain examples. The continued fraction of Gauss is the subject of Chapter XVIII, and a number of examples illustrating the Stieltjes theory are given in Chapter XIX. We have included here a list of some formal continued fraction expansions for particular functions.

The final chapter of the book treats of the Padé table of rational approximants for power series. This is largely formal in character.

At the ends of the chapters we have included a number of exercises, accompanied, in some instances, by references to the literature. These exercises frequently supplement the material of the text.

Part I

CONVERGENCE THEORY

Chapter I

THE CONTINUED FRACTION AS A PRODUCT OF LINEAR FRACTIONAL TRANSFORMATIONS

1. Definitions and Formulas. Let

$$\tau = \tau_p(w) = \frac{\alpha_p w + \beta_p}{\gamma_p w + \delta_p}, \quad \gamma_p \neq 0, \quad p = 0, 1, 2, \cdots,$$

be an infinite sequence of linear transformations of the variable w into the variable τ, and consider the product $\tau_0\tau_1 \cdots \tau_n(w)$ of the first $n + 1$ of these transformations, given by

$$\tau_0\tau_1(w) = \tau_0[\tau_1(w)], \quad \tau_0\tau_1\tau_2(w) = \tau_0\tau_1[\tau_2(w)],$$

$$\tau_0\tau_1\tau_2\tau_3(w) = \tau_0\tau_1\tau_2[\tau_3(w)], \quad \cdots.$$

If we write

$$\tau_p(w) = \frac{\alpha_p}{\gamma_p} - \frac{\Delta_p/\gamma_p^2}{\delta_p/\gamma_p + w}, \quad \Delta_p = \alpha_p\delta_p - \beta_p\gamma_p,$$

then the required product is

$$\tau_0\tau_1\tau_2 \cdots \tau_n(w)$$

$$= \frac{\alpha_0}{\gamma_0} - \cfrac{\Delta_0/\gamma_0^2}{\cfrac{\delta_0}{\gamma_0} + \cfrac{\alpha_1}{\gamma_1} - \cfrac{\Delta_1/\gamma_1^2}{\cfrac{\delta_1}{\gamma_1} + \cfrac{\alpha_2}{\gamma_2} - \cdots - \cfrac{\Delta_{n-1}/\gamma_{n-1}^2}{\cfrac{\delta_{n-1}}{\gamma_{n-1}} + \cfrac{\alpha_n}{\gamma_n} - \cfrac{\Delta_n/\gamma_n^2}{\cfrac{\delta_n}{\gamma_n} + w}}}}.$$

If we now put $w = \infty$ and then let n tend to ∞, the resulting infinite expression which is generated is called a **continued fraction.**

In case at most a finite number of the quantities $\tau_0\tau_1 \cdots \tau_n(\infty)$ are meaningless, and the limit

$$\lim_{n=\infty} \tau_0\tau_1 \cdots \tau_n(\infty) = v$$

exists and is finite, then the continued fraction is said to **converge,** and v is called its **value.** Thus, the value of a continued fraction is the limit of an infinite sequence of images, under certain transformations, of a fixed point $w = \infty$.

A glance at the above expression for $\tau_0\tau_1 \cdots \tau_n(w)$ will show immediately that the transformations $\tau_p(w)$ may as well be replaced by the simpler transformations

$$t_0(w) = b_0 + w, \quad t_p(w) = \frac{a_p}{b_p + w}, \quad p = 1, 2, 3, \cdots. \tag{1.1}$$

We observe that $t_0t_1 \cdots t_n(0) = t_0t_1 \cdots t_{n+1}(\infty)$. The continued fraction which is generated is

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}}, \tag{1.2}$$

and the value of the continued fraction is

$$\lim_{n=\infty} t_0t_1 \cdots t_n(0) = \lim_{n=\infty} t_0t_1 \cdots t_n(\infty).$$

We shall introduce some definitions with a view toward making these ideas more precise. The numbers a_p and b_p, called **elements,** may be any complex numbers; a_p/b_p is called the pth **partial quotient,** a_p is the pth **partial numerator,** and b_p is the pth **partial denominator.** The quantity

$$t_0t_1 \cdots t_n(0) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_n}{b_n}}}$$

is called the nth **approximant.**[1] The 0-th approximant is $t_0(0) = b_0$. We shall exhibit some properties of the approximants.

By mathematical induction it is readily shown that

$$t_0 t_1 \cdots t_n(w) = \frac{A_{n-1}w + A_n}{B_{n-1}w + B_n}, \quad n = 0, 1, 2, \cdots, \tag{1.3}$$

where the quantities A_{n-1}, A_n, B_{n-1}, B_n are independent of w and may be computed by means of the **fundamental recurrence formulas:**

$$A_{-1} = 1, \quad B_{-1} = 0, \quad A_0 = b_0, \quad B_0 = 1;$$

$$\begin{aligned} A_{p+1} &= b_{p+1}A_p + a_{p+1}A_{p-1}, \\ B_{p+1} &= b_{p+1}B_p + a_{p+1}B_{p-1}, \end{aligned} \quad p = 0, 1, 2, \cdots. \tag{1.4}$$

In fact, this may be verified directly for $n = 0$. If true for $n = k$, then

$$\begin{aligned} t_0 t_1 \cdots t_{k+1}(w) &= t_0 t_1 \cdots t_k\left(\frac{a_{k+1}}{b_{k+1} + w}\right) \\ &= \frac{A_k w + (b_{k+1}A_k + a_{k+1}A_{k-1})}{B_k w + (b_{k+1}B_k + a_{k+1}B_{k-1})} \\ &= \frac{A_k w + A_{k+1}}{B_k w + B_{k+1}}, \end{aligned}$$

so that the statement is true for $n = k + 1$ and therefore for all n.

We call A_n the nth **numerator** and B_n the nth **denominator.** The nth approximant is given by

$$t_0 t_1 \cdots t_n(0) = \frac{A_n}{B_n}.$$

The determinant of the transformation $t = t_0 t_1 \cdots t_n(w)$ is

$$\begin{aligned} \begin{vmatrix} A_{n-1}, & A_n \\ B_{n-1}, & B_n \end{vmatrix} &= \begin{vmatrix} A_{n-1}, & b_nA_{n-1} + a_nA_{n-2} \\ B_{n-1}, & b_nB_{n-1} + a_nB_{n-2} \end{vmatrix} \\ &= -a_n \begin{vmatrix} A_{n-2}, & A_{n-1} \\ B_{n-2}, & B_{n-1} \end{vmatrix}, \end{aligned}$$

[1] This is sometimes called the nth **convergent.**

so that

$$A_{n-1}B_n - A_nB_{n-1} = (-1)^n a_0 a_1 \cdots a_n, \quad n = 0, 1, 2, \cdots, \tag{1.5}$$

where a_0 must be taken equal to unity. The formula (1.5) is called the **determinant formula.**

We are now prepared to make the following definition.

DEFINITION 1.1. *The continued fraction* (1.2) *is said to converge or to be convergent if at most a finite number of its denominators* B_p *vanish, and if the limit of its sequence of approximants*

$$\lim_{n=\infty} \frac{A_n}{B_n}, \tag{1.6}$$

exists and is finite. Otherwise, the continued fraction is said to diverge or to be divergent. The value of a continued fraction is defined to be the limit (1.6) *of its sequence of approximants. No value is assigned to a divergent continued fraction.*

We remark that if the partial numerators a_p are all different from zero so that, by (1.5), A_n and B_n cannot both vanish, then the existence of the finite limit (1.6) insures that but a finite number of the denominators B_n can vanish. Hence, in this important case, the continued fraction converges if (and only if) the limit (1.6) exists and is finite.

Frequently, the elements a_p and b_p of the continued fraction depend upon one or more parameters, or may themselves be regarded as independent variables. In such cases, one is naturally concerned with the question of uniform convergence. We make the following definition.

DEFINITION 1.2. *If the elements* a_p *and* b_p *of a continued fraction are functions of one or more variables over a certain domain* D, *then the continued fraction is said to converge uniformly over* D *if it converges for all values of the variable or variables in* D, *and if its sequence of approximants converges uniformly over* D.

The first part of the book is concerned largely with the problem of determining conditions upon the elements a_p and b_p of the continued fraction which are sufficient to insure convergence. This **convergence problem** is essentially more complex and interesting than the corresponding problem for infinite series.

We have adopted the natural notation for a continued fraction. Other notations in more or less common use are

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots,$$

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots,$$

and

$$b_0 + K_{p=1}^{p=\infty} \frac{a_p}{b_p}.$$

2. Continued Fractions and Series. The following theorem establishes a connection between certain continued fractions and infinite series.

THEOREM 2.1. *If the denominators B_p of the continued fraction*

$$\cfrac{1}{1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{2.1}$$

are all different from zero, and if we put

$$\rho_p = -\frac{a_{p+1}B_{p-1}}{B_{p+1}}, \quad p = 1, 2, 3, \cdots, \tag{2.2}$$

then the continued fraction (2.1) *is equivalent to the continued fraction*

$$\cfrac{1}{1 - \cfrac{\rho_1}{1 + \rho_1 - \cfrac{\rho_2}{1 + \rho_2 - \cfrac{\rho_3}{1 + \rho_3 - \cdots}}}} \tag{2.3}$$

in the sense that the nth *approximants of* (2.1) *and* (2.3) *are equal to one another for all values of* n. *Moreover, for arbitrary numbers* ρ_p, *the* nth *numerator of* (2.3) *is equal to the sum of the first n terms of the infinite series*

$$1+\sum_{p=1}^{\infty}\rho_1\rho_2\cdots\rho_p, \tag{2.4}$$

and the nth *denominator is equal to unity.*[2]

Proof. The sum of the first n terms of the infinite series

$$1+\sum_{p=1}^{\infty}\left(\frac{A_{p+1}}{B_{p+1}}-\frac{A_p}{B_p}\right) \tag{2.5}$$

is A_n/B_n, the nth approximant of (2.1). By the determinant formula (1.5), this infinite series may be written as

$$1-\frac{a_2}{B_1B_2}+\frac{a_2a_3}{B_2B_3}-\frac{a_2a_3a_4}{B_3B_4}+\cdots,$$

which, by (2.2), is the series (2.4). Now, the linear transformation $s = s_p(w) = 1 + \rho_p w$ may be written in the form

$$s = s_p(w) = \cfrac{1}{1-\cfrac{\rho_p}{\rho_p+\cfrac{1}{w}}}, \qquad (p = 1, 2, 3, \cdots).$$

If we apply the first n of these in succession, and then put $w = 0$, we obtain as the product, on the one hand, the sum of the first n terms of the series (2.4), and, on the other hand, the nth approximant of the continued fraction (2.3). Consequently, the nth approximants of (2.1) and (2.3) are equal to the sum of the first n terms of the series (2.4), and hence to each other, for all values of n. One may readily verify by means of the fundamental recurrence formulas that the nth denominator of (2.3) is unity, and therefore the nth numerator is equal to the sum of the first n terms of (2.4).

This completes the proof of Theorem 2.1.

[2] This equivalence between series and continued fractions goes back to Euler [11]. Cf. Szász [98] for a somewhat different formulation.

We note for future reference that if $b_p = 1$, $p = 2, 3, 4, \cdots$, then we have the formulas

$$\rho_p = \frac{-a_{p+1}(1 + \rho_{p-1})}{1 + a_{p+1}(1 + \rho_{p-1})}, \quad p = 1, 2, 3, \cdots, \tag{2.6}$$

where ρ_0 must be taken equal to zero; and

$$\rho_p = \frac{-a_{p+1}}{1 + a_p + a_{p+1} + a_p\rho_{p-2}}, \quad p = 1, 2, 3, \cdots, \tag{2.7}$$

where we must take $a_1 = 0$, $\rho_{-1} = \rho_0 = 0$. These may be readily verified by means of (2.2) and (1.4).

3. Equivalence Transformations. It is often convenient to throw the continued fraction (1.2) into another form by means of a so-called **equivalence transformation.** This consists in multiplying numerators and denominators of successive fractions by numbers different from zero:

$$b_0 + \cfrac{c_1a_1}{c_1b_1 + \cfrac{c_1c_2a_2}{c_2b_2 + \cfrac{c_2c_3a_3}{c_3b_3 + \cdots}}}, \qquad (c_p \neq 0). \tag{3.1}$$

One may easily show by mathematical induction that this continued fraction has precisely the approximants of (1.2). In fact, the pth numerator and denominator of (3.1) are

$$c_1c_2 \cdots c_pA_p \quad \text{and} \quad c_1c_2 \cdots c_pB_p,$$

respectively, where A_p and B_p are the pth numerator and denominator of (1.2). This can be readily verified by means of the fundamental recurrence formulas (1.4).

If, conversely, two continued fractions with nonvanishing partial numerators have a common sequence of approximants, then either can be transformed into the other by means of an equivalence transformation. In fact, if A_p' and B_p' are the pth numerator and denominator of one continued fraction, and A_p and B_p

are those of the other, then there must exist constants $C_p \neq 0$ such that

$$A_p' = C_p A_p, \quad B_p' = C_p B_p, \quad p = 1, 2, 3, \cdots. \tag{3.2}$$

Let

$$\begin{aligned} A_p &= b_p A_{p-1} + a_p A_{p-2}, \\ B_p &= b_p B_{p-1} + a_p B_{p-2}, \end{aligned} \quad p = 1, 2, 3, \cdots,$$

$$A_{-1} = 1, \quad B_{-1} = 0, \quad A_0 = b_0, \quad B_0 = 1.$$

Then, since $A_{p-1}B_{p-2} - A_{p-2}B_{p-1} \neq 0$, by virtue of (1.5), we conclude that the elements a_p and b_p are uniquely determined by the A_p and B_p. Similarly, the elements of the other continued fraction are uniquely determined by the $A_p'$and B_p'. Let

$$\begin{aligned} A_p' &= b_p' A_{p-1}' + a_p' A_{p-2}', \\ B_p' &= b_p' B_{p-1}' + a_p' B_{p-2}', \end{aligned} \quad p = 1, 2, 3, \cdots,$$

$$A_{-1}' = 1, \quad B_{-1}' = 0, \quad A_0' = b_0, \quad B_0' = 1,$$

so that, by (3.2),

$$A_p = b_p' \frac{C_{p-1}}{C_p} A_{p-1} + a_p' \frac{C_{p-2}}{C_{p-1}} \cdot \frac{C_{p-1}}{C_p} A_{p-2},$$

with a like relation for the B_n. Here we must take $C_{-1} = C_0 = 1$. Consequently, by the preceding, we must have

$$b_p' = c_p b_p, \quad a_p' = c_{p-1} c_p a_p, \quad p = 1, 2, 3, \cdots,$$

where $c_p = C_p / C_{p-1}$. Thus, the two continued fractions are the same up to an equivalence transformation [89].

We note the following important special cases. If $b_p \neq 0$, $p = 1, 2, 3, \cdots$, and we take $c_p = 1/b_p$, then (3.1) takes a form in which all the partial denominators are equal to unity. Likewise, if $a_p \neq 0$, $p = 1, 2, 3, \cdots$, and we take $c_0 = 1$, and determine the other c_p recurrently by the equations $c_{p-1} c_p a_p = 1$, $p = 1, 2, 3, \cdots$, then (3.1) takes a form in which the partial numerators are all equal to unity.

4. Even and Odd Parts of a Continued Fraction. By the **even part** of a continued fraction we shall understand the continued fraction whose sequence of approximants is the sequence of even approximants of the given continued fraction. Similarly, the **odd part** of a continued fraction is the continued fraction whose sequence of approximants is the sequence of odd approximants of the given continued fraction. For the sake of simplicity, we shall write these down for the continued fraction

$$\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots}}} \tag{4.1}$$

rather than for (1.2). The even part of (4.1) is

$$\cfrac{1}{1+a_2-\cfrac{a_2a_3}{1+a_3+a_4-\cfrac{a_4a_5}{1+a_5+a_6-\cdots}}}, \tag{4.2}$$

and the odd part is

$$1-\cfrac{a_2}{1+a_2+a_3-\cfrac{a_3a_4}{1+a_4+a_5-\cfrac{a_5a_6}{1+a_6+a_7-\cdots}}} \tag{4.3}$$

The even and odd parts of (1.2) may be obtained from (4.2) and (4.3) by multiplying them by a_1, adding b_0, and then replacing a_1 by a_1/b_1, and a_p by $a_p/b_{p-1}b_p$, $p = 2, 3, 4, \cdots$.

To prove that (4.2) is the even part of (4.1), let us regard (4.1) as being generated by the sequence of transformations [8]

$$t = t_1(w) = w, \quad t = t_p(w) = \frac{1}{1+a_pw}, \quad p = 2, 3, 4, \cdots,$$

so that $t_1t_2 \cdots t_n(1) = A_n/B_n$, the nth approximant. Let $s_p(w)$

[8] This device for obtaining (4.2) was used by Stieltjes [95].

$= t_p t_{p+1}(w)$, $p = 1, 2, 3, \cdots$. Then

$$s_1(w) = \frac{1}{1 + a_2 w}, \quad s_p(w) = 1 - \frac{a_{2p-1}}{1 + a_{2p-1} + a_{2p}w},$$

$$p = 2, 3, 4, \cdots,$$

and $s_1 s_2 \cdots s_p(1) = t_1 t_2 \cdots t_{2p}(1) = A_{2p}/B_{2p}$, the $2p$th approximant of (4.1). Since

$$s_1 s_2 \cdots s_p(1) = \cfrac{1}{1 + a_2 - \cfrac{a_2 a_3}{1 + a_3 + a_4 - \cdots - \cfrac{a_{2p-2}a_{2p-1}}{1 + a_{2p-1} + a_{2p}}}},$$

it therefore follows that (4.2) is the even part of (4.1). The proof that (4.3) is the odd part of (4.1) can be made in an analogous way.

More general "contraction formulas" and also "extension formulas" will be found in Perron [**69**, pp. 197–205]. Next to the equivalence transformation, the transformations represented by (4.2) and (4.3) are perhaps the most useful continued fraction transformations.

Exercise 1

1.1. Let A_n/B_n, $n = 0, 1, 2, \cdots$, be the sequence of approximants of the continued fraction

$$b_0 + \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cdots}}$$

where $a_p \neq 0$, $a_{2p-1} \neq -1$, $p = 1, 2, 3, \cdots$. Form a continued fraction

$$s_0 + \cfrac{r_1}{1 + \cfrac{r_2}{1 + \cdots}}$$

having the sequence of approximants $A_1/B_1, A_0/B_0, A_3/B_3, A_2/B_2, A_5/B_5, A_4/B_4, \cdots$. *Ans.* $s_0 = b_0 + a_1$, $r_1 = -a_1$, $r_2 = (1 + a_3)/a_2$, $r_{2p-1} = a_{2p-1}$, $r_{2p} = (1 + a_{2p-1})(1 + a_{2p+1})/a_{2p}$. [**58, 124.**]

1.2. Let c be any number such that the denominators $B_p(z)$ of the continued fraction

$$\cfrac{1}{1+\cfrac{a_2 z}{1+\cfrac{a_3 z}{1+\cdots}}}$$

are all different from zero for $z = -c$. Put

$$g_{p-1} = 1 - \frac{B_p(-c)}{B_{p-1}(-c)}, \quad p = 2, 3, 4, \cdots.$$

Then, the continued fraction may be written in the form

$$\cfrac{1}{1+\cfrac{g_1\zeta}{1+\cfrac{(1-g_1)g_2\zeta}{1+\cfrac{(1-g_2)g_3\zeta}{1+\cdots}}}},$$

where $\zeta = z/c$.

1.3. Show how to transform the continued fraction of 1.2 into the continued fraction

$$\cfrac{1}{\xi + a_2 - \cfrac{a_2 a_3}{\xi + a_3 + a_4 - \cfrac{a_4 a_5}{\xi + a_5 + a_6 - \cdots}}}.$$

1.4. Let u and v be the two roots of the quadratic equation

$$x^2 - bx - a = 0, \quad (a \neq 0, b \neq 0),$$

and suppose that $|u| \geq |v|$. Show that the nth approximant of the periodic continued fraction

$$b + \cfrac{a}{b+\cfrac{a}{b+\cdots}}$$

can be written in the form

$$u + v - \cfrac{v}{\cfrac{v}{u} + \cfrac{1}{\sum_{p=0}^{n-1}(v/u)^p}},$$

and hence deduce the facts concerning convergence and divergence of the continued fraction.

Suggestion. Since $a = -uv, b = u + v$, the continued fraction may be written in the form

$$u + v - \cfrac{uv}{u + v - \cfrac{uv}{u + v - \cfrac{uv}{u + v - \cdots}}}$$

Now apply Theorem 2.1, after making a suitable equivalence transformation.

1.5. Show that

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

1.6. Show that if $a_p \neq 0$ is real for $p \geq n$, then all the approximants of the continued fraction (4.1) from and after the nth approximant lie upon a circle (or straight line) in the complex plane.

1.7. Let $u_0, u_1, u_2, \cdots$ be numbers all except possibly the first different from zero, and put $U_n = u_0 + u_1 + \cdots + u_n$. Let

$$b_0 = u_0, \quad b_1 = 1, \quad a_1 = \frac{u_1}{2},$$

$$b_{2p} = 2, \quad b_{2p+1} = \frac{u_p + u_{p+1}}{u_p}, \quad p = 1, 2, 3, \cdots,$$

$$a_{2p} = -1, \quad a_{2p+1} = -\frac{u_{p+1}}{u_p},$$

Show that the $2n$th and $(2n + 1)$th approximants of the continued fiaction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}}$$

are U_n and $U_n + u_{n+1}/2$, respectively [58].

Chapter II

CONVERGENCE THEOREMS

This chapter contains, among other things, an example indicating in some way the complexity of the convergence problem, some necessary conditions for convergence of continued fractions, and a complete treatment of the question of convergence for periodic continued fractions.

5. Some General Remarks on the Convergence Problem. By the **convergence problem** we shall understand the problem of determining conditions upon the elements a_p and b_p of a continued fraction which are sufficient to insure its convergence. In case the partial numerators a_p are different from zero, we may confine our attention to continued fractions of the form

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}}, \tag{5.1}$$

whose partial numerators are all equal to unity; and in case the partial denominators b_p are all different from zero, we may confine our attention to continued fractions of the form

$$b_0 + \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \ddots}}}, \tag{5.2}$$

whose partial denominators are all equal to unity (cf. § 3).

In case some partial numerator is zero, we have this theorem.

THEOREM 5.1. *Let the continued fraction* (1.2) *satisfy the following conditions, for some integer* $m \geq 1$:

$$a_m = 0 \quad \textit{while, if} \quad m > 1, \quad a_n \neq 0 \quad \textit{for} \quad n = 1, 2, 3, \cdots, \quad m - 1.$$

Then, the continued fraction converges if, and only if, there exists an index k *such that the denominators* $B_{n-1} \neq 0$ *for* $n \geq k$. *When convergent, the value of the continued fraction is* A_{m-1}/B_{m-1}.

Proof. By Definition 1.1, it is necessary for convergence that such an index k exist. Suppose, conversely, that such an index k exists. Then we see from the fundamental recurrence formulas (1.4) that $B_{m-1} \neq 0$, for otherwise $B_m = B_{m+1} = B_{m+2} = \cdots = 0$. Now,

$$\begin{aligned} A_pB_{m-1} &- A_{m-1}B_p \\ &= (b_pA_{p-1} + a_pA_{p-2})B_{m-1} - A_{m-1}(b_pB_{p-1} + a_pB_{p-2}) \\ &= b_p(A_{p-1}B_{m-1} - A_{m-1}B_{p-1}) + a_p(A_{p-2}B_{m-1} - A_{m-1}B_{p-2}). \end{aligned}$$

On putting p successively equal to m, $m + 1$, $m + 2$, $\cdots$ in this equation, and using the fact that $a_m = 0$, we find immediately that

$$A_pB_{m-1} - A_{m-1}B_p = 0 \quad \text{for} \quad p \geq m - 1.$$

Consequently, if $p \geq m - 1$ and $p \geq k - 1$,

$$\frac{A_p}{B_p} = \frac{A_{m-1}}{B_{m-1}}.$$

The continued fraction is therefore convergent, and its value is A_{m-1}/B_{m-1}.

In view of this theorem, the convergence problem for a continued fraction with a vanishing partial numerator does not involve the investigation of a limiting process, but is the algebraic problem of establishing the system of inequalities $B_p \neq 0$ for $p > k$. Since the value of such a continued fraction, when it converges, is a rational function of a finite number of the elements a_p and b_p, such a continued fraction is not especially interesting from the point of view of analysis. All other continued fractions can be thrown into the form (5.1).

We note the following theorem.

THEOREM 5.2. *The continued fraction* (5.1) *diverges if its odd partial denominators* b_{2p-1}, $p = 1, 2, 3, \cdots$, *are all zero.*

In fact, $B_1 = b_1 = 0$, $B_3 = b_3B_2 + a_3B_1 = 0$, $B_5 = b_5B_4 + a_5B_3 = 0, \cdots$, so that infinitely many of the denominators are zero. The continued fraction is therefore divergent.

The complex nature of the general convergence problem is indicated by the following example (cf. Leighton and Wall [**58**]). In the continued fraction of Exercise 1.7, take

$$u_0 = 0, \quad u_{2n} = \frac{2s_n + 1}{k_n}, \quad u_{2n-1} = \frac{-2s_n}{k_n}, \quad n = 1, 2, 3, \cdots,$$

where $s_1, s_2, s_3, \cdots$ is an arbitrary sequence of numbers different from 0 and $-\frac{1}{2}$, and where the k_n are chosen so that the series Σu_n converges and such that $u_n + u_{n+1} \neq 0$, $n = 1, 2, 3, \cdots$. By means of an equivalence transformation we may throw the continued fraction into the form (5.2), where

$$b_0 = 0, \quad a_1 = \frac{u_1}{2}, \quad a_2 = -\frac{1}{2},$$

$$a_{2n+1} = -\frac{u_{n+1}}{2(u_n + u_{n+1})}, \quad a_{2n+2} = -\frac{u_n}{2(u_n + u_{n+1})},$$
$$n = 1, 2, 3, \cdots.$$

Then, $a_{4n} = s_n$. The continued fraction converges (cf. Exercise 1.7) and its value is Σu_n. Thus, an infinite subsequence of the partial numerators of the continued fraction (5.2), namely, a_{4n}, $n = 1, 2, 3, \cdots$, may be chosen arbitrarily, two values excepted, and the continued fraction converges provided the remaining partial numerators are suitably chosen.

Questions arising from this example have been investigated by Leighton [**52**].

6. Necessary Conditions for Convergence. The theorems of this section deal with the continued fraction (5.1). Since convergence and divergence are not affected by the value of the additive term b_0, the latter may as well be omitted. We shall now prove the following theorem of von Koch [**115**].

THEOREM 6.1. *If the series* $\Sigma| b_p |$ *converges, then the continued fraction*

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_2 + \cdots}}} \tag{6.1}$$

diverges. The sequences of its even and odd numerators and denominators, $\{A_{2p}\}$, $\{A_{2p+1}\}$, $\{B_{2p}\}$, $\{B_{2p+1}\}$, *converge to finite limits* F_0, F_1, G_0, G_1, *respectively, where*

$$F_1G_0 - F_0G_1 = 1. \tag{6.2}$$

Proof. From (1.4), with the a_n now equal to unity, we have

$$\begin{aligned} A_{2p} &= b_{2p}A_{2p-1} + A_{2p-2} \\ &= b_{2p}A_{2p-1} + b_{2p-2}A_{2p-3} + A_{2p-4} \\ &= \cdots \\ &= b_{2p}A_{2p-1} + b_{2p-2}A_{2p-3} + \cdots + b_2A_1, \end{aligned}$$

so that A_{2p} is the sum of the first p terms of the infinite series $\Sigma b_{2r}A_{2r-1}$. Since, by hypothesis, the series $\Sigma| b_{2r} |$ is convergent, the convergence of the sequence $\{A_{2p}\}$ will be established if we show that $| A_{2p-1} | \leq C$, where C is a constant independent of p. But if M is the larger of $| A_{-1} |$ and $| A_0 |$, then

$$\begin{aligned} | A_1 | &\leq | b_1 |\cdot| A_0 | + | A_{-1} | \leq M(1 + | b_1 |), \\ | A_2 | &\leq | b_2 |\cdot| A_1 | + | A_0 | \leq M | b_2 |(1 + | b_1 |) + M \\ &\leq M(1 + | b_1 |)(1 + | b_2 |), \end{aligned}$$

and, by mathematical induction,

$$| A_n | \leq M(1 + | b_1 |)(1 + | b_2 |) \cdots (1 + | b_n |).$$

This holds for $n = 1, 2, 3, \cdots$. We may then take

$$C = M\prod_1^\infty(1 + | b_p |),$$

the infinite product being convergent inasmuch as the series $\Sigma| b_p |$ is convergent by hypothesis. The proof that the other three sequences converge can be made in the same way. Since, by the determinant formula, (1.5),

$$A_{2p+1}B_{2p} - A_{2p}B_{2p+1} = 1,$$

we conclude that (6.2) holds. Therefore, the continued fraction diverges since its sequence of approximants oscillates between the two distinct limit-points F_0/G_0 and F_1/G_1. One of these may be ∞.

Remark 1. From the results of Exercise 1.4, the reader will find that the continued fraction

$$\cfrac{1}{i + \cfrac{1}{i + \cfrac{1}{i + \cdots}}} \qquad (i = \sqrt{-1}), \tag{6.3}$$

is divergent. Thus, although the divergence of the series $\Sigma | b_p |$ is necessary for the convergence of (6.1), this condition is not, in general, sufficient.

Remark 2. If, in Theorem 6.1, the b_p are functions of any variables over a domain G in which the series $\Sigma | b_p |$ is uniformly convergent, then the above proof shows that the sequences $\{A_{2p}\}$, $\cdots$, $\{B_{2p+1}\}$ converge uniformly over G.

We shall now turn our attention to a generalization of Theorem 6.1, due to Scott and Wall [**88a**].

THEOREM 6.2.[4] *If the series*

$$\Sigma | b_{2p+1} | \tag{6.4}$$

and

$$\Sigma | b_{2p+1}s_p^2 |, \quad \textit{where} \quad s_p = b_2 + b_4 + \cdots + b_{2p}, \tag{6.5}$$

converge, and

$$\liminf | s_p | < \infty, \tag{6.6}$$

then the continued fraction (6.1) *diverges. The sequences of its odd numerators and denominators, $\{A_{2p+1}\}$ and $\{B_{2p+1}\}$, converge to finite limits F_1 and G_1, respectively. Moreover, if s is a finite limit-point of the sequence $\{s_p\}$, and* $\lim s_p = s$ *as p tends to ∞ over a certain sequence P of indices,*

[4] This theorem was proved under certain conditions upon the b_p by Hamburger [**26**], Mall [**59**], and Wall [**137**].

then A_{2p} and B_{2p} converge to finite limits $F(s)$ and $G(s)$, respectively, as p tends to ∞ over P, and

$$F_1G(s) - G_1F(s) = 1. \tag{6.7}$$

If the sequence $\{s_p\}$ has two different finite limit-points s and t, then

$$F(s)G(t) - F(t)G(s) = t - s. \tag{6.8}$$

Finally, corresponding to values of p for which $\lim s_p = \infty$, *we have*

$$\lim \frac{A_{2p}}{B_{2p}} = \frac{F_1}{G_1}, \quad (\textit{finite or infinite}). \tag{6.9}$$

Proof. Since the series (6.4) and (6.5) converge, it follows that the series

$$\Sigma | b_{2p+1}s_p | \tag{6.10}$$

converges, and therefore there exists an index $n \geq 1$ such that

$$| b_{2p+1}s_p | < 1 \quad \text{for} \quad p \geq n. \tag{6.11}$$

Hence the quantities

$$\pi_k = \prod_{p=1}^{k} (1 + b_{2n+2p+1}s_{n+p}), \quad k = 1, 2, 3, \cdots,$$

are different from zero, and the infinite product

$$\lim_{k=\infty} \pi_k = \prod_{p=1}^{\infty} (1 + b_{2n+2p+1}s_{n+p}) \tag{6.12}$$

converges and its value is not zero.

Lemma 6.1. *Let*

$$U_{2k} = \frac{A_{2n+2k+1}}{\pi_k}, \quad V_{2k} = \frac{B_{2n+2k+1}}{\pi_k},$$

$$U_{2k+1} = (A_{2n+2k+2} - s_{n+k+1}A_{2n+2k+1})\pi_k,$$

$$V_{2k+1} = (B_{2n+2k+2} - s_{n+k+1}B_{2n+2k+1})\pi_k,$$

$$k = 0, 1, 2, \cdots, \quad (\pi_0 = 1);$$

$$c_{2k} = \frac{b_{2n+2k+1}}{\pi_{k-1}\pi_k}, \quad c_{2k+1} = -b_{2n+2k+1}s_{n+k}{}^2\pi_{k-1}\pi_k,$$

$$k = 1, 2, 3, \cdots.$$

Then[5]

$$U_k = c_k U_{k-1} + U_{k-2}, \qquad k = 2, 3, 4, \cdots. \tag{6.13}$$
$$V_k = c_k V_{k-1} + V_{k-2},$$

Let us assume for the moment that the lemma is true. From the hypothesis that (6.4) and (6.5) converge, and from the convergence of (6.12), we conclude that the series $\Sigma | c_p |$ is convergent. It then follows, as in the proof of Theorem 6.1, that the sequences $\{U_{2k}\}$, $\{U_{2k+1}\}$, $\{V_{2k}\}$, $\{V_{2k+1}\}$, converge, and therefore the limits

$$\lim_{p=\infty} A_{2p+1} = F_1, \quad \lim_{p=\infty} B_{2p+1} = G_1, \tag{6.14}$$

and

$$\lim_{p=\infty} (A_{2p} - s_p A_{2p-1}) = X,$$
$$\lim_{p=\infty} (B_{2p} - s_p B_{2p-1}) = Y, \tag{6.15}$$

exist and are finite. Inasmuch as, by (1.5),

$$A_{2p-1}(B_{2p} - s_p B_{2p-1}) - B_{2p-1}(A_{2p} - s_p A_{2p-1}) = 1,$$

it follows that

$$F_1 Y - G_1 X = 1. \tag{6.16}$$

Let s be a finite limit-point of the sequence $\{s_p\}$, and suppose that $\lim s_p = s$ as $p \to \infty$ over a certain sequence P of indices. We then conclude immediately from (6.14) and (6.15) that

$$\lim A_{2p} = sF_1 + X = F(s),$$
$$\lim B_{2p} = sG_1 + Y = G(s),$$

as $p \to \infty$ over P. Moreover, by (6.16), we see that (6.7) holds. The relation (6.8) may now be readily verified. Inasmuch as

$$\frac{A_{2n+2p}}{B_{2n+2p}} = \frac{A_{2n+2p-1} + \dfrac{U_{2p-1}}{\pi_{p-1} s_{n+p}}}{B_{2n+2p-1} + \dfrac{V_{2p-1}}{\pi_{p-1} s_{n+p}}}, \tag{6.17}$$

the relation (6.9) is now evident.

[5] Formulas of this type can be found, for instance, in Leighton and Wall [58], and Mall [59].

From the preceding we conclude that the sequence of approximants of the continued fraction oscillates among the distinct limits F_1/G_1 and $F(s)/G(s)$, where the range of s is the set of finite limit-points of the sequence $\{s_p\}$. The continued fraction is therefore divergent.

We shall now establish the lemma, and the proof of the theorem will then be complete.

From (1.4), with a_m now equal to unity, we have

$$\begin{aligned} A_{2n+2p+1} &= b_{2n+2p+1}A_{2n+2p} + A_{2n+2p-1} \\ &= b_{2n+2p+1}(A_{2n+2p} - s_{n+p}A_{2n+2p-1}) \\ &\qquad + (1 + b_{2n+2p+1}s_{n+p})A_{2n+2p-1}. \end{aligned}$$

That is, if we introduce the quantities U_m,

$$\pi_p U_{2p} = b_{2n+2p+1}\frac{U_{2p-1}}{\pi_{p-1}} + \pi_p U_{2p-2},$$

or

$$U_{2p} = c_{2p}U_{2p-1} + U_{2p-2}.$$

This is the first relation (6.13) with $k = 2p$. The second relation (6.13) with $k = 2p$ results if we replace the A_m by the B_m in the preceding. To verify (6.13) for odd values of k, we have:

$$\begin{aligned} &(1 + b_{2n+2p+1}s_{n+p})(A_{2n+2p+2} - s_{n+p+1}A_{2n+2p+1}) \\ &\quad = (1 + b_{2n+2p+1}s_{n+p}) \\ &\qquad (A_{2n+2p+2} - b_{2n+2p+2}A_{2n+2p+1} - s_{n+p}A_{2n+2p+1}) \\ &\quad = (1 + b_{2n+2p+1}s_{n+p})(A_{2n+2p} - s_{n+p}A_{2n+2p+1}) \\ &\quad = A_{2n+2p} - s_{n+p}(A_{2n+2p+1} - b_{2n+2p+1}A_{2n+2p}) \\ &\qquad - b_{2n+2p+1}s_{n+p}{}^2A_{2n+2p+1} \\ &\quad = - b_{2n+2p+1}s_{n+p}{}^2A_{2n+2p+1} + (A_{2n+2p} - s_{n+p}A_{2n+2p-1}). \end{aligned}$$

Thus,

$$\frac{U_{2p+1}}{\pi_{p-1}} = -b_{2n+2p+1}s_{n+p}{}^2\pi_p U_{2p} + \frac{U_{2p-1}}{\pi_{p-1}},$$

or

$$U_{2p+1} = c_{2p+1}U_{2p} + U_{2p-1}.$$

This is the first relation (6.13) for $k = 2p + 1$. The second results if we replace the A_m by the B_m in the preceding.

If the series $\Sigma|\,b_p\,|$ converges, then the hypothesis of Theorem 6.2 is clearly satisfied, and $\lim s_p$ exists and is finite. Hence, Theorem 6.1 is contained in Theorem 6.2.

Let A_p' and B_p' denote the pth numerator and denominator of the continued fraction

$$\cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}}, \tag{6.18}$$

obtained from (6.1) by the removal of the first partial quotient. Then it is easy to see that

$$\begin{aligned} A_{p+1} &= B_p', \\ B_{p+1} &= b_1 B_p' + A_p', \end{aligned} \qquad p = 0, 1, 2, \cdots.$$

Hence, we conclude, on applying Theorem 6.2 to the continued fraction (6.18), that the continued fraction (6.1) diverges if the series

$$\Sigma|\,b_{2p+2}\,|$$

and

$$\Sigma|\,b_{2p+2}t_p^{\,2}\,|, \quad \text{where} \quad t_p = b_3 + b_5 + \cdots + b_{2p+1},$$

converge and if, moreover, the sequence $\{t_p\}$ has a finite limit-point. In this case, the sequences $\{A_{2p}\}$ and $\{B_{2p}\}$ have finite limits, and A_{2p+1} and B_{2p+1} converge to finite limits as $p \to \infty$ over a set of indices for which t_p has a finite limit.

On combining this result with Theorem 6.2, we obtain the following refinement of von Koch's theorem.

THEOREM 6.3. *If the series Σb_{2p} and Σb_{2p+1} converge, and at least one of them is absolutely convergent,*[6] *then the continued fraction* (6.1) *is divergent. The sequences $\{A_{2p}\}$, $\{B_{2p}\}$, $\{A_{2p+1}\}$, $\{B_{2p+1}\}$ of its even and odd numerators and denominators have finite limits F_0, F_1, G_0, G_1, respectively, where*

$$F_1 G_0 - F_0 G_1 = 1.$$

7. A Sufficient Condition for Convergence. It follows from the proof of Theorem 6.2 that if the series (6.4) and (6.5) are con-

[6] One may raise the question as to whether or not the simple convergence of the series Σb_p is sufficient for the divergence of the continued fraction.

vergent and $\lim s_p = \infty$, then the sequence of approximants of the continued fraction (6.1) has the limit F_1/G_1. If this is finite, i.e., if $G_1 \neq 0$, then the continued fraction is convergent. In order to insure that $G_1 \neq 0$, we need to impose some additional restrictions upon the b_p. We have the following theorem.

THEOREM 7.1. *Let the partial denominators b_p of the continued fraction* (6.1) *be complex numbers such that*

$$\Re(b_1) > 0, \quad \Re(b_p) \geq 0, \quad p = 2, 3, 4, \cdots, \tag{7.1}$$

and such that the series (6.4) *and* (6.5) *are convergent, while* $\lim s_p = \infty$. *Then, the continued fraction converges, and its value, v, satisfies the inequality*

$$\left| v - \frac{1}{2\Re(b_1)} \right| \leq \frac{1}{2\Re(b_1)}. \quad \text{[88a.]} \tag{7.2}$$

Proof. Consider the linear transformations (cf. § 1)

$$t = t_p(w) = \frac{1}{b_p + w} = \frac{\bar{b}_p + \bar{w}}{|b_p + w|^2},$$

of the variable w into the variable t. By (7.1), it follows that $t = t_p(w)$ maps the right half-plane $\Re(w) \geq 0$ into the right half-plane $\Re(t) \geq 0$. In particular, $t = t_1(w)$ maps $\Re(w) \geq 0$ upon the circular region

$$\left| t - \frac{1}{2\Re(b_1)} \right| \leq \frac{1}{2\Re(b_1)}. \tag{7.3}$$

It follows immediately that the transformation $t = t_1 t_2 \cdots t_p(w)$ maps $\Re(w) \geq 0$ into this same circular region. Therefore, since $t_1 t_2 \cdots t_p(0) = A_p/B_p$, the pth approximant of the continued fraction, it follows that

$$\left| \frac{A_p}{B_p} \right| \leq \frac{1}{\Re(b_1)}, \quad (p = 1, 2, 3, \cdots).$$

Since, as we have seen, $\lim (A_p/B_p) = F_1/G_1$, a finite number or ∞, provided the series (6.4) and (6.5) converge and $\lim s_p = \infty$, we now conclude that this limit is finite. Inasmuch as $t = A_p/B_p$ satisfies (7.3), it follows that the value v of the continued fraction satisfies (7.2).

This completes the proof of Theorem 7.1.

In a later chapter (Chapter VI) we shall find that if the b_p are restricted still further, then the continued fraction converges if at least one of the series (6.4) or (6.5) is divergent.

It is important to note that the hypothesis (7.1) in Theorem 7.1 can be weakened somewhat, as follows:

$$b_1 = b_3 = \cdots = b_{2n-1} = 0, \quad \Re(b_{2n+1}) > 0,$$
$$\Re(b_p) \geq 0 \quad \text{for} \quad p > 2n+1. \tag{7.4}$$

In fact, the continued fraction (6.1) and the continued fraction

$$b_2 + b_4 + \cdots + b_{2n} + \cfrac{1}{b_{2n+1} + \cfrac{1}{b_{2n+2} + \cfrac{1}{b_{2n+3} + \cdots}}}$$

have all but a finite number of their approximants in common when (7.4) holds. Moreover, it is easy to see that the series (6.4) and (6.5) formed for the latter continued fraction converge if, and only if, they converge for the continued fraction (6.1) and the sequences $\{s_p\}$ for the two continued fraction tend to ∞ together. Hence, we have

THEOREM 7.2. *Let the partial denominators b_p of the continued fraction* (6.1) *be complex numbers such that* (7.4) *holds, and such that the series* (6.4) *and* (6.5) *are convergent, while* $\lim s_p = \infty$. *Then, the continued fraction converges.*

8. Convergence of Periodic Continued Fractions. The continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_k}{b_k + \cfrac{a_1}{b_1 + \cdots + \cfrac{a_k}{b_k + \cfrac{a_1}{b_1 + \cdots}}}}}}, \quad (k \geq 1), \tag{8.1}$$

is called a **periodic continued fraction.** We shall suppose that $a_1, a_2, \cdots, a_k$ are different from zero, and shall obtain necessary and sufficient conditions for convergence and the value of the continued fraction in case it converges. (See Exercise 1.4 for the case $k = 1$.) The method followed here is that of Lane [**48**]. For other proofs cf. Stolz [**96**], Pringsheim [**74**], Perron [**69, 70**], and Schwerdtfeger [**84a**]. An interesting application of periodic continued fractions to "pseudo-elliptic" integrals was given by Abel [**1**].

We shall regard the periodic continued fraction as being generated by the linear fractional transformation

$$s = s(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_k}{b_k + w}}}. \tag{8.2}$$

This may be written in the form (cf. § 1)

$$s = s(w) = \frac{A_{k-1}w + A_k}{B_{k-1}w + B_k}, \tag{8.3}$$

where A_p and B_p denote, as usual, the pth numerator and aenominator of the continued fraction. The determinant of this transformation is $A_{k-1}B_k - A_kB_{k-1} = (-1)^k a_1 a_2 \cdots a_k$, and is different from zero by assumption.

The **fixed points** of the transformation are, by definition, points x such that

$$x = \frac{A_{k-1}x + A_k}{B_{k-1}x + B_k},$$

and are, in general, the two roots x_1 and x_2 of the quadratic equation

$$B_{k-1}x^2 + (B_k - A_{k-1})x - A_k = 0.$$

If x_1 and x_2 are finite, so that $f = s(\infty) = A_{k-1}/B_{k-1}$ is finite, then the transformation $s = s(w)$ can be written as [**65**, § 17]

$$\frac{1}{s - x_1} = \frac{1}{w - x_1} + \frac{1}{f - x_1}, \quad \text{if} \quad x_1 = x_2,$$

and as

$$\frac{s - x_1}{s - x_2} = \frac{f - x_1}{f - x_2} \cdot \frac{w - x_1}{w - x_2}, \quad \text{if} \quad x_1 \neq x_2.$$

The transformation $s = s^n(w)$ obtained by iteration of the transformation $s = s(w)$ n times can then be written as

$$\frac{1}{s - x_1} = \frac{1}{w - x_1} + \frac{n}{f - x_1}, \quad \text{if} \quad x_1 = x_2, \tag{8.4}$$

and as

$$\frac{s - x_1}{s - x_2} = \left(\frac{f - x_1}{f - x_2}\right)^n \cdot \frac{w - x_1}{w - x_2}, \quad \text{if} \quad x_1 \neq x_2. \tag{8.5}$$

We are now prepared to prove the following theorem.

THEOREM 8.1. *Let x_1 and x_2 be the fixed points of the transformation* (8.2), *where $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ are any complex numbers, the a_p being different from zero. Let F_n be the* nth *approximant of the periodic continued fraction* (8.1). *Then,* (8.1) *converges if and only if x_1 and x_2 are finite numbers satisfying one of the following two conditions:*

$$x_1 = x_2, \tag{8.6}$$

or

$$|F_{k-1} - x_2| > |F_{k-1} - x_1|, \quad F_p \neq x_2,$$
$$p = 0, 1, 2, \cdots, k - 1. \tag{8.7}$$

If the continued fraction converges, its value is x_1.

Proof. We observe that

$$F_{nk+p} = s^n(F_p), \quad p = 0, 1, 2, \cdots, k - 1,$$
$$n = 1, 2, 3, \cdots. \tag{8.8}$$

From this it follows that $F_p = x_1$ if and only if $F_{nk+p} = x_1$. We have to consider four cases.

Case 1. The point at ∞ is a fixed point of $s(w)$. From (8.3) and the determinant formula we see that in this case $B_{k-1} = 0$, $A_{k-1} \neq 0$, so that $F_{k-1} = \infty$. Hence, by (8.8), $F_{nk+k-1} = \infty$, $n = 1, 2, 3, \cdots$, so that the continued fraction diverges.

Case 2. The fixed points x_1 and x_2 are finite, and $x_1 = x_2$. In this case, we have, by (8.4) and (8.8),

$$\frac{1}{F_{nk+p} - x_1} = \frac{1}{F_p - x_1} + \frac{n}{F_{k-1} - x_1}. \tag{8.9}$$

Here, $F_{k-1} = s(\infty)$ is finite and different from x_1. If $F_p = x_1$, then we know that $F_{nk+p} = x_1$. If $F_p \neq x_1$, then $F_{nk+p} \neq x_1$, but, on allowing n to increase to ∞ in (8.9) we see that

$$\lim_{n=\infty} F_{nk+p} = x_1.$$

The continued fraction is therefore convergent and its value is x_1.

Case 3. The fixed points x_1 and x_2 are finite and $|F_{k-1} - x_2| > |F_{k-1} - x_1|$. In this case we have, by (8.5) and (8.8),

$$\frac{F_{nk+p} - x_1}{F_{nk+p} - x_2} = K^n \cdot \frac{F_p - x_1}{F_p - x_2}, \quad \text{where} \quad K = \frac{F_{k-1} - x_1}{F_{k-1} - x_2}. \qquad (8.10)$$

By hypothesis, $|K| < 1$. If $F_p \neq x_2$, then (8.10) shows that

$$F_{nk+p} - x_1 = \epsilon_n(F_{nk+p} - x_2),$$

where $\lim \epsilon_n = 0$. Thus, if $|\epsilon_n| < 1$,

$$F_{nk+p} - x_1 = \frac{\epsilon_n(x_1 - x_2)}{1 - \epsilon_n},$$

so that $\lim_{n=\infty} F_{nk+p} = x_1$. On the other hand, if $F_p = x_2$, then $F_{nk+p} = s^n(F_p) = s^n(x_2) = x_2$. Since F_{k-1} is not a fixed point of the transformation, then $F_{k-1} \neq x_2$ and, by the preceding,

$$\lim_{n=\infty} F_{nk+k-1} = x_1 \neq x_2.$$

The continued fraction therefore diverges by oscillation.

Case 4. The fixed points x_1 and x_2 are finite, $x_1 \neq x_2$, and $|F_{k-1} - x_1| = |F_{k-1} - x_2|$. In this case the equation (8.10) applies, where now $|K| = 1$, $K \neq 1$. The sequence $K, K^2, K^3, \cdots$ must have at least two different limit-points. On putting $p = k - 1$ in (8.10), we then conclude that the sequence F_{nk-1}, $n = 1, 2, 3, \cdots$, must have at least two different limit-points, and again the continued fraction diverges.

Since these four cases include all possibilities, the proof of the theorem is complete.

In the special case where $k = 1$ in (8.1), the transformation $s = s(w)$ is

$$s = \frac{a}{1 + w}, \quad (a = a_1).$$

Its fixed points are

$$\frac{-1 \pm (1 + 4a)^{1/2}}{2}.$$

By Theorem 8.1, the continued fraction converges when $a \neq 0$ if, and only if, these fixed points are equal to one another or else have unequal moduli. When $a = 0$, one may verify directly that the continued fraction converges by computing its approximants. On adding 1 to the continued fraction we then have the following theorem.

THEOREM 8.2. *The continued fraction*

$$1 + \cfrac{a}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \cdots}}} \tag{8.11}$$

converges excepting when $a = -\frac{1}{4} - c$, *where* c *is real and positive. When convergent, its value is* $\frac{1}{2}$ *if* $a = -\frac{1}{4}$, *and is equal to the one of the quantities*

$$\frac{1 \pm (1 + 4a)^{1/2}}{2}$$

having the larger modulus if $a \neq -\frac{1}{4}$.

EXERCISE 2

2.1. Let $z = x + iy$, $\bar{z} = x - iy$, where x and y are real. The continued fraction

$$\cfrac{1}{1 + \cfrac{z}{1 + \cfrac{\bar{z}}{1 + \cfrac{z}{1 + \cfrac{\bar{z}}{1 + \cdots}}}}}$$

whose partial numerators after the first are $z, \bar{z}, z, \bar{z}, z, \bar{z}, z, \cdots$, converges if, and only if, $y^2 \leq x + \frac{1}{4}$, i.e., if, and only if,

$$|z| - \Re(z) \leq \tfrac{1}{2}.$$

2.2. As $z = x + iy$ ranges over the parabola $y^2 = x + \frac{1}{4}$, then the value v of the continued fraction of 2.1 ranges over the circle $(x - 1)^2 + y^2 = 1$.

2.3. Show that the odd part of the continued fraction of 2.1 diverges if $y^2 > x + \frac{1}{4}$.

2.4. Show that the continued fraction (5.2) diverges if $a_p \neq 0$, and there exists a constant $k > 1$ such that $|a_{n+1}/a_n| \geq k$ for all n sufficiently large [52].

Chapter III

CONVERGENCE OF CONTINUED FRACTIONS WHOSE PARTIAL DENOMINATORS ARE EQUAL TO UNITY

In this chapter we shall obtain convergence criteria for continued fractions whose partial quotients are of the form

$$\frac{a_p}{1}.$$

It should be emphasized that any continued fraction whose partial denominators are not zero can be thrown into this form by means of an equivalence transformation (cf. § 3). The basis of the method used here is simply the comparison test for infinite series. We seek to find a majorant series for the series $1 + \Sigma\rho_1\rho_2 \cdots \rho_p$ of Theorem 2.1.

9. The First Interpretation of the Fundamental Inequalities. The continued fraction

$$\cfrac{1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdots}}} \tag{9.1}$$

is said to satisfy the **fundamental inequalities** if there exist numbers $r_n \geq 0$, such that

$$r_p|\,1 + a_p + a_{p+1}\,| \geq r_p r_{p-2}|\,a_p\,| + |\,a_{p+1}\,|, \quad p = 1, 2, 3, \ldots, \tag{9.2}$$

where we shall put

$$a_1 = 0, \quad r_0 = 0, \quad r_{-1} = 0.$$

THEOREM 9.1. *If the continued fraction* (9.1) *satisfies the fundamental inequalities* (9.2), *then its denominators* B_p *are different from* 0, *and the numbers*

$$\rho_p = \frac{-a_{p+1}B_{p-1}}{B_{p+1}}$$

satisfy the inequalities

$$|\rho_p| \leq r_p, \quad p = 1, 2, 3, \cdots \quad [86.] \tag{9.3}$$

Proof. By (9.2) for $p = 1, 2$, we have

$$r_1|1 + a_2| \geq |a_2|, \quad r_2|1 + a_2 + a_3| \geq |a_3|.$$

Therefore, $B_2 = 1 + a_2 \neq 0$, $B_3 = 1 + a_2 + a_3 \neq 0$, and

$$|\rho_1| = \left|\frac{a_2}{1 + a_2}\right| \leq r_1, \quad |\rho_2| = \left|\frac{a_3}{1 + a_2 + a_3}\right| \leq r_2.$$

Using induction, we now suppose that $B_{p+1} \neq 0$, $|\rho_p| \leq r_p$, for $p = 1, 2, 3, \ldots, k$, where $k \geq 2$, and shall establish them for $p = k + 1$. We have to distinguish two cases according as $a_{k+2} = 0$ or $a_{k+2} \neq 0$. If $a_{k+2} = 0$, then, by the fundamental recurrence formulas, $B_{k+2} = B_{k+1} + a_{k+2}B_k = B_{k+1}$, which is different from zero by hypothesis; and

$$|\rho_{k+1}| = \left|\frac{a_{k+2}B_k}{B_{k+2}}\right| = 0 \leq r_{k+1}.$$

If, on the other hand, $a_{k+2} \neq 0$, then it follows from (9.2), with $p = k + 1$, that $r_{k+1} > 0$. Moreover, from the fundamental recurrence formulas we obtain

$$B_{k+2} = (1 + a_{k+1} + a_{k+2})B_k - a_k a_{k+1} B_{k-2},$$

so that, by the hypothesis of the induction and (9.2),

$$\left|\frac{B_{k+2}}{a_{k+2}B_k}\right| = \left|\frac{1 + a_{k+1} + a_{k+2}}{a_{k+2}} - \frac{a_{k+1}}{a_{k+2}} \cdot \frac{a_k B_{k-2}}{B_k}\right|$$

$$\geq \left|\frac{1 + a_{k+1} + a_{k+2}}{a_{k+2}}\right| - \left|\frac{a_{k+1}}{a_{k+2}}\right| r_{k-1} \geq \frac{1}{r_{k+1}} > 0.$$

Therefore, $B_{k+2} \neq 0$ and $|\rho_{k+1}| \leq r_{k+1}$. This completes the induction and the proof of Theorem 9.1.

When the continued fraction (9.1) satisfies the fundamental inequalities (9.2), then the series $1 + \Sigma r_1 r_2 \cdots r_p$ is a majorant for the series

$$\sum_{p=1}^{\infty}\left(\frac{A_p}{B_p} - \frac{A_{p-1}}{B_{p-1}}\right) = 1 + \Sigma \rho_1 \rho_2 \cdots \rho_p,$$

which is equivalent to the continued fraction (cf. § 2). That is,

$$\left|\frac{A_{p+1}}{B_{p+1}} - \frac{A_p}{B_p}\right| = |\rho_1\rho_2 \cdots \rho_p| \leq r_1 r_2 \cdots r_p, \quad p = 1, 2, 3, \cdots. \tag{9.4}$$

This is what we shall call the **first interpretation** of the fundamental inequalities.

From Theorems 9.1 and 5.1 we have

THEOREM 9.2. *If the continued fraction* (9.1) *satisfies the fundamental inequalities, and if some partial numerator a_p vanishes, then the continued fraction converges.*

10. Worpitzky's Theorem. In what appears to be the earliest published paper treating of the convergence of continued fractions with complex elements (cf. Van Vleck [**112**, p. 147] and Szász [**98**, p. 160] for comments), Worpitzky [**143**] showed that the continued fraction (9.1) converges if the partial numerators a_2, a_3, a_4, $\cdots$ all have moduli less than $\frac{1}{4}$.

We shall now prove

THEOREM 10.1. *Let $a_2, a_3, a_4, \cdots$ be functions of any variables over a domain D in which*

$$|a_{p+1}| \leq \tfrac{1}{4}, \quad p = 1, 2, 3, \cdots. \tag{10.1}$$

Then the following statements hold.

(a) *The continued fraction* (9.1) *converges uniformly over D.*

(b) *The values of the continued fraction and of its approximants are in the circular domain*

$$|z - \tfrac{4}{3}| \leq \tfrac{2}{3}. \tag{10.2}$$

(c) *The constant $\frac{1}{4}$ is the "best" constant that can be used in* (10.1), *and* (10.2) *is the "best" domain of values of the approximants.*

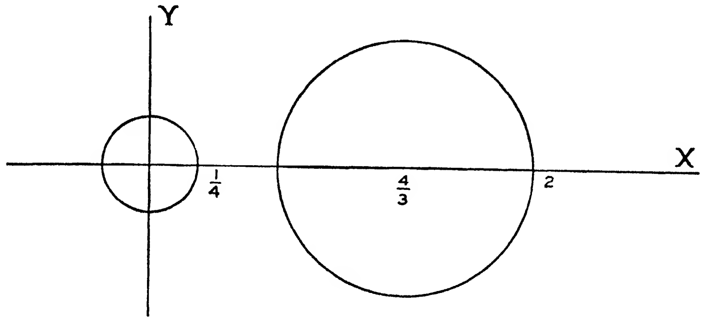

FIG. 1.

Proof. (a) By (10.1),

$$\tfrac{1}{3}|\,1 + a_2\,| \geq \tfrac{1}{3}(1 - \tfrac{1}{4}) = \tfrac{1}{4} \geq |\,a_2\,|,$$

$$\tfrac{2}{4}|\,1 + a_2 + a_3\,| \geq \tfrac{2}{4}(1 - \tfrac{1}{4} - \tfrac{1}{4}) = \tfrac{1}{4} \geq |\,a_3\,|,$$

$$\frac{p}{p+2}|\,1 + a_p + a_{p+1}\,| \geq \frac{p}{2(p+2)} = \frac{p(p-2)}{(p+2)p}\cdot\frac{1}{4} + \frac{1}{4}$$

$$\geq \frac{p}{p+2}\cdot\frac{p-2}{p}|\,a_p\,| + |\,a_{p+2}\,|, \quad p = 3, 4, 5, \cdots.$$

Consequently, the continued fraction satisfies the fundamental inequalities with $r_p = p/(p+2)$, $p = 1, 2, 3, \cdots$. Since

$$1 + \Sigma r_1 r_2 \cdots r_p = 1 + \Sigma\left(\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdots\frac{p}{p+2}\right)$$

$$= 1 + \Sigma\frac{2}{(p+1)(p+2)}$$

$$= 1 + 2\sum_{p=1}^{\infty}\left(\frac{1}{p+1} - \frac{1}{p+2}\right) = 2,$$

we therefore conclude that the continued fraction converges uniformly for $|\,a_{p+1}\,| \leq \frac{1}{4}$, $p = 1, 2, 3, \cdots$, and that the modulus of its value does not exceed 2. Inasmuch as the a_p can have the value 0, it follows that the moduli of the values of the approximants do not exceed 2.

(b) We now write the continued fraction in the form

$$z = \frac{1}{1 + \frac{1}{4}w}, \tag{10.3}$$

where

$$w = \cfrac{x_1}{1 + \cfrac{a_3}{1 + \cfrac{a_4}{1 + \cdots}}}, \qquad |x_1| \leq 1.$$

It is clear that $|w| \leq 2$. Any approximant of the continued fraction can, of course, be written in the form (10.3). Then

$$|w| = 4\left|\frac{z-1}{z}\right| \leq 2,$$

or

$$|z - \tfrac{4}{3}| \leq \tfrac{2}{3},$$

which is (10.2).

(c) To see that $\frac{1}{4}$ is the "best" constant that can be used in (10.1), we need but note that if $c > \frac{1}{4}$, then the continued fraction diverges if $a_{p+1} = -c$, $p = 1, 2, 3, \cdots$, by Theorem 8.2. To show that (10.2) is the "best" domain, it suffices to note that the values of the particular continued fraction

$$z = \cfrac{1}{1 + \cfrac{a_2}{1 - \cfrac{(\frac{1}{4})}{1 - \cfrac{(\frac{1}{4})}{1 - \cdots}}}} = \frac{1}{1 + 2a_2}$$

fill the domain (10.2) as a_2 ranges over the domain $|a_2| \leq \frac{1}{4}$.

The statements (b) and (c) were proved by Paydon and Wall [**68**]. They showed, more generally, that when $|a_{p+1}| \leq t(1-t)$, $0 < t \leq \frac{1}{2}$, then the values z of the continued fraction fill the domain (cf. § 38)

$$\left|z - \frac{1}{1-t^2}\right| \leq \frac{t}{1-t^2}.$$

DEFINITION 10.1. *A set of points D of the complex plane will be called a convergence set for the continued fraction* (9.1) *if the continued fraction converges for all values of the partial numerators* a_2, a_3, a_4, $\cdots$ *in D.*

Theorem 10.1 shows that the largest circular neighborhood of the origin, with center at the origin, which is a convergence set for the continued fraction (9.1) is the one with radius $\frac{1}{4}$. The question naturally arises as to whether or not there are larger neighborhoods which are not circular which are convergence sets. This question will be considered in § 14. Another question which arises is the following: Given a point a in the complex plane, does there exist a neighborhood of a which is a convergence set for (9.1)? This question will be considered in §§ 15 and 32.

An important type of continued fraction is

$$\cfrac{1}{1+\cfrac{c_1z}{1+\cfrac{c_2z}{1+\cdots}}},$$

where c_1, c_2, c_3, $\cdots$ are constants and z is a complex variable. From Theorem 10.1 and a well-known theorem of Weierstrass, it follows that if $|c_p| \le M$, $p = 1, 2, 3, \cdots$, then this continued fraction converges uniformly and represents an analytic function of z for $|z| < 1/4M$ (cf. § 54).

11. Convergence of Continued Fractions Whose Partial Quotients Are of the Form

$$\frac{(1 - g_{p-1})g_px_p}{1}.$$

The continued fraction of Theorem 10.1 has partial quotients which can be written in the above form with $g_n = \frac{1}{2}$, $|x_p| \le 1$. We shall now prove this theorem.

THEOREM 11.1. *Let* $g_1, g_2, g_3, \cdots$ *be constants which satisfy one or the other of the conditions*

$$0 \le g_p < 1, \quad p = 1, 2, 3, \cdots, \tag{11.1}$$

or

$$0 < g_p \le 1, \quad p = 1, 2, 3, \cdots. \tag{11.2}$$

Then, the following three statements hold.

(a) *The continued fraction*

$$\cfrac{g_1}{1+\cfrac{(1-g_1)g_2x_2}{1+\cfrac{(1-g_2)g_3x_3}{1+\cdot_{\cdot_{\cdot}}}}} \tag{11.3}$$

converges uniformly for $|x_p| \leq 1$, $p = 2, 3, 4, \cdots$.

(b) *The values of the continued fraction and of its approximants are in the circular domain*

$$|z| \leq 1 - \frac{1}{S}, \tag{11.4}$$

where

$$S = 1 + \sum_{p=1}^{\infty} \frac{g_1 g_2 \cdots g_p}{(1-g_1)(1-g_2)\cdots(1-g_p)}, \tag{11.5}$$

(*possibly* ∞). *The value of the continued fraction for* $x_p = -1$, $p = 2, 3, 4, \cdots$, *is* $1 - \dfrac{1}{S}$.

(c) *The values of the continued fraction and of its approximants are in the circular domain*

$$\left| z - \frac{1}{2-g_1} \right| \leq \frac{1-g_1}{2-g_1}. \tag{11.6}$$

Proof. Consider the continued fraction

$$\cfrac{g_1}{1-\cfrac{(1-g_1)g_2}{1-\cfrac{(1-g_2)g_3}{1-\cdot_{\cdot_{\cdot}}}}} \tag{11.7}$$

For the sake of simplicity, let $\theta_{p+1} = (1 - g_p)g_{p+1}$, $p = 1, 2, 3, \cdots$. We denote by P_n and Q_n the nth numerator and denominator of (11.7). These satisfy the recurrence formulas

$$\begin{aligned} P_n &= P_{n-1} - \theta_n P_{n-2}, \\ Q_n &= Q_{n-1} - \theta_n Q_{n-2} \end{aligned} \quad n = 2, 3, 4, \cdots,$$

$$P_0 = 0, \quad P_1 = g_1, \quad Q_0 = 1, \quad Q_1 = 1. \tag{11.8}$$

With the aid of these formulas one may readily verify by mathematical induction that

$$\begin{aligned} Q_n = {} & (1 - g_1)(1 - g_2)(1 - g_3) \cdots (1 - g_n) \\ & + g_1(1 - g_2)(1 - g_3) \cdots (1 - g_n) \\ & + g_1g_2(1 - g_3) \cdots (1 - g_n) + \cdots \qquad (11.9) \\ & + g_1g_2 \cdots g_{n-1}(1 - g_n) + g_1g_2 \cdots g_n, \end{aligned}$$

$$P_n = Q_n - (1 - g_1)(1 - g_2) \cdots (1 - g_n).$$

Hence, it follows that under either of the hypotheses (11.1) or (11.2), $Q_n > 0$, $n = 1, 2, 3, \cdots$. Now, we find by (11.8) that

$$Q_{n+2} = (1 - \theta_{n+1} - \theta_{n+2})Q_n - \theta_n\theta_{n+1}Q_{n-2}, \quad n = 2, 3, 4, \cdots.$$

Hence, if we put

$$r_p = \frac{\theta_{p+1}Q_{p-1}}{Q_{p+1}}, \quad p = 1, 2, 3, \cdots, \qquad (11.10)$$

then

$$\theta_{p+1} = (1 - \theta_p - \theta_{p+1})r_p - r_pr_{p-2}\theta_p.$$

Therefore, if $|\, a_{p+1} \,| \leq \theta_{p+1}$, $p = 1, 2, 3, \cdots$, then

$$\begin{aligned} r_p|\, 1 + a_p + a_{p+1} \,| \geq r_p(1 - \theta_p - \theta_{p+1}) &= r_pr_{p-2}\theta_p + \theta_{p+1} \\ &\geq r_pr_{p-2}|\, a_p \,| + |\, a_{p+1} \,|, \end{aligned}$$

$$p = 1, 2, 3, \cdots, \quad r_0 = r_{-1} = 0, \quad \theta_1 = a_1 = 0.$$

That is, the continued fraction (11.3) satisfies the fundamental inequalities (9.2) if $|\, x_{p+1} \,| \leq 1$, $p = 1, 2, 3, \cdots$, the parameters r_p being given by (11.10). To prove (a), it is therefore sufficient to prove that the series

$$g_1(1 + \Sigma r_1r_2 \cdots r_p)$$

converges. This follows at once from Theorem 2.1, applied to (11.7), inasmuch as

$$g_1(1 + \Sigma r_1r_2 \cdots r_p) = \lim_{n=\infty} \frac{P_n}{Q_n} = 1 - \frac{1}{S},$$

where S is given by (11.5).

The proof of (b) is contained in the preceding.

To prove (c), we may write the value of the continued fraction or of any of its approximants in the form

$$z = \frac{g_1}{1 + (1 - g_1)w},$$

where $|w| \leq 1$. Then, it follows at once that (11.6) holds.
This completes the proof of Theorem 11.1.

The preceding proof is based upon that of Scott and Wall [**86**]. A different proof, with the above formulation, was given by Paydon and Wall [**68**]. Part (a), under the hypothesis $0 < g_p < 1$, $p = 1, 2, 3, \cdots$, was proved by Perron [**69**, p. 262], starting with less a complete result of Pringsheim [**73**]. Perron attributes the theorem to Van Vleck. However, it seems that Perron confused this theorem with the following theorem of Van Vleck [**108**].

THEOREM 11.2. *If* $0 \leq g_p < 1$, $p = 1, 2, 3, \cdots$, *and if the series*

$$S = 1 + \sum_{p=1}^{\infty} \frac{g_1 g_2 \cdots g_p}{(1 - g_1)(1 - g_2) \cdots (1 - g_p)}$$

converges, then the continued fraction

$$\cfrac{1}{1 + \cfrac{g_1 x_1}{1 + \cfrac{(1 - g_1) g_2 x_2}{1 + \cfrac{(1 - g_2) g_3 x_3}{1 + \cdots}}}} \tag{11.11}$$

converges uniformly for $|x_p| \leq 1$, $p = 1, 2, 3, \cdots$, *the modulus of its value does not exceed* S, *and its value for* $x_p = -1$, $p = 1, 2, 3, \cdots$, *is* S.

Proof.[7] The continued fraction (11.11) can be regarded as obtained from (11.3) by multiplying the latter by x_1, adding 1, and then taking the reciprocal. When S is finite, the moduli of (11.3) and its approximants do not exceed $1 - 1/S < 1$. It is therefore obvious that (11.11) converges uniformly for $|x_p| \leq 1$, $p = 1, 2,$

[7] This proof shows that Theorem 11.2 is an easy corollary to Theorem 11.1. One may show, conversely, that Theorem 11.1 is a consequence of Theorem 11.2 (cf. [**68**]).

3, $\cdots$, in this case, and the statements concerning the value of the continued fraction are also at once evident.

The continued fraction (11.11) diverges for $x_p = -1, p = 1, 2, 3, \cdots$, in case $S = \infty$. However, we have this theorem [68].

THEOREM 11.3. *If* $0 < g_p < 1, p = 1, 2, 3, \cdots$, *and the series*

$$1 + \sum_{p=1}^{\infty} \frac{g_1 g_2 \cdots g_p}{(1 - g_1)(1 - g_2) \cdots (1 - g_p)}$$

diverges, then the continued fraction (11.11) *converges for* $|x_p| \leq 1$, $p = 1, 2, 3, \cdots$, *provided* $x_p \neq -1$ *for at least one value of* p.

Proof. Let $x_1, x_2, x_3, \cdots$ have values with moduli not greater than unity, for which (11.11) diverges. Inasmuch as (11.3) converges to a value z_1 with modulus not greater than unity, it is obvious, since (11.11) diverges, that $x_1 z_1 = -1$. Inasmuch as $|x_1| \leq 1$, $|z_1| \leq 1$, then we must have $|x_1| = 1$, $|z_1| = 1$. But, by (11.6), $|z_1| = 1$ only when $z_1 = 1$; and since $x_1 z_1 = -1$, then $x_1 = -1$. We now put

$$z_1 = \frac{g_1}{1 + (1 - g_1)x_2 z_2},$$

where

$$z_2 = \cfrac{g_2}{1 + \cfrac{(1 - g_2)g_3 x_3}{1 + \cfrac{(1 - g_3)g_4 x_4}{1 + \cdots}}}$$

Since $z_1 = 1$, this shows that $x_2 z_2 = -1$. On repeating the argument used before, we conclude that $z_2 = 1, x_2 = -1$. The proof may now be completed by mathematical induction.

If $|x_1| \leq r < 1$, $|x_p| \leq 1, p = 2, 3, 4, \cdots$, then (11.11) evidently converges uniformly inasmuch as (11.3) converges uniformly and has modulus not greater than unity. Hence we have

THEOREM 11.4. *If* $0 \leq g_p < 1$, $p = 1, 2, 3, \cdots$, *or* $0 < g_p \leq 1$, $p = 1, 2, 3, \cdots$, *then the continued fraction* (11.11) *converges uniformly for* $|x_1| \leq r$, $|x_p| \leq 1$, $p = 2, 3, 4, \cdots$, *for every constant* r *less than unity.*

Interesting cases of the preceding theorems result by specializing the g_p in different ways. We mention, in particular, the cases $g_p = \frac{1}{2}$ and $g_p = p/(2p + 1)$.

We may of course apply the theorems of this chapter to continued fractions of the form

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}} \tag{11.12}$$

if we first reduce the partial denominators to unity by means of an equivalence transformation. For instance, (11.12) converges uniformly if there exist constants g_p such that

$$\left|\frac{a_{p+1}}{b_p b_{p+1}}\right| \le (1 - g_p)g_{p+1}, \quad 0 < g_p < 1, \quad p = 1, 2, 3, \cdots.$$

If we put $g_n = 1 - (1/p_n)$, this condition takes the form preferred by Perron [69], namely:

$$\left|\frac{a_{n+1}}{b_n b_{n+1}}\right| \le \frac{p_{n+1} - 1}{p_n p_{n+1}}, \quad p_n > 1, \quad n = 1, 2, 3, \cdots.$$

An interesting special case is obtained by taking

$$a_{n+1} = 1, \quad p_{2n-1} = p_{2n} = |b_{2n}|, \quad n = 1, 2, 3, \cdots.$$

The continued fraction

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

is therefore convergent if

$$\left|\frac{1}{b_{2n-1}}\right| + \left|\frac{1}{b_{2n}}\right| \le 1, \quad n = 1, 2, 3, \cdots.$$

This criterion goes back to Pringsheim [73].

12. A Convergence Theorem of von Koch. If the series Σa_p of partial numerators of the continued fraction (9.1) is absolutely convergent, then it is easy to show, by means of the same type of argument which we used in establishing the limits in Theorem

6.1, that the sequences $\{A_p\}$ and $\{B_p\}$ of numerators and denominators of the continued fraction converge to finite limits A and B, respectively. Therefore, if $B \neq 0$, the continued fraction converges, and A/B is its value. H. von Koch [**116**] showed that the continued fraction converges if infinitely many of the a_p are different from zero and if $\Sigma | a_p | < 1$. Szász [**99**, **100**] and Verbeek [**113**] have contributed to this problem (cf. Perron [**69**, p. 259], footnote). The result of Szász will be considered later on (§ 35). The following somewhat improved form of von Koch's theorem (cf. Dennis and Wall [**9**]) will now be established by means of Theorems 11.1, 11.2 and 11.3.

THEOREM 12.1. *The continued fraction* (9.1) *converges if*

$$\sum_{p=1}^{n} | a_{p+1} | < 1, \quad n = 1, 2, 3, \cdots. \tag{12.1}$$

We shall first prove the following lemma.

Lemma 12.1. *Let $c_1, c_2, c_3, \cdots$ be nonnegative real numbers such that*

$$\sum_{p=1}^{n} c_p < 1, \quad n = 1, 2, 3, \cdots. \tag{12.2}$$

Then, there exist numbers g_p such that $0 \leq g_p < 1, p = 1, 2, 3, \cdots$, and such that

$$c_1 = g_1, \quad c_p = (1 - g_{p-1})g_p, \quad p = 2, 3, 4, \cdots.$$

Since $c_p \geq 0$, $p = 1, 2, 3, \cdots$, it follows from (12.2) that $0 \leq c_1 < 1$. Hence, $g_1 = c_1$ has the required property. Let us assume that $g_1, g_2, \cdots, g_k$ have been determined. Then

$$\begin{aligned}
c_{k+1} &< 1 - c_1 - c_2 \cdots - c_k \\
&= 1 - g_1 - (1 - g_1)g_2 - (1 - g_2)g_3 - \cdots - (1 - g_{k-1})g_k \\
&= (1 - g_1)(1 - g_2) - (1 - g_2)g_3 - \cdots - (1 - g_{k-1})g_k \\
&\quad \leq (1 - g_2) - (1 - g_2)g_3 - (1 - g_3)g_4 - \cdots - (1 - g_{k-1})g_k \\
&= (1 - g_2)(1 - g_3) - (1 - g_3)g_4 - \cdots - (1 - g_{k-1})g_k \leq \cdots \\
&= (1 - g_{k-2})(1 - g_{k-1}) - (1 - g_{k-1})g_k \\
&\quad \leq (1 - g_{k-1}) - (1 - g_{k-1})g_k \\
&= (1 - g_{k-1})(1 - g_k) \leq (1 - g_k).
\end{aligned}$$

Consequently, there exists a number g_{k+1} such that $0 \leq g_{k+1} < 1$, and such that

$$c_{k+1} = (1 - g_k)g_{k+1}.$$

The lemma now follows by mathematical induction.

By (12.1) and the lemma we may now write

$$a_2 = g_1 x_1, \quad a_{p+1} = (1 - g_{p-1})g_p x_p, \quad p = 1, 2, 3, \cdots,$$

where $0 \leq g_p < 1$, and $|x_p| = 1$, $(p = 1, 2, 3, \cdots)$. Hence, if one or more of the a_p vanishes, then some g_p vanishes, and hence the series (11.5) converges. The convergence of the continued fraction in this case then follows from Theorem 11.2.

If $a_p \neq 0$, $p = 2, 3, 4, \cdots$, let

$$g_{n+1} = 1 - \sum_{p=n+2}^{\infty} |a_p|, \quad n = 0, 1, 2, \cdots,$$

so that $0 \leq g_1 < 1$, $0 < g_p < 1$, $p = 1, 2, 3, \cdots$. Then

$$(1 - g_n)g_{n+1} = \left(\sum_{p=n+1}^{\infty} |a_p|\right)\left(1 - \sum_{p=n+2}^{\infty} |a_p|\right)$$

$$= |a_{n+1}| + \left(\sum_{p=n+2}^{\infty} |a_p|\right)\left(1 - \sum_{p=n+1}^{\infty} |a_p|\right),$$

and, consequently,

$$(1 - g_1)g_2 \geq |a_2|,$$

$$(1 - g_n)g_{n+1} > |a_{n+1}| \quad \text{for} \quad n > 1.$$

Therefore, if we put

$$a_{n+1} = (1 - g_n)g_{n+1}x_{n+1}, \quad n = 1, 2, 3, \cdots,$$

we have $|x_2| \leq 1$, $|x_p| < 1$, $p = 3, 4, 5, \cdots$. The convergence of the continued fraction then follows from Theorem 11.2 or Theorem 11.3 if $g_1 = 0$, and from Theorem 11.1 in case $g_1 > 0$.

13. Second Interpretation of the Fundamental Inequalities. We shall now introduce two parameters k_1 and k_2, greater than or equal to zero, and write the fundamental inequalities (9.2) in the form

$$\begin{aligned} r_1|1 + a_2| &\geq (1 + k_1)|a_2|, \\ r_1|1 + a_2 + a_3| &\geq (1 + k_2)|a_3|, \\ r_p|1 + a_p + a_{p+1}| &\geq r_p r_{p-2}|a_p| + |a_{p+1}|, \quad p = 3, 4, 5, \cdots. \end{aligned} \tag{13.1}$$

We thus write the first two of the inequalities in a form which will enable us to conveniently distinguish two cases, according as $k_1 = k_2 = 0$, or at least one of k_1 or k_2 is positive.

THEOREM 13.1. *Let the continued fraction*

$$\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots}}} \tag{13.2}$$

satisfy the fundamental inequalities (13.1). *If* $k_1 > 0 (k_2 > 0)$, *then the even* (*odd*) *part of the continued fraction converges* [86].

Proof. If some partial numerator vanishes, then the continued fraction, and, of course, its even and odd parts, converge, by Theorem 9.2. We assume, then, that $a_p \neq 0$, $p = 2, 3, 4, \cdots$. Then, it follows from (13.1) that $r_p > 0$, $|1 + a_p + a_{p+1}| > 0$, and $r_p|1 + a_p + a_{p+1}| - |a_{p+1}| > 0$, $(p = 1, 2, 3, \cdots)$, where a_1 is to be taken equal to zero.

Let us suppose that $k_1 > 0$. Put

$$g_p = \frac{r_{2p+1}|1 + a_{2p+1} + a_{2p+2}| - |a_{2p+2}|}{r_{2p+1}|1 + a_{2p+1} + a_{2p+2}|}, \quad p = 1, 2, 3, \cdots.$$

Then, $0 < g_p < 1$. Also, by (13.1),

$$g_1 = \frac{r_3|1 + a_3 + a_4| - |a_4|}{r_3|1 + a_3 + a_4|} \geq \frac{r_1|a_3|}{|1 + a_3 + a_4|}$$

$$\geq \frac{(1 + k_1)|a_2 a_3|}{|(1 + a_2)(1 + a_3 + a_4)|};$$

$$(1 - g_{p-1})g_p \geq \frac{|a_{2p}a_{2p+1}|}{|(1 + a_{2p-1} + a_{2p})(1 + a_{2p+1} + a_{2p+2})|},$$
$$p = 2, 3, 4, \cdots,$$

so that

$$\frac{-a_{2p}a_{2p+1}}{(1 + a_{2p-1} + a_{2p})(1 + a_{2p+1} + a_{2p+2})} = (1 - g_{p-1})g_p x_p,$$
$$p = 1, 2, 3, \cdots,$$

where $g_0 = 0$, $|x_1| \leq 1/(1 + k_1)$, $|x_p| \leq 1$, $p = 2, 3, 4, \cdots$. After making a suitable equivalence transformation upon the even part of (13.2), namely, upon (4.2), we see that the latter is equal to

$$\frac{1}{1+a_2}\cdot\cfrac{1}{1+\cfrac{g_1x_1}{1+\cfrac{(1-g_1)g_2x_2}{1+\cfrac{(1-g_2)g_3x_3}{1+\cdots}}}}$$

This is convergent by Theorem 11.4.

The convergence of the odd part, in case $k_2 > 0$, can be established in a similar way.

Theorem 13.1 states that if the continued fraction (13.2) satisfies the fundamental inequalities, and if actual inequality holds in the first two, then the even and odd parts of the continued fraction converge. This gives us a **second interpretation** of the fundamental inequalities. This will enable us, in certain cases, to establish the convergence of the continued fraction even when the majorant series $1 + \Sigma r_1 r_2 \cdots r_p$, furnished by the **first interpretation** of the fundamental inequalities, is divergent. We shall now proceed to show how this can be done.

THEOREM 13.2. *Let the continued fraction* (13.2) *satisfy the fundamental inequalities* (13.1), *with* $k_1 > 0$, $k_2 > 0$, *and suppose that* $a_p \neq 0$, $p = 2, 3, 4, \cdots$. *Let* $b_1, b_2, b_3, \cdots$ *be defined by the formulas*

$$b_1 = 1, \quad a_{p+1} = \frac{1}{b_p b_{p+1}}, \quad p = 1, 2, 3, \cdots, \tag{13.3}$$

so that the continued fraction is equivalent to

$$\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\cdots}}} \tag{13.4}$$

Let Q_p *be the* pth *denominator of* (13.4), *so that*

$$Q_p = b_1 b_2 \cdots b_p B_p, \tag{13.5}$$

where B_p is the pth *denominator of* (13.2). *Then the following inequalities hold:*

$$r_1r_3 \cdots r_{2p-1}|Q_{2p}| \geq k(1 + r_1|b_2| + r_1^2r_3|b_4| + \cdots + r_1^2r_3^2 \cdots r_{2p-3}^2r_{2p-1}|b_{2p}|),$$

$$r_2r_4 \cdots r_{2p}|Q_{2p+1}| \geq k(1 + r_2|b_3| + r_2^2r_4|b_5| + \cdots + r_2^2r_4^2 \cdots r_{2p-2}^2r_{2p}|b_{2p+1}|), \tag{13.6}$$

$$p = 1, 2, 3, \cdots,$$

where k is a positive constant. (*For $p = 1$, the inequalities are to be interpreted according to* (13.9), *below.*) [86.]

Proof. On making the substitution (13.3) in (13.1), we obtain

$$r_1|1 + b_1b_2| \geq 1 + k_1, \quad r_2|b_1b_2b_3 + b_1 + b_3| \geq 1 + k_2, \tag{13.7}$$

$$r_p|b_{p-1}b_pb_{p+1} + b_{p-1} + b_{p+1}| \geq r_pr_{p-2}|b_{p+1}| + |b_{p-1}|,$$

$$p = 3, 4, 5, \cdots.$$

Inasmuch as $Q_0 = 1$, $Q_1 = b_1 = 1$, $Q_2 = 1 + b_1b_2$, $Q_3 = b_1b_2b_3 + b_1 + b_3$, the first two of these inequalities can be written

$$|Q_2| - \frac{1}{r_1}|Q_0| \geq k|b_2|,$$

$$|Q_3| - \frac{1}{r_2}|Q_1| \geq k|b_3|, \tag{13.8}$$

where k is the smaller of

$$\frac{k_1}{r_1|b_2|}, \quad \frac{k_2}{r_2|b_3|}, \quad \text{and} \quad 1.$$

Thus,

$$r_1|Q_2| \geq k(1 + r_1|b_2|),$$

$$r_2|Q_3| \geq k(1 + r_2|b_3|). \tag{13.9}$$

Now, by (13.7) and the fundamental recurrence formulas, we have

$$\begin{aligned}
&| Q_{p+3} | \\
&\quad \geq | b_{p+1}b_{p+2}b_{p+3} + b_{p+1} + b_{p+3} | \cdot \left| \frac{Q_{p+1}}{b_{p+1}} \right| - \left| \frac{b_{p+3}}{b_{p+1}} \right| \cdot | Q_{p-1} | \\
&\quad \geq \frac{r_{p+2}r_p| b_{p+3} | + | b_{p+1} |}{r_{p+2}} \cdot \left| \frac{Q_{p+1}}{b_{p+1}} \right| - \left| \frac{b_{p+3}}{b_{p+1}} \right| \cdot | Q_{p-1} |,
\end{aligned}$$

so that

$$| Q_{p+3} | - \frac{1}{r_{p+2}} | Q_{p+1} | \geq r_p \left| \frac{b_{p+3}}{b_{p+1}} \right| \left(| Q_{p+1} | - \frac{1}{r_p} | Q_{p-1} | \right).$$

On applying this inequality for $p = 1, 3, 4, \cdots$, and then for $p = 2, 4, 6, \cdots$, and making use of (13.8), we then find that

$$\begin{aligned}
r_{2p-1}| Q_{2p} | &\geq | Q_{2p-2} | + r_1r_3 \cdots r_{2p-1}k| b_{2p} |, \\
r_{2p}| Q_{2p+1} | &\geq | Q_{2p-1} | + r_2r_4 \cdots r_{2p}k| b_{2p+1} |.
\end{aligned} \tag{13.10}$$

From these inequalities and (13.9), the inequalities (13.6) follow.

From the determinant formula and (13.3), (13.5), we find that

$$\left| \frac{A_{p+1}}{B_{p+1}} - \frac{A_p}{B_p} \right| = \frac{1}{| Q_pQ_{p+1} |}. \tag{13.11}$$

In view of Theorem 13.1, this shows that the continued fraction converges if it satisfies the hypothesis of Theorem 13.2 and if

$$\limsup | Q_pQ_{p+1} | = \infty. \tag{13.12}$$

In the next section we shall consider in detail the case where the fundamental inequalities are satisfied with $r_p = 1, p = 1, 2, 3, \cdots$. In this case it follows at once from (13.6) that (13.12) holds if the series $\Sigma| b_p |$ is divergent. *Thus, the continued fraction may converge although the majorant series* $1 + \Sigma r_1r_2 \cdots r_p$ *diverges.*

14. The Parabola Theorem. We shall suppose that the fundamental inequalities (13.1) hold with $r_p = 1, p = 1, 2, 3, \cdots$, and with $k_1 > 0$, $k_2 > 0$, i.e., that

$$\begin{aligned}
| 1 + a_2 | &> | a_2 |, \\
| 1 + a_2 + a_3 | &> | a_3 |, \\
| 1 + a_p + a_{p+1} | &\geq | a_p | + | a_{p+1} |, \quad p = 3, 4, 5, \cdots.
\end{aligned} \tag{14.1}$$

If some a_p vanishes, then the continued fraction converges by Theorem 9.2 (even when $|1 + a_2| = |a_2|$ or $|1 + a_2 + a_3| = |a_3|$). If $a_p \neq 0$, $p = 2, 3, 4, \cdots$, then, as remarked at the end of § 13, the continued fraction converges if the series $\Sigma|b_p|$, given by (13.3), diverges. If this series converges, then the continued fraction diverges by virtue of Theorem 6.1.

Let $m_1, m_2, m_3, \cdots$ be positive numbers such that

$$m_1 < 1, \qquad m_p + m_{p+1} \leq 1, \quad p = 1, 2, 3, \cdots. \tag{14.2}$$

Then, if

$$|a_{p+1}| - \Re(a_{p+1}) \leq m_p, \quad p = 1, 2, 3, \cdots, \tag{14.3}$$

we can show that the inequalities (14.1) are satisfied (except that $|1 + a_2 + a_3|$ may equal $|a_3|$ if $a_2 = 0$). We have, in fact,

$$|1 + a_2| \geq 1 + \Re(a_2) \geq 1 + |a_2| - m_1 > |a_2|,$$

$$\begin{aligned} |1 + a_p + a_{p+1}| &\geq 1 + \Re(a_p) + \Re(a_{p+1}) \\ &\geq 1 - m_{p-1} - m_p + |a_p| + |a_{p+1}| \\ &\geq |a_p| + |a_{p+1}|, \quad p = 2, 3, 4, \cdots. \end{aligned}$$

We therefore have the following theorem.

THEOREM 14.1. *Let $a_2, a_3, a_4, \cdots$ satisfy the inequalities* (14.3), *where $m_1, m_2, m_3, \cdots$ are positive numbers satisfying* (14.2). *Then, the continued fraction*

$$\cfrac{1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdot_{\cdot_{\cdot}}}}} \tag{14.4}$$

converges if, and only if, (a) *some a_p vanishes, or* (b) *$a_p \neq 0$ for $p = 2, 3, 4, \cdots$, and the series $\Sigma|b_p|$, given by* (13.3), *diverges.*

This theorem has a simple geometrical interpretation. The condition (14.3) states, in fact, that for each p, a_{p+1} has its value within or upon the parabola with focus at the origin and vertex at $-m_p/2$. In Figure 2, the parabolas are drawn for $p = 1, 2, 3,$

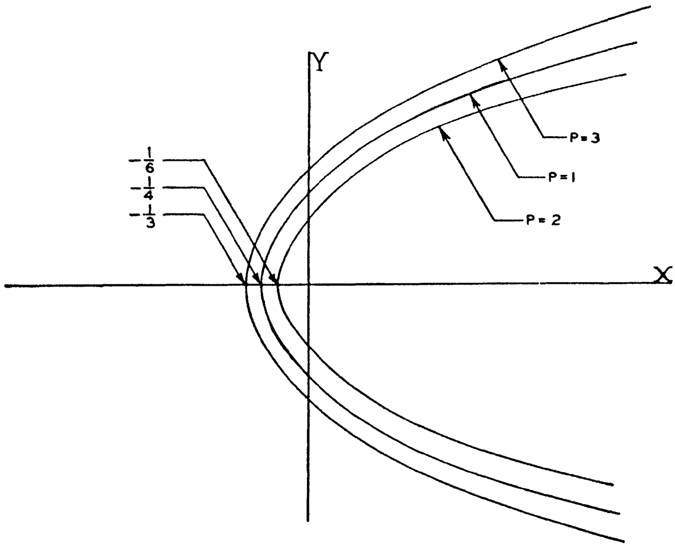

FIG. 2.

and with $m_1 = \frac{1}{2}$, $m_2 = \frac{1}{3}$, $m_3 = \frac{2}{3}$. We note that if the pth parabola contains real values less than $-\frac{1}{4}$, then the $(p+1)$th parabola *cannot* contain the value $-\frac{1}{4}$. *Thus, all the parabolas contain the Worpitzky circle,* $|z| = \frac{1}{4}$, *if, and only if,* $m_p = \frac{1}{2}$, $p = 1, 2, 3, \cdots$. If $|a_{p+1}| \leq \frac{1}{4}, p = 1, 2, 3, \cdots$, then, obviously, one or the other of the conditions (a) or (b) of Theorem 14.1 holds. Hence, we see that Theorem 14.1, with $m_p = \frac{1}{2}$, furnishes a generalization of the theorem of Worpitzky (§ 10).

We shall now prove, more generally, the following **parabola theorem** (Scott and Wall [86]).

THEOREM 14.2. *A set of points, S, in the complex plane, which is symmetrical with respect to the real axis, is a convergence set* (*Definition* 10.1) *for the continued fraction* (14.4) *if, and only if, S is a bounded set contained in the parabolic domain*

$$|z| - \Re(z) \leq \tfrac{1}{2}. \tag{14.5}$$

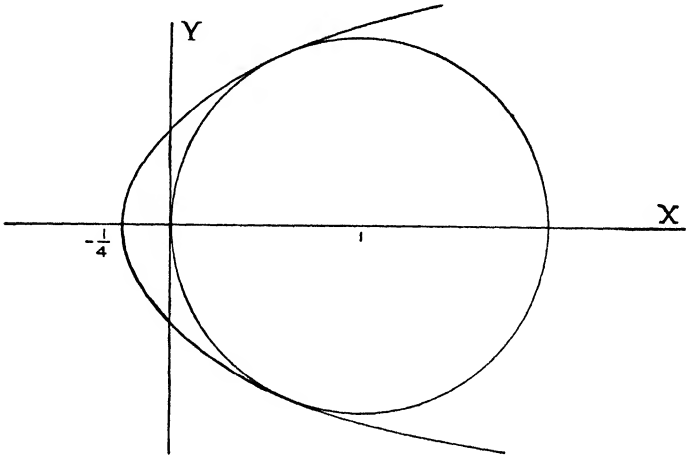

FIG. 3.

Moreover, if $a_2, a_3, a_4, \cdots$ *are values contained in this parabolic domain, then* (14.4) *converges if, and only if,* (a) *some* a_p *vanishes or* (b) $a_p \neq 0$, $p = 2, 3, 4, \cdots$, *and the series* $\Sigma| b_p |$, *given by* (13.3), *is divergent.*

Proof. The last part of the theorem is the special case $m_p = \frac{1}{2}$ of Theorem 14.1. The *sufficiency* of the conditions imposed upon S is contained in the last part of the theorem. It therefore remains to be proved that the conditions are *necessary.*

It is necessary for a convergence set to be bounded. For otherwise $a_2, a_3, a_4, \cdots$ can be chosen in S such that the series $\Sigma| b_p |$ converges. The continued fraction then diverges by Theorem 6.1. It is necessary for S to be contained in the parabolic domain (14.5). For if S contains a point $z = a$ outside this domain, then, inasmuch as, by hypothesis, S is symmetrical with respect to the real axis, it contains the complex conjugate $\bar{a}$ of a. But if $a_{2p} = a$, $a_{2p+1} = \bar{a}$, $p = 1, 2, 3, \cdots$, then the continued fraction diverges by Exercise 2.1.

In Theorem 10.1 we found that the values of the continued fraction (14.4) fill the circular domain $| z - \frac{4}{3} | \leq \frac{2}{3}$ as $a_2, a_3, a_4, \cdots$ range independently over all values with moduli not greater than $\frac{1}{4}$. The corresponding result when the circular region $| z | \leq \frac{1}{4}$ is replaced by the parabolic domain (14.5) is as follows.

THEOREM 14.3. *The values of the continued fraction* (14.4), *and of its approximants, fill the domain*

$$|z - 1| \leq 1, \quad z \neq 0, \tag{14.6}$$

as $a_2, a_3, a_4, \cdots$ *range independently over the parabolic domain* (14.5), *taking on values such that* (a) *or* (b) *of Theorem* 14.2 *holds.*

Proof. From the recurrence formula (2.6) we find that

$$1 + \rho_p = \cfrac{1}{1 + \cfrac{a_{p+1}}{1 + \cfrac{a_p}{1 + \cdots + \cfrac{a_3}{1 + \cfrac{a_2}{1}}}}},$$

so that, by (9.3), remembering that the fundamental inequalities are satisfied here with $r_p = 1$, we have

$$|\rho_p| = \left| \cfrac{1}{1 + \cfrac{a_{p+1}}{1 + \cfrac{a_p}{1 + \cdots + \cfrac{a_2}{1}}}} - 1 \right| \leq 1.$$

Since $a_2, a_3, a_4, \cdots$ are **independent variables** over (14.5), it follows that every approximant of the continued fraction has its value in (14.6) when the a_n are in (14.5). Thus the values of (14.4) and of its approximants are all in (14.6).

Now, let z be any value in (14.6), and consider the continued fraction

$$\cfrac{1}{1 - \cfrac{(1 - 1/z)(1/\bar{z})}{1 - \cfrac{(1/z)(1 - 1/\bar{z})}{1 - \cfrac{(1 - 1/z)(1/\bar{z})}{1 - \cdots}}}} \tag{14.7}$$

This is of the form (14.4) with

$$a_{2p} = \frac{1 - z}{z\bar{z}}, \quad a_{2p+1} = \frac{1 - \bar{z}}{z\bar{z}}, \quad p = 1, 2, 3, \cdots.$$

We shall prove that (14.7) converges to the value z. From the fact that the continued fraction converges, it will then follow by Exercise 2.1 that the a_p are in (14.5). By Theorem 2.1, we see that (14.7) is equivalent to the expression

$$z\left(1 - \cfrac{1}{1 + \cfrac{1}{z-1} + \cfrac{1}{(z-1)(\bar{z}-1)} + \cfrac{1}{(z-1)(\bar{z}-1)(z-1)} + \cdots}\right).$$

Inasmuch as $|z - 1| \leq 1$, $z \neq 0$, it readily follows that the continued fraction converges to the value z.

This completes the proof of Theorem 14.3.

Theorem 14.3 was proved by Scott and Wall [**87**] by means of a geometric argument. The above simple proof is due to R. E. Lane [**47**].

Note. The example used in the above proof was obtained in the following way. In view of Exercise 2.1, one tries to find a continued fraction of the form

$$\cfrac{1}{1 + \cfrac{a}{1 + \cfrac{\bar{a}}{1 + \cfrac{a}{1 + \cfrac{\bar{a}}{1 + \cdots}}}}} \tag{14.8}$$

which takes on the value z. Both the odd and the even parts must then have the value z, i.e.,

$$z = 1 - \cfrac{a}{1 + 2\Re(a) - \cfrac{|a|^2}{1 + 2\Re(a) - \cfrac{|a|^2}{1 + 2\Re(a) - \cdots}}},$$

and

$$z = \cfrac{1}{a + 1 - \cfrac{|a|^2}{1 + 2\Re(a) - \cfrac{|a|^2}{1 + 2\Re(a) - \cdots}}}$$

Thus, z has simultaneously the forms

$$z = 1 - \frac{a}{t}, \quad \text{and} \quad z = \frac{1}{a + s},$$

where s and t are real. If we determine the intersection of the maps of the real axis under these transformations of t and s into z, we find that $a = (1 - z)/z\bar{z}$.

Remark. In a later section (§ 34) we shall return to Theorem 14.1, and connect it with a more general theorem, in which m_p is replaced by $2(1 - g_{p-1})g_p$, $0 \leq g_{p-1} \leq 1$, $(p = 1, 2, 3, \cdots)$.

15. "Convergence Neighborhoods" of a Point (1). If z is a given point in the complex plane, does there exist a neighborhood of c which is a convergence set for the continued fraction (14.4)? Inasmuch as (14.4) diverges if $a_{p+1} = c$, $p = 1, 2, 3, \cdots$, in case c is on the real interval $-\infty < x < -\frac{1}{4}$ (Theorem 8.2), we see that a *necessary condition* is that c be not real, or that it be real and greater than $-\frac{1}{4}$. In a later section (§ 32) we shall prove that this necessary condition is also *sufficient*. The methods developed thus far enable us to establish the existence of convergence neighborhoods of c only when $\Re(c) > -\frac{1}{4}$. We have the following theorem.

THEOREM 15.1. *The continued fraction* (14.4) *converges if*

$$|a_{p+1} - c| \leq \frac{|1 + 2c| - 2|c|}{4}, \quad p = 1, 2, 3, \cdots. \qquad (15.1)$$

(*Note that* $|1 + 2c| - 2|c| > 0$ *if, and only if,* $\Re(c) > -\frac{1}{4}$). [86.]

Proof. Let $a_{p+1} = c + R_{p+1}e^{i\theta_{p+1}}$, $R_{p+1} \geq 0$. Then, it is easy to verify that the fundamental inequalities (14.1) are satisfied (except that $|1 + a_2 + a_3|$ may equal $|a_3|$ if $a_2 = 0$), provided $R_{p+1} \leq (|1 + 2c| - 2|c|)/4$. Hence, it follows that (14.4) converges when (15.1) holds.

We note that when $c = 0$, Theorem 15.1 reduces to Worpitzky's theorem.

EXERCISE 3

3.1. By specializing the g_p in Theorem 11.2, show that the continued fraction

$$\cfrac{1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdots}}}$$

converges uniformly for $|a_{p+1}| \leq \frac{1}{4}$, $p = 1, 2, 3, \cdots$.

3.2. Show that Theorem 11.1 can be derived from Theorem 11.2. [**68.**]

3.3. Show that the values of the continued fraction of Theorem 15.1 are in the circular domain $|z - 1| \leq 1$.

3.4. Let $r_n = |a_{n+1}|$ in the fundamental inequalities (9.2), and these inequalities lead to the inequalities

$$|1 + a_2| \geq 1,$$
$$|1 + a_2 + a_3| \geq 1,$$
$$|1 + a_p + a_{p+1}| \geq 1 + |a_{p-1}a_p|, \quad p = 3, 4, 5, \cdots.$$

If these inequalities hold and some a_p vanishes, then the continued fraction of 3.1 converges. If $a_p \neq 0$, $p = 2, 3, 4, \cdots$, the inequalities hold with actual inequality in the first, and $\lim \sup |a_p| < \infty$, then the continued fraction converges [**51, 86**].

3.5. If $|1 + a_2| > 1$, $|a_3| \geq (2 + r)/(1 - r)$, $|a_{2p}| \leq r$, $|a_{2p+3}| \geq 2 + r + r|a_{2p+1}|$, $p = 1, 2, 3, \cdots$, where $0 < r < 1$, then the continued fraction of 3.1 is convergent [**51**].

3.6. If (13.1) holds with $k_1 > 0$, $k_2 > 0$, then any one of the following conditions is sufficient for the convergence of the continued fraction of 3.1.

(a) $\lim \inf |a_n| = 0$;

(b) $|a_p| < M$, $p = 2, 3, 4, \cdots$, M finite;

(c) $\lim \inf r_1r_2 \cdots r_n < \infty$, $\Sigma(1/|a_p|)$ diverges;

(d) $r_1r_3 \cdots r_{2p-1} < M$, $r_2r_4 \cdots r_{2p} < M$, $\Sigma|b_p|$ diverges;

(e) $\lim \inf r_1r_2 \cdots r_n < \infty$, $r_1r_3 \cdots r_{2p-1}$ and $r_2r_4 \cdots r_{2p}$ bounded away from zero, $\Sigma|b_p|$ diverges [**86**].

3.7. If the series $\Sigma|a_p|$ converges, then the sequences $\{A_n\}$ and $\{B_n\}$ converge to finite limits [**116**].

3.8. The continued fraction of 3.1 converges and the modulus of its value does not exceed 3 if $|a_n| \leq n^2/(4n^2 - 1)$, $n = 2, 3, 4, \cdots$. Moreover, the value of the continued fraction is 3 if $a_p = -p^2/(4p^2 - 1)$, $p = 2, 3, 4, \cdots$.

3.9. If $a_n = (2n - 3 + 4x_n)/4$, $n = 2, 3, 4, \cdots$, then the continued fraction of 3.1 converges uniformly for $|x_2| \leq \frac{1}{8}$, $|x_n| \leq \frac{1}{4} - (1/4(n - 1))$, $n = 3, 4, 5, \cdots$.

3.10. Show that the continued fraction (9.1) converges if $|a_{2n+1}| \leq \frac{1}{4}$, $|a_{2n}| \geq \frac{25}{4}$, $n = 1, 2, 3, \cdots$. *Hint.* Use the result of Exercise 1.1. [**58.**] (For extensions or refinements of this result, see [**124, 41, 101a, 7a**]).

Chapter IV

INTRODUCTION TO THE THEORY OF POSITIVE DEFINITE CONTINUED FRACTIONS

In the preceding chapter we have approached the convergence problem for a continued fraction by investigating a certain infinite series, $1 + \Sigma\rho_1\rho_2 \cdots \rho_p$, equivalent to the continued fraction. We now propose to approach the problem by determining a class of continued fractions whose denominators are different from zero. To do this we shall express the denominators as determinants, and then use the theorem of algebra which states that a system of n homogeneous linear equations in n variables has only the trivial solution (all variables equal to zero) if, and only if, the determinant of the system is different from zero. We shall be led in this way to connect the continued fraction with a certain quadratic form. This method will enable us to carry the convergence problem much farther than we have been able to carry it by means of the methods of Chapter III.

16. Definition of a Positive Definite Continued Fraction. We shall now consider continued fractions of the form

$$\cfrac{1}{b_1 + z_1 - \cfrac{{a_1}^2}{b_2 + z_2 - \cfrac{{a_2}^2}{b_3 + z_3 - \ddots}}}, \tag{16.1}$$

in which the a_p and b_p are complex constants, and $z_1, z_2, z_3, \cdots$ are complex variables. It will often be convenient to let z stand for the set $(z_1, z_2, z_3, \cdots)$ and 0 for the set $(0, 0, 0, \cdots)$. Thus, the pth numerator and denominator of (16.1) will be denoted by $A_p(z)$ and $B_p(z)$, respectively; and $A_p(0)$, $B_p(0)$ will denote the values of $A_p(z)$, $B_p(z)$ for $z_1 = z_2 = z_3 = \cdots = 0$. We use $a_p{}^2$ instead of a_p in order to avoid using the exponent $\frac{1}{2}$ later on.

Lemma 16.1. *The denominators of the continued fraction* (16.1) *are given by the formulas*

$$B_p(z) =$$
$$\begin{vmatrix} b_1+z_1, & -a_1, & 0, & 0, & 0, & \cdots, & 0 \\ -a_1, & b_2+z_2, & -a_2, & 0, & 0, & \cdots, & 0 \\ 0, & -a_2, & b_3+z_3, & -a_3, & 0, & \cdots, & 0 \\ & & & \cdot\ \cdot\ \cdot & & & \\ 0, & 0, & \cdots, & 0, & -a_{p-2}, & b_{p-1}+z_{p-1}, & -a_{p-1} \\ 0, & 0, & \cdots, & 0, & 0, & -a_{p-1}, & b_p+z_p \end{vmatrix},$$
$$(p = 1, 2, 3, \cdots). \qquad (16.2)$$

Proof. We need but note that these determinants satisfy the recurrence formulas

$$B_p(z) = (b_p + z_p)B_{p-1}(z) - a_{p-1}{}^2 B_{p-2}(z),$$
$$p = 1, 2, 3, \cdots, \qquad (16.3)$$

with the initial values $B_{-1}(z) = 0$, $B_0(z) = 1$, $(a_0 = 1)$, which are satisfied by the denominators of (16.1).

We now turn to the system of homogeneous linear equations in the variables x_n, whose determinant is $B_p(z)$, namely,

$$(b_1 + z_1)x_1 - a_1x_2 = 0,$$
$$-a_1x_1 + (b_2 + z_2)x_2 - a_2x_3 = 0, \qquad (16.4)$$
$$\cdot\ \cdot\ \cdot$$
$$-a_{p-1}x_{p-1} + (b_p + z_p)x_p = 0.$$

This has no solution excepting the trivial solution $x_1 = x_2 = \cdots = x_p = 0$ if, and only if, $B_p(z) \neq 0$. Let us multiply the equations (16.4) by $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p$, respectively, and add the resulting equations. This gives

$$\sum_{r=1}^{p}(b_r + z_r)|\, x_r\,|^2 - \sum_{r=1}^{p-1} a_r(x_r\bar{x}_{r+1} + \bar{x}_r x_{r+1}) = 0. \quad (16.5)$$

We shall now put

$$\beta_r = \Im(b_r), \quad y_r = \Im(z_r), \quad \alpha_r = \Im(a_r),$$

and suppose that the imaginary part of the left-hand member of (16.5) is *positive definite* for $y_r > 0$, i.e., that

$$\sum_{r=1}^{p}(\beta_r + y_r)|\, x_r\,|^2 - \sum_{r=1}^{p-1} \alpha_r(x_r\bar{x}_{r+1} + \bar{x}_r x_{r+1}) > 0 \quad (16.6)$$

for $y_r > 0$, $r = 1, 2, 3, \cdots, p, \sum_{r=1}^{p} |\, x_r\,|^2 > 0$. It then follows that (16.5) and (16.4) can hold when the y_r are positive if, and only if, $x_1 = x_2 = \cdots = x_p = 0$, and therefore $B_p(z) \neq 0$ for $y_r > 0$ $(r = 1, 2, 3, \cdots, p)$.

Lemma 16.2. *If the real quadratic form*

$$\sum_{r=1}^{p}\beta_r\xi_r^2 - 2\sum_{r=1}^{p-1}\alpha_r\xi_r\xi_{r+1} \geq 0 \quad (16.7)$$

for all real values of the ξ_r, *then* (16.6) *holds, and conversely.*

Proof. If (16.6) holds, then (16.7) obviously holds. Suppose that (16.7) holds, and put $x_r = u_r + iv_r$, u_r, v_r real. We then find that the quadratic form in the left-hand member of (16.6) is equal to

$$\left(\sum_{r=1}^{p}\beta_r u_r^2 - 2\sum_{r=1}^{p-1}\alpha_r u_r u_{r+1}\right) + \left(\sum_{r=1}^{p}\beta_r v_r^2 - 2\sum_{r=1}^{p-1}\alpha_r v_r v_{r+1}\right)$$
$$+ \sum_{r=1}^{p} y_r|\, x_r\,|^2 \geq \sum_{r=1}^{p} y_r|\, x_r\,|^2,$$

so that (16.6) holds.

We now make the following definition.

Definition 16.1. *The continued fraction* (16.1) *is said to be positive definite if the quadratic form*

$$\sum_{r=1}^{p} \beta_r \xi_r^2 - 2\sum_{r=1}^{p-1} \alpha_r \xi_r \xi_{r+1} \geq 0$$

for $p = 1, 2, 3, \cdots$, *and for all real values of* $\xi_1, \xi_2, \xi_3, \cdots$, *where* $\beta_r = \mathfrak{I}(b_r)$, $\alpha_r = \mathfrak{I}(a_r)$.

In view of Lemma 16.2, we conclude from the preceding that this theorem holds.

Theorem 16.1. *If the continued fraction* (16.1) *is positive definite, then its denominators* $B_p(z)$ *are different from zero for* $\mathfrak{I}(z_r) > 0$, $(r = 1, 2, 3, \cdots)$. [**138.**]

We shall now prove the following theorem of Wall and Wetzel [**139**], which furnishes a parametric representation for the coefficients a_p and b_p of a positive definite continued fraction.

Theorem 16.2. *The continued fraction* (16.1) *is positive definite if, and only if, both the following conditions are satisfied.*

(a) *The imaginary parts* $\beta_n = \mathfrak{I}(b_n)$ *of the numbers* b_n *are all non-negative:*

$$\beta_n \geq 0, \quad n = 1, 2, 3, \cdots. \tag{16.8}$$

(b) *There exist numbers* $g_0, g_1, g_2, \cdots$ *such that*

$$0 \leq g_{n-1} \leq 1, \quad \alpha_n^2 = \beta_n\beta_{n+1}(1 - g_{n-1})g_n, \quad n = 1, 2, 3, \cdots, \tag{16.9}$$

where $\alpha_n = \mathfrak{I}(a_n)$. [**139.**]

Proof. On putting all the ξ_r equal to 0 with the exception of ξ_n, we see from (16.7), with $p > n$, that the condition (16.8) is necessary for (16.7) to hold. Let $\xi_3 = \xi_4 = \xi_5 = \cdots = 0$. Then (16.7) becomes

$$\beta_1\xi_1^2 - 2\alpha_1\xi_1\xi_2 + \beta_2\xi_2^2 \geq 0.$$

Consequently, it is necessary that $\alpha_1^2 \leq \beta_1\beta_2$, i.e., that $\alpha_1^2 = \beta_1\beta_2(1 - g_0)g_1$, where $g_0 = 0$ and $0 \leq g_1 \leq 1$. We shall agree that $g_1 = 0$ if $\beta_1 = 0$. Let $\xi_4 = \xi_5 = \xi_6 = \cdots = 0$, and we may now write (16.7) in the form

$$[(\beta_1(1 - g_0))^{1/2}\xi_1 - (\beta_2 g_1)^{1/2}\xi_2]^2 + \beta_2(1 - g_1)\xi_2^2 - 2\alpha_2\xi_2\xi_3 + \beta_3\xi_3^2 \geq 0.$$

If $\beta_1 = 0$ so that, by agreement, $g_1 = 0$, then the first term on the left is equal to 0. If $\beta_1 > 0$ and we put

$$\xi_1 = \frac{(\beta_2 g_1)^{1/2}\xi_2}{(\beta_1(1 - g_0))^{1/2}},$$

then again the first term is zero. Since ξ_2 and ξ_3 are unrestricted, we must therefore have

$$\alpha_2^2 \leq \beta_2\beta_3(1 - g_1), \quad \text{or} \quad \alpha_2^2 = \beta_2\beta_3(1 - g_1)g_2,$$

where $0 \leq g_2 \leq 1$. We shall agree that $g_2 = 0$ if $\beta_2 = 0$ or if $g_1 = 1$. On setting $\xi_5 = \xi_6 = \xi_7 = \cdots = 0$, we find next that

$$\begin{aligned}[(\beta_1(1 - g_0))^{1/2}\xi_1 - (\beta_2 g_1)^{1/2}\xi_2]^2 \\ + [(\beta_2(1 - g_1))^{1/2}\xi_2 - (\beta_3 g_2)^{1/2}\xi_3]^2 \\ + \beta_3(1 - g_2)\xi_3^2 - 2\alpha_3\xi_3\xi_4 + \beta_4\xi_4^2 \geq 0,\end{aligned}$$

and, consequently, we conclude that $\alpha_3^2 \leq \beta_3\beta_4(1 - g_2)$, or $\alpha_3^2 = \beta_3\beta_4(1 - g_2)g_3$ where $0 \leq g_3 \leq 1$ and $g_3 = 0$ if $\beta_3 = 0$ or if $g_2 = 1$. Continuing in this manner, we see that the condition (16.9) is necessary for the continued fraction to be positive definite.

If we suppose, conversely, that (16.8) and (16.9) hold, then we have the obvious identity

$$\begin{aligned}\sum_{r=1}^{p}\beta_r\xi_r^2 - 2\sum_{r=1}^{p-1}\alpha_r\xi_r\xi_{r+1} = g_0\beta_1\xi_1^2 + (1 - g_{p-1})\beta_p\xi_p^2 \\ + \sum_{r=1}^{p-1}[(\beta_r(1 - g_{r-1}))^{1/2}\xi_r - (\beta_{r+1}g_r)^{1/2}\xi_{r+1}]^2, \quad (16.10)\end{aligned}$$

from which (16.7) follows.

This completes the proof of Theorem 16.2.

Corollary 16.1. *We may assume that* $g_0 = 0$ *in Theorem* 16.2.

If (16.7) holds, then

$$\sum_{r=1}^{p}\beta_r\xi_r^2 - 2\sum_{r=1}^{p-1}\alpha_r'\xi_r\xi_{r+1} \geq 0$$

for all real values of the ξ_r, provided $|\alpha_r'| \leq |\alpha_r|$, $(r = 1, 2, 3, \cdots, p - 1)$. Hence, we have

COROLLARY 16.2. *In Theorem* 16.2, *we may replace the condition* (16.9) *by the condition*

$$0 \leq g_{n-1} \leq 1, \quad \alpha_n^2 \leq \beta_n\beta_{n+1}(1 - g_{n-1})g_n, \qquad n = 1, 2, 3, \cdots. \tag{16.11}$$

Inasmuch as $|a_n^2| - \Re(a_n^2) = 2\alpha_n^2$, this condition may be written

$$0 \leq g_{n-1} \leq 1, \quad |a_n^2| - \Re(a_n^2) \leq 2\beta_n\beta_{n+1}(1 - g_{n-1})g_n, \qquad n = 1, 2, 3, \cdots. \tag{16.12}$$

By means of this formulation of the condition of positive definiteness, we can show that the continued fractions studied in Chapter III belong to the class of positive definite continued fractions. In fact, with the aid of an equivalence transformation we find that

$$\cfrac{1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdot_{\cdot_{\cdot}}}}} = \cfrac{i}{i - \cfrac{c_1^2}{i - \cfrac{c_2^2}{i - \cdot_{\cdot_{\cdot}}}}},$$

where we have put $a_{p-1} = c_p^2$, $p = 1, 2, 3, \cdots$. Then, in the notation introduced above, $\beta_p = 1$. The condition (16.12) now reads

$$0 \leq g_{n-1} \leq 1, \quad |c_n^2| - \Re(c_n^2) \leq 2(1 - g_{n-1})g_n, \qquad n = 1, 2, 3, \cdots. \tag{16.13}$$

If we put $g_n = \frac{1}{2}$, this reduces to the condition (14.5) of the parabola theorem. If the g_n satisfy the additional requirement that $(1 - g_0)g_1 < \frac{1}{2}$, $(1 - g_{n-1})g_n + (1 - g_n)g_{n+1} \leq \frac{1}{2}$, $(n = 1, 2, 3, \cdots)$, the above condition reduces to (14.3). If the c_n are pure imaginary, then (16.13) reduces to $|c_n^2| \leq (1 - g_{n-1})g_n$, which is the condition used in § 11. In any case, (16.13) implies that $|c_n^2| \leq (1 - g_{n-1})g_n$.

It should be noted that (16.1) is positive definite if the a_p and b_p are real; or if the a_p are real and the b_p have nonnegative imaginary parts (cf. Hellinger and Wall **[35]**). The class of positive definite continued fractions was first introduced and investigated by Wall and Wetzel [**138, 139**].

17. The Nest of Circles. We saw in § 14 that the values of the approximants of the continued fraction of the parabola theorem are all in a certain circle. We shall now show that the same is true of a positive definite continued fraction, provided that $\mathfrak{I}(z_1) > 0$, $\mathfrak{I}(z_r) \geq 0$, $r = 2, 3, 4, \cdots$. For this purpose we shall need the following lemma.

Lemma 17.1. *Let*

$$t = t_p(w) = b_p + z_p - \frac{a_p^2}{w}, \quad p = 1, 2, 3, \cdots, \tag{17.1}$$

where

$$\begin{gathered} \beta_p = \mathfrak{I}(b_p) \geq 0, \quad \alpha_p^2 = \mathfrak{I}(a_p)^2 \leq \beta_p\beta_{p+1}(1 - g_{p-1})g_p, \\ 0 \leq g_{p-1} \leq 1, \quad p = 1, 2, 3, \cdots. \end{gathered} \tag{17.2}$$

If $\mathfrak{I}(w) \geq \beta_{p+1}g_p$ *then* $\mathfrak{I}(t) \geq \beta_p g_{p-1} + y_p$, *where* $y_p = \mathfrak{I}(z_p)$, $(p = 1, 2, 3, \cdots)$. [9.]

Proof. Since, by hypothesis, $\mathfrak{I}(w) \geq \beta_{p+1}g_p$, we have, for $y_p > 0$,

$$\mathfrak{I}(w) \geq \beta_{p+1}g_p \frac{\beta_p(1 - g_{p-1})}{\beta_p(1 - g_{p-1}) + y_p} \geq \frac{\alpha_p^2}{\beta_p(1 - g_{p-1}) + y_p}.$$

Therefore,

$$\left| w + \frac{ia_p^2}{2[\beta_p(1 - g_{p-1}) + y_p]} \right| \geq \frac{|a_p^2|}{2[\beta_p(1 - g_{p-1}) + y_p]}.$$

This says, in fact, that w lies outside or upon a circle which is tangent to the line $\mathfrak{I}(w) = \alpha_p^2/[\beta_p(1 - g_{p-1}) + y_p]$ from below, and is equivalent to the inequality $\beta_p + y_p - \mathfrak{I}(a_p^2/w) \geq \beta_p g_{p-1}$ or $\mathfrak{I}(t) \geq \beta_p g_{p-1}$. Since this holds for $y_p > 0$, it must hold for $y_p = 0$. Thus,

$$\beta_p - \mathfrak{I}\left(\frac{a_p^2}{w}\right) \geq \beta_p g_{p-1},$$

$$\beta_p + y_p - \mathfrak{I}\left(\frac{a_p^2}{w}\right) \geq \beta_p g_{p-1} + y_p,$$

or

$$\mathfrak{I}(t) \geq \beta_p g_{p-1} + y_p,$$

as was to be proved.

We now apply the transformation $t = t_0(w) = 1/w$ to the half-plane $\Im(w) \geq \beta_1 g_0 + y_1$, and find that it is transformed into the proper circular region

$$\left| t + \frac{i}{2(g_0\beta_1 + y_1)} \right| \leq \frac{1}{2(g_0\beta_1 + y_1)}, \quad \text{for} \quad g_0\beta_1 + y_1 > 0. \tag{17.3}$$

We shall denote this region by $K_0(z)$. This is tangent to the real axis from below at the origin.

We next define $K_1(z)$ as the map of the half-plane $\Im(w) \geq g_1\beta_2$ under the transformation

$$t = t_0 t_1(w) = \frac{1}{b_1 + z_1 - \cfrac{a_1^2}{w}}.$$

By the lemma, $\Im[t_1(w)] \geq g_0\beta_1 + y_1$ for $\Im(w) \geq g_1\beta_2$, and therefore $K_1(z)$ is contained in $K_0(z)$. Since $w = \infty$ is on the boundary of the half-plane $\Im(w) \geq g_1\beta_2$, it follows that $t_0t_1(\infty)$, the first approximant of (16.1), is on the boundary of $K_1(z)$. We note that if $a_1 = 0$, then $K_1(z)$ is the single point

$$\frac{1}{b_1 + z_1} = \frac{A_1(z)}{B_1(z)}.$$

For arbitrary $p \geq 1$, we now define $K_p(z)$ as the map of the half-plane $\Im(w) \geq \beta_{p+1}g_p$ under the transformation

$$t = t_0t_1 \cdots t_p(w) = \cfrac{1}{b_1 + z_1 - \cfrac{a_1^2}{b_2 + z_2 - \cdots - \cfrac{a_{p-1}^2}{b_p + z_p - \cfrac{a_p^2}{w}}}}, \tag{17.4}$$

where we suppose that $\Im(z_r) \geq 0$, $r = 2, 3, 4, \cdots$. From the lemma it follows immediately that $K_p(z)$ is contained in $K_{p-1}(z)$. As before, we see that the pth approximant of (16.1), namely, $t_0t_1 \cdots t_p(\infty)$, is upon the boundary of $K_p(z)$. If some a_p vanishes, and a_m is the first which vanishes, then all the circular regions $K_p(z)$ reduce to the point $t_0t_1 \cdots t_m(\infty)$ for $p \geq m$.

On referring to the inequality (17.3) defining $K_0(z)$, we now conclude from the fact that

$$K_0(z) \supset K_1(z) \supset K_2(z) \supset \cdots, \tag{17.5}$$

that for all points t of these circular domains:

$$\begin{aligned} \mathfrak{J}(t) \leq 0, \quad |t| \leq \frac{1}{g_0\beta_1 + y_1}, \quad \text{for} \quad g_0\beta_1 + y_1 > 0, \\ y_r \geq 0, \quad r = 2, 3, 4, \cdots, \end{aligned} \tag{17.6}$$

and that, in particular, the approximants of (16.1) satisfy:

$$\begin{aligned} \mathfrak{J}\left(\frac{A_p(z)}{B_p(z)}\right) \leq 0, \quad \left|\frac{A_p(z)}{B_p(z)}\right| \leq \frac{1}{g_0\beta_1 + y_1}, \\ \text{for } g_0\beta_1 + y_1 > 0, \quad y_r \geq 0, \quad r = 2, 3, 4, \cdots. \end{aligned} \tag{17.7}$$

Thus, the approximants of a positive definite continued fraction have all their values in the lower half-plane, and have moduli not greater than $1/y_1$ for $\mathfrak{J}(z_1) = y_1 > 0$, $\mathfrak{J}(z_r) = y_r \geq 0$, $r = 2, 3, 4, \cdots$. More precisely, the values of the approximants are all in the circular domain $K_0(z)$.

We note the following theorem, which supplements Theorem 16.1.

THEOREM 17.1. *If in* (17.2), $g_{p-1}\beta_p > 0$, $p = 1, 2, 3, \cdots$, *then the denominators of the continued fraction* (16.1) *are different from zero for* $\mathfrak{J}(z_r) \geq 0$, $r = 1, 2, 3, \cdots$. [9.]

Proof. In case the a_p are different from 0, this follows at once from the determinant formula and (17.7). If some a_p vanishes, then it suffices to observe that, by (16.2), the denominators can be written as products of factors, each of which may be regarded as a denominator of a positive definite continued fraction, with nonvanishing partial numerators and bounded approximants.

From Theorems 5.1, 16.1 and 17.1 we have:

THEOREM 17.2. *Let* (16.1) *be a positive definite continued fraction having one or more vanishing partial numerators. The continued fraction converges for* $\mathfrak{J}(z_r) > 0$, $r = 1, 2, 3, \cdots$; *and, if* $g_{p-1}\beta_p > 0$, $p = 1, 2, 3, \cdots$, *the continued fraction converges for* $\mathfrak{J}(z_r) \geq 0$, $r = 1, 2, 3, \cdots$.

We shall assume from this point on, unless the contrary is explicitly stated, that

$$a_p \neq 0, \quad p = 1, 2, 3, \cdots. \tag{17.8}$$

From (17.5) and the fact that the pth approximant of (16.1) is in $K_p(z)$, it follows that for any particular values of the z_r with $\mathfrak{J}(z_1) > 0$, $\mathfrak{J}(z_r) \geq 0$, $r = 2, 3, 4, \cdots$, there are just two possible cases.

Case 1. The limit-point case. The circular regions $K_p(z)$ have one and only one value $f(z)$ in common; the radius of $K_p(z)$ has the limit 0 for $p = \infty$; the continued fraction converges and its value is $f(z)$.

Case 2. The limit-circle case. The circular regions $K_p(z)$ have a circular region in common; the radius of $K_p(z)$ has a *positive* limit for $p = \infty$; the continued fraction may conceivably converge or diverge.

The decision as to which of the two cases holds rests upon a knowledge of the behavior of the radius $r_p(z)$ of $K_p(z)$ for $p = \infty$. Accordingly, we shall proceed to determine a formula for $r_p(z)$.

It will be convenient to use, instead of the polynomials $A_p(z)$ and $B_p(z)$, the polynomials

$$X_{p+1}(z) = \frac{A_p(z)}{a_1 a_2 \cdots a_p}, \quad Y_{p+1}(z) = \frac{B_p(z)}{a_1 a_2 \cdots a_p}, \tag{17.9}$$
$$p = 1, 2, 3, \cdots.$$

We define $X_1(z) = 0$, $Y_1(z) = 1$, $X_0(z) = -1$, $Y_0(z) = 0$. The polynomials $X_p = X_p(z)$ and $Y_p = Y_p(z)$ satisfy the recurrence formulas

$$\begin{aligned} -a_{p-1}X_{p-1} + (b_p + z_p)X_p - a_p X_{p+1} &= 0, \\ -a_{p-1}Y_{p-1} + (b_p + z_p)Y_p - a_p Y_{p+1} &= 0, \end{aligned} \quad p = 1, 2, 3, \cdots, \tag{17.10}$$

where we must put $a_0 = 1$. The determinant formula becomes

$$X_{p+1}Y_p - X_p Y_{p+1} = \frac{1}{a_p}, \quad p = 1, 2, 3, \cdots. \tag{17.11}$$

We shall suppose now that the parameters g_p have been chosen so that, in (17.2),

$$\alpha_p^2 = \beta_p\beta_{p+1}(1 - g_{p-1})g_p, \quad p = 1, 2, 3, \cdots. \tag{17.12}$$

Let us recall that the circular region $K_p(z)$ is the map of the half-plane $\mathfrak{I}(w) \geq \beta_{p+1}g_p$, under the transformation (17.4). We now write the latter in the form

$$t = t_0 t_1 \cdots t_p(w) = \frac{A_p(z)w - a_p^2 A_{p-1}(z)}{B_p(z)w - a_p^2 B_{p-1}(z)}.$$

On introducing the polynomials (17.9), we then find that

$$t = \frac{X_{p+1}w - a_p X_p}{Y_{p+1}w - a_p Y_p}. \tag{17.13}$$

This transformation carries the point $w = a_p Y_p / Y_{p+1}$ into the point $t = \infty$. Inasmuch as the center C_p of $K_p(z)$ and ∞ are inverse points with respect to $K_p(z)$, and inasmuch as the points

$$\frac{\bar{a}_p \overline{Y}_p}{\overline{Y}_{p+1}} + 2ig_p\beta_{p+1} \tag{17.14}$$

and $a_p Y_p / Y_{p+1}$ are inverse points with respect to the line $\mathfrak{I}(w) = \beta_{p+1}g_p$, it follows that C_p is the image under the transformation (17.13) of the point (17.14), that is,

$$C_p = \frac{X_{p+1}\left(\dfrac{\bar{a}_p \overline{Y}_p}{\overline{Y}_{p+1}} + 2ig_p\beta_{p+1}\right) - a_p X_p}{Y_{p+1}\left(\dfrac{\bar{a}_p \overline{Y}_p}{\overline{Y}_{p+1}} + 2ig_p\beta_{p+1}\right) - a_p Y_p}. \tag{17.15}$$

Inasmuch as the pth approximant of the continued fraction lies upon the boundary of $K_p(z)$, we therefore have:

$$r_p(z) = \left| C_p - \frac{X_{p+1}}{Y_{p+1}} \right|.$$

On substituting here the value of C_p from (17.15), and using (17.11), we then find that

$$2r_p(z) = \frac{1}{\left| g_p\beta_{p+1} |Y_{p+1}|^2 - \mathfrak{I}(a_p Y_p Y_{p+1}) \right|}. \tag{17.16}$$

In order to throw this expression into a more convenient form, we multiply the second recurrence formula (17.10) by $\overline{Y}_p$, and get, on replacing p by r,

$$-a_{r-1}Y_{r-1}\overline{Y}_r + (b_r + z_r)|\,Y_r\,|^2 - a_rY_{r+1}\overline{Y}_r = 0,$$
$$r = 1, 2, 3, \cdots.$$

Remembering that $Y_0 = 0$, we then find, on summing over r from 1 to p:

$$\sum_{r=1}^{r=p}(b_r + z_r)|\,Y_r\,|^2 - \sum_{r=1}^{r=p}a_r(Y_{r+1}\overline{Y}_r + \overline{Y}_{r+1}Y_r) = -a_p\overline{Y}_{p+1}Y_p.$$

Hence, if we consider only the imaginary parts, we get

$$\mathfrak{J}(a_pY_p\overline{Y}_{p+1}) = \sum_{r=1}^{p}\alpha_r(Y_{r+1}\overline{Y}_r + \overline{Y}_{r+1}Y_r) - \sum_{r=1}^{p}(\beta_r + y_r)|\,Y_r\,|^2. \tag{17.17}$$

With the aid of this formula and (17.16) we then have:

$$2r_p(z) = \frac{1}{\left[\sum_{r=1}^{p+1}\beta_r|\,Y_r\,|^2 - \sum_{r=1}^{p}\alpha_r(Y_{r+1}\overline{Y}_r + \overline{Y}_{r+1}Y_r) + \sum_{r=1}^{p}y_r|\,Y_r\,|^2 - (1 - g_p)\beta_{p+1}|\,Y_{p+1}\,|^2\right]}. \tag{17.18}$$

Finally, if we employ the identity (16.10), this formula can be written, if we now use (17.12),

$$2r_p(z) = \frac{1}{g_0\beta_1 + \sum_{r=1}^{p}(y_r|\,Y_r\,|^2 + |\,(\beta_r(1 - g_{r-1}))^{1/2}Y_r - (\beta_{r+1}g_r)^{1/2}Y_{r+1}\,|^2)}. \tag{17.19}$$

This holds for $p = 1, 2, 3, \cdots$. It should be emphasized that this formula was developed under the hypothesis (17.12). The formulas prior to (17.19) hold under the hypothesis (17.2).

18. Positive Definite Continued Fractions and the Parabola Theorem. We pointed out in § 16 that the continued fraction of the parabola theorem (Theorem 14.2) is a particular case of the positive definite continued fraction. The question naturally arises as to whether or not the machinery which we have set up in the preceding section can be used to obtain the condition for convergence given in the parabola theorem.

To answer this question, we shall first express the center C_p (cf. (17.15)) of the circular region $K_p(z)$ in terms of the polynomials $A_p = A_p(z)$ and $B_p = B_p(z)$ of (17.9). We find immediately that

$$C_p = \frac{\bar{a}_p{}^2 A_p \bar{B}_{p-1} - a_p{}^2 A_{p-1}\bar{B}_p + 2ig_p\beta_{p+1}A_p\bar{B}_p}{\bar{a}_p{}^2 B_p \bar{B}_{p-1} - a_p{}^2 B_{p-1}\bar{B}_p + 2ig_p\beta_{p+1}B_p\bar{B}_p}.$$

Hence, the radius $r_p(z) = | C_p - A_p/B_p |$ is given by

$$r_p(z) = \frac{| a_1 a_2 \cdots a_p |^2}{| B_p + i a_p{}^2 B_{p-1} |^2 - | a_p{}^2 B_{p-1} |^2 + (2g_p\beta_{p+1} - 1)| B_p |^2}. \tag{18.1}$$

In (16.1) we now put $z_p = 0$, $b_p = i$, $p = 1, 2, 3, \cdots$, so that the continued fraction becomes

$$\cfrac{1}{i - \cfrac{a_1{}^2}{i - \cfrac{a_2{}^2}{i - \cdot_{\textstyle\cdot_{\textstyle\cdot}}}}}, \tag{18.2}$$

and $B_p + i a_p{}^2 B_{p-1} = -iB_{p+1}$. On taking $g_p = \frac{1}{2}$, $p = 1, 2, 3, \cdots$, we then find that

$$r_p = r_p(0) = \frac{| a_1 a_2 \cdots a_p |^2}{| B_{p+1} |^2 - | a_p{}^2 B_{p-1} |^2}.$$

We now make the substitution

$$c_1 = 1, \quad a_p{}^2 = \frac{1}{c_p c_{p+1}}, \quad p = 1, 2, 3, \cdots,$$

$$Q_p = c_1 c_2 \cdots c_p B_p,$$

and this expression for r_p becomes

$$r_p = \frac{| c_{p+1} |}{| Q_{p+1} | - | Q_{p-1} |} \cdot \frac{1}{| Q_{p+1} | + | Q_{p-1} |}. \tag{18.3}$$

At the same time, the continued fraction (18.2) becomes, under the above substitution,

$$\cfrac{1}{c_1 i - \cfrac{1}{c_2 i - \cfrac{1}{c_3 i - \cdots}}}, \tag{18.4}$$

which is equivalent to

$$\cfrac{-i}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cdots}}}.$$

Of course, this substitution can be made only when the a_p are different from zero. The convergence of (18.2) when some a_p vanishes is covered by Theorem 17.2.

We wish to show that $\lim_{p=\infty} r_p = 0$ if the series $\Sigma | c_p |$ diverges. That is, the limit-point case (cf. § 17) holds, so that the continued fraction converges if the series $\Sigma | c_p |$ diverges. The question raised at the beginning of this section will then have been answered in the affirmative.

The means for the proof are at hand. In fact, since

$$| a_p^2 | - \Re(a_p^2) \leq \tfrac{1}{2}, \quad p = 1, 2, 3, \cdots,$$

then, as in §14,

$$| 1 + a_1^2 | \geq \tfrac{1}{2} + | a_1^2 |,$$

$$| 1 + a_p^2 + a_{p+1}^2 | \geq | a_p^2 | + | a_{p+1}^2 |, \quad p = 1, 2, 3, \cdots,$$

or

$$| 1 + c_1 c_2 | \geq \tfrac{1}{2} | c_1 c_2 | + 1,$$

$$| c_p c_{p+1} c_{p+2} + c_{p+2} + c_p | \geq | c_{p+2} | + | c_p |, \quad p = 1, 2, 3, \cdots.$$

Therefore, as in (13.8),

$$| Q_2 | - | Q_0 | \geq \tfrac{1}{2} | c_2 |,$$

$$| Q_3 | - | Q_1 | \geq | c_3 | \geq \tfrac{1}{2} | c_3 |,$$

or

$$|Q_2| \geq \tfrac{1}{2}(1 + |c_2|),$$
$$|Q_3| \geq \tfrac{1}{2}(1 + |c_3|); \tag{18.5}$$

and, as in (13.10),

$$|Q_{2p}| \geq |Q_{2p-2}| + \tfrac{1}{2}|c_{2p}|,$$
$$|Q_{2p+1}| \geq |Q_{2p-1}| + \tfrac{1}{2}|c_{2p+1}|. \tag{18.6}$$

On applying (18.5) and (18.6) to (18.3), we get

$$r_{2p-1} \leq \frac{2}{1 + \sum_{r=1}^{p-1} |c_{2r}|}, \quad r_{2p} \leq \frac{2}{1 + \sum_{r=1}^{p-1} |c_{2r+1}|}. \tag{18.7}$$

Inasmuch as $r_1 > r_2 > r_3 > \cdots$, it then follows that $r_p \to 0$ if the series $\Sigma|c_p|$ diverges.

This proof gives us some new information. Since all the approximants from and after the pth approximant are in the circular region K_p, we can make an assertion concerning uniform convergence. Let $a_1^2, a_2^2, a_3^2, \cdots$ have their values so restricted in the parabolic domain $|z| - \Re(z) \leq \frac{1}{2}$ that the quantity R_p defined below tends *uniformly* to 0 for all these values of a_1^2, $a_2^2, a_3^2, \cdots$. We take

$$R_p = \begin{cases} \dfrac{1}{\sum_{r=1}^{p} |c_r|} & \text{if} \quad a_1 a_2 \cdots a_{p-1} \neq 0; \\ 0 \quad \text{if} \quad a_k = 0 & \text{for some} \quad k \leq p - 1. \end{cases}$$

Then, the continued fraction converges uniformly over the specified set of values of $a_1^2, a_2^2, a_3^2, \cdots$. This is true, in particular, if these values are restricted to a bounded portion of the parabolic domain. Hence, we have, if we change the notation,

THEOREM 18.1. *The continued fraction*

$$\cfrac{1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdot_{\,\cdot}}}}$$

converges uniformly for $a_2, a_3, a_4, \cdots$ *in the domain*

$$|z| - \Re(z) \leq \tfrac{1}{2}, \quad |z| < M,$$

for every constant M.

The above type of argument was used by Paydon and Wall **[68]** before the notion of a positive definite continued fraction had been introduced. They obtained the nest of circles in this particular case by mapping the domain $|w - 1| \leq 1$ by means of the transformations $t = t_1 t_2 \cdots t_p(w)$, where $t_p(w) = 1/(1 + a_{p+1}w)$.

19. Chain Sequences. In the present chapter, and also in Chapter III, we have seen that sequences of the form

$$(1 - g_0)g_1, \quad (1 - g_1)g_2, \quad (1 - g_2)g_3, \cdots, \tag{19.1}$$

where $0 \leq g_p \leq 1$, $p = 0, 1, 2, \cdots$, play a fundamental role. Some of the properties of these sequences have been revealed during the course of the preceding developments. We shall pause here to give the subject a somewhat more systematic treatment.

The sequences in question will be called **chain sequences;** and the numbers g_p will be called **parameters** of the sequences.

The constant term sequence $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \cdots$ is a chain sequence with parameters $g_p = \frac{1}{2}, p = 0, 1, 2, \cdots$. In the following theorem we characterize all constant term chain sequences.

THEOREM 19.1. *A constant term sequence* $a, a, a, \cdots$ *is a chain sequence if, and only if,* $0 \leq a \leq \frac{1}{4}$.

Proof. To show that the condition is necessary, we show that any chain sequence whose terms are greater than $\frac{1}{4}$ is not a constant term sequence. In fact, if

$$(1 - g_{p-1})g_p > \tfrac{1}{4}, \quad p = 1, 2, 3, \cdots, \tag{19.2}$$

then

$$\frac{(1 - g_{p-1}) + g_p}{2} \geq \sqrt{(1 - g_{p-1})g_p} > \frac{1}{2},$$

so that $g_p > g_{p-1}$. Hence, the increasing sequence $g_0 < g_1 < g_2 < \cdots < 1$, must converge to a limit g, and, by (19.2), $(1 - g)g \geq \frac{1}{4}$. That is, the chain sequence converges to the limit $\frac{1}{4}$, inasmuch as $(1 - g)g$ cannot exceed $\frac{1}{4}$, and is therefore not a constant term sequence. To show that the condition is sufficient, one need but note the identity $(1 - g)g = a$ where $g = (1 + \sqrt{1 - 4a})/2$, $0 \leq a \leq \frac{1}{4}$.

A chain sequence does not, in general, determine its parameters uniquely. For instance, the sequence $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \cdots$ has the parameters

$$m_p = \frac{1}{2}\left(1 - \frac{1}{p+1}\right), \quad \text{and} \quad M_p = \frac{1}{2}, \quad p = 0, 1, 2, \cdots. \tag{19.3}$$

We shall now prove

THEOREM 19.2. *Every chain sequence has minimal parameters m_p and maximal parameters M_p, such that*

$$m_p \leq g_p \leq M_p, \quad p = 0, 1, 2, \cdots, \tag{19.4}$$

for all other parameters g_p of the sequence [139, 9].

Proof. Let $\{a_p\}$ be a given chain sequence with parameters g_p. We shall define the minimal parameters recurrently as follows.

$$m_0 = 0, \quad m_{p+1} = \begin{cases} 0 & \text{if} \quad m_p = 1, \\ \dfrac{a_{p+1}}{1 - m_p} & \text{if} \quad m_p < 1, \end{cases} \quad p = 0, 1, 2, \cdots. \tag{19.5}$$

Since $m_0 = 0$, then $m_0 \leq g_0$. Using induction, we suppose that for some $k \geq 0$, $m_k \leq g_k$. If $m_k = 1$ or $m_k < 1$ and $a_{k+1} = 0$, then $m_{k+1} = 0 \leq g_{k+1}$; if $m_k = 1$ and $a_{k+1} > 0$, then, by (19.5) and the induction hypothesis,

$$m_{k+1} = \frac{a_{k+1}}{1 - m_k} = \frac{(1 - g_k)g_{k+1}}{1 - m_k} \leq \frac{(1 - g_k)g_{k+1}}{1 - g_k} = g_{k+1}.$$

Thus, $m_p \leq g_p$ for $p = 0, 1, 2, \cdots$.

From the definition of the m_p, it follows that $m_p \geq 0$; and from what we have just proved, it follows that $m_p \leq 1$.

That $(1 - m_p)m_{p+1} = a_{p+1}$ is obvious from (19.5) if $m_p < 1$. If $m_p = 1$, then g_p, being greater than or equal to m_p, must equal 1, so that $a_{p+1} = (1 - g_p)g_{p+1} = 0$. Hence, $(1 - m_p)m_{p+1} = a_{p+1}$ in this case.

We have completed the proof of existence of the minimal parameters.

We now define the maximal parameters by the formula

$$M_p = 1 - \cfrac{a_{p+1}}{1 - \cfrac{a_{p+2}}{1 - \cfrac{a_{p+3}}{1 - \cdots}}}, \qquad p = 0, 1, 2, \cdots, \tag{19.6}$$

where we adopt the convention that in case some partial numerator of the continued fraction vanishes, then the continued fraction shall terminate with the first vanishing partial quotient. Thus, if $a_{p+1} = 0$, then $M_p = 1 \geq g_p$; if $a_{p+2} = 0$, $a_{p+1} > 0$, then $M_p = 1 - a_{p+1} = 1 - (1 - g_p)g_{p+1} \geq g_p$. If a_{p+1}, a_{p+2}, $\cdots$, a_{p+k} are positive and $a_{p+k+1} = 0$, $(k > 0)$, then

$$M_p = 1 - (1 - g_p) \cdot \cfrac{g_{p+1}}{1 - \cfrac{(1 - g_{p+1})g_{p+2}}{1 - \cfrac{(1 - g_{p+2})g_{p+3}}{1 - \cdots - \cfrac{(1 - g_{p+k-2})g_{p+k-1}}{1 - \cfrac{(1 - g_{p+k-1})g_{p+k}}{1}}}}}.$$

If $g_{p+k} = 1$, this obviously reduces to $M_p = g_p$; and if $g_{p+k} < 1$, then we may write, by Theorem 2.1,

$$M_p = 1 - (1 - g_p)\left(1 - \frac{1}{T_p}\right), \tag{19.7}$$

where

$$T_p = 1 + \sum_{r=p+1}^{p+k} \frac{g_{p+1}g_{p+2} \cdots g_r}{(1 - g_{p+1})(1 - g_{p+2}) \cdots (1 - g_r)}. \tag{19.8}$$

In this case, $M_p > g_p$. Finally, if $a_{p+r} > 0$, $r = 1, 2, 3, \cdots$, then M_p is again given by (19.7), but with T_p now defined by

$$T_p = 1 + \sum_{r=p+1}^{\infty} \frac{g_{p+1}g_{p+2} \cdots g_r}{(1 - g_{p+1})(1 - g_{p+2}) \cdots (1 - g_r)} \leq \infty. \tag{19.9}$$

Hence, again $M_p \geq g_p$. Equality holds here if, and only if, $T_p = \infty$.

It follows from the definition of the M_p, and from what we have just proved, that $0 \leq M_p \leq 1$ for $p = 0, 1, 2, \cdots$. Finally, if $a_{p+1} = 0$, then $M_p = 1$ and $(1 - M_p)M_{p+1} = 0 = a_{p+1}$. If $a_{p+1} > 0$ so that $0 < g_{p+1} \leq M_{p+1}$, then it follows from (19.6) that

$$M_p = 1 - \frac{a_{p+1}}{M_{p+1}}, \quad a_{p+1} = (1 - M_p)M_{p+1}.$$

This completes the proof of Theorem 19.2.

We note that in case $a_p > 0$, $p = 1, 2, 3, \cdots$, then $M_p = g_p$ for all values of p if, and only if, the series

$$1 + \sum_{r=1}^{\infty} \frac{g_1 g_2 \cdots g_r}{(1 - g_1)(1 - g_2) \cdots (1 - g_r)} \tag{19.10}$$

diverges. In particular, we have

THEOREM 19.3. *Let a_p, $p = 1, 2, 3, \cdots$, be a positive-term chain sequence, with minimal parameters m_p. Then, the maximal parameters are equal to the corresponding minimal parameters, so that the chain sequence determines its parameters uniquely if, and only if, the series*

$$1 + \sum_{r=1}^{\infty} \frac{m_1 m_2 \cdots m_r}{(1 - m_1)(1 - m_2) \cdots (1 - m_r)} \tag{19.11}$$

diverges.

COROLLARY 19.1. *A positive-term chain sequence determines its parameters uniquely if, and only if, its* 0th *maximal parameter M_0 is equal to* 0.

It is important in certain connections (cf. Theorem 17.1) to know conditions under which the maximal parameters of a chain sequence are all positive. We have this theorem, which readily follows from the definition of the maximal parameters.

THEOREM 19.4. *Let a_p, $p = 1, 2, 3, \cdots$, be a chain sequence whose minimal parameters satisfy the inequalities $0 \leq m_p < 1$, $p = 0, 1, 2, \cdots$. Then, its maximal parameters are all positive if, and only if, all the continued fractions*

$$\cfrac{1}{1-\cfrac{a_{p+1}}{1-\cfrac{a_{p+2}}{1-\cdots}}},\qquad p = 0, 1, 2, \cdots,$$

are convergent.

We may illustrate Theorem 19.2 by means of the constant term sequence $a, a, a, \cdots$, where $0 \leq a \leq \frac{1}{4}$ (cf. Theorem 19.1). In this case, the minimal parameter m_p is given by the pth approximant of the continued fraction

$$\cfrac{a}{1-\cfrac{a}{1-\cfrac{a}{1-\cdots}}},$$

so that

$$m_p = \frac{1-\sqrt{1-4a}}{2}\left[1-\frac{1}{\displaystyle\sum_{r=0}^{p}\left(\frac{1+\sqrt{1-4a}}{1-\sqrt{1-4a}}\right)^r}\right], \tag{19.12}$$

$$p = 0, 1, 2, \cdots;$$

and the maximal parameters are given by

$$M_p = 1-\cfrac{a}{1-\cfrac{a}{1-\cdots}} = \frac{1+\sqrt{1-4a}}{2},\quad p = 0, 1, 2, \cdots. \tag{19.13}$$

These formulas reduce to (19.3) when $a = \frac{1}{4}$.

We shall now investigate the question as to when a chain sequence $\{b_p\}$ can be found which dominates a given chain sequence $\{a_p\}$. We have this theorem.

THEOREM 19.5. *Given a chain sequence $\{a_p\}$ with minimal and maximal parameters m_p and M_p. Then, there exists a chain sequence $\{b_p\}$ such that*

$$b_p > a_p,\quad p = 1, 2, 3, \cdots, \tag{19.14}$$

if, and only if,

$$m_p < M_p,\quad p = 0, 1, 2, \cdots. \tag{19.15}$$

Proof. Suppose first that (19.15) holds, and put

$$g_p = rm_p + (1 - r)M_p, \quad p = 0, 1, 2, \cdots,$$

where $0 < r < 1$. Then, clearly, $0 < g_p < 1$. Let

$$b_p = (1 - g_{p-1})g_p.$$

Then, (19.14) holds. In fact, the inequality (19.14) is equivalent to the inequality

$$(1 - rm_{p-1} - (1 - r)M_{p-1})(rm_p + (1 - r)M_p)$$
$$> r(1 - m_{p-1})m_p + (1 - r)(1 - M_{p-1})M_p.$$

This, in turn, is equivalent to the inequality

$$(M_{p-1} - m_{p-1})(M_p - m_p) > 0,$$

which holds because of (19.15).

To prove that, conversely, (19.14) implies (19.15), we prove the following more precise result.

THEOREM 19.6. *Let $\{a_p\}$ be a chain sequence with minimal and maximal parameters m_p and M_p, respectively. Let $\{b_p\}$ be a second chain sequence with parameters h_p, such that*

$$b_p \geq a_p, \quad p = 1, 2, 3, \cdots. \tag{19.16}$$

Then,

$$m_p \leq h_p \leq M_p, \quad p = 0, 1, 2, \cdots. \tag{19.17}$$

If, moreover, for any particular $k \geq 0$, $b_{k+1} > a_{k+1}$, then $m_{k+1} < h_{k+1}$, and $h_k < M_k$.

Proof. Since $m_0 = 0$, then $m_0 \leq h_0$. Using induction, let us assume that $m_k \leq h_k$ for some $k \geq 0$. In case $a_{k+1} = 0$, so that $m_{k+1} = 0$, then $m_{k+1} \leq h_{k+1}$; while if $a_{k+1} > 0$, then it follows from the relations $(1 - h_k)h_{k+1} \geq (1 - m_k)m_{k+1} > 0$ and $1 - h_k \leq 1 - m_k$, that $m_{k+1} \leq h_{k+1}$. If $(1 - h_k)h_{k+1} > (1 - m_k)m_{k+1}$, then it is clear that $m_{k+1} < h_{k+1}$. This proves the part of the theorem which relates to the minimal parameters.

We now prove the part relating to the maximal parameters. By (19.6), if $a_{p+1} = 0$, then $M_p = 1 \geq h_p$, and if $(1 - h_p)h_{p+1} > a_{p+1}$, so that $h_p < 1$, then $M_p > h_p$. If $a_{p+1} > 0$, $a_{p+2} = 0$, then $h_{p+1} > 0$, $h_p < 1$, so that if $(1 - h_p)h_{p+1} \geq a_{p+1}$, we have

$$h_p \leq 1 - \frac{a_{p+1}}{h_{p+1}} \leq 1 - a_{p+1} = M_p,$$

and if $(1 - h_p)h_{p+1} > a_{p+1}$, then

$$h_p < 1 - \frac{a_{p+1}}{h_{p+1}} \leq 1 - a_{p+1} = M_p.$$

For some $k \geq 2$, let $a_{p+1}, a_{p+2}, \cdots, a_{p+k}$ be positive and $a_{p+k+1} = 0$. If (19.16) holds, then

$$h_p \leq 1 - \frac{a_{p+1}}{h_{p+1}},$$

$$h_{p+1} \leq 1 - \frac{a_{p+2}}{h_{p+2}},$$

$$\cdot \quad \cdot \quad \cdot$$

$$h_{p+k-1} \leq 1 - \frac{a_{p+k}}{h_{p+k}} \leq 1 - a_{p+k},$$

and therefore

$$h_p \leq 1 - \cfrac{a_{p+1}}{1 - \cfrac{a_{p+2}}{1 - \cdots - \cfrac{a_{p+k-1}}{1 - a_{p+k}}}} = M_p.$$

If $(1 - h_p)h_{p+1} > a_{p+1}$, then it is evident that $h_p < M_p$. Finally, if $a_{p+r} > 0$, $r = 1, 2, 3, \cdots$, and (19.16) holds, then we have, successively,

$$h_p \leq 1 - \frac{a_{p+1}}{h_{p+1}} < 1 - a_{p+1},$$

$$h_p \leq 1 - \frac{a_{p+1}}{h_{p+1}} \leq 1 - \cfrac{a_{p+1}}{1 - \cfrac{a_{p+2}}{h_{p+2}}} < 1 - \frac{a_{p+1}}{1 - a_{p+2}} < 1 - a_{p+1},$$

$$h_p \leq 1 - \frac{a_{p+1}}{h_{p+1}} \leq 1 - \cfrac{a_{p+1}}{1 - \cfrac{a_{p+2}}{h_{p+2}}} \leq 1 - \cfrac{a_{p+1}}{1 - \cfrac{a_{p+2}}{1 - \cfrac{a_{p+3}}{h_{p+3}}}}$$

$$< 1 - \cfrac{a_{p+1}}{1 - \cfrac{a_{p+2}}{1 - a_{p+3}}} < 1 - \frac{a_{p+1}}{1 - a_{p+2}} < 1 - a_{p+1},$$

$$\cdot \quad \cdot \quad \cdot,$$

so that $h_p \leq M_p$. If $(1 - h_p)h_{p+1} > a_{p+1}$, then actual inequality holds in the first place in each line of the above system, so that $h_p < M_p$.

This completes the proof of Theorem 19.6.

In the proof of Theorem 19.5, we used a chain sequence whose parameters are certain means between m_p and M_p. It is interesting to note that if

$$g_p = \sqrt{m_p M_p}, \quad m_p < M_p, \quad m_{p+1} > 0, \quad p = 0, 1, 2, \cdots,$$

and $b_p = (1 - g_{p-1})g_p$, then $b_p > a_p$.

In Lemma 12.1 we have an example of a chain sequence, namely, any sequence $\{c_p\}$ of nonnegative numbers such that $\sum_{p=1}^{n} c_p < 1$, $n = 1, 2, 3, \cdots$. Hence, $r, r^2, r^3, \cdots$ is a chain sequence if $0 \leq r \leq \frac{1}{2}$. The sequence $\{m_p\}$ of (14.2) has the property that $\{m_p/2\}$ is a chain sequence. In this case the parameters may all be taken less than $\frac{1}{2}$.

20. Quadratic Forms and Chain Sequences. In Theorem 16.2, we have a remarkable characterization of chain sequences in terms of a special type of quadratic form. This may be stated as follows.

THEOREM 20.1. *The sequence $\{\alpha_p^2\}$ is a chain sequence if, and only if, for all real numbers $\xi_1, \xi_2, \xi_3, \cdots$,*

$$\sum_{p=1}^{n} \xi_p^2 - 2\sum_{p=1}^{n-1} \alpha_p \xi_p \xi_{p+1} \geq 0, \quad n = 2, 3, 4, \cdots. \tag{20.1}$$

From this theorem it follows immediately that if $\{\alpha_p^2\}$ is a chain sequence, and if $\beta_p^2 \leq \alpha_p^2$, $p = 1, 2, 3, \cdots$, then $\{\beta_p^2\}$ is a chain sequence.

The minimal and maximal parameters of a chain sequence can be expressed in terms of values of the quadratic form of Theorem 20.1. We have this theorem.

THEOREM 20.2. *Let $\{\alpha_p^2\}$ be a given chain sequence with minimal and maximal parameters m_p and M_p, respectively. Then, for $p = 0, 1, 2, \cdots$, we have*

$$m_p = 1 - \min\left[\sum_{r=1}^{p+1} \xi_r^2 - 2\sum_{r=1}^{p} \alpha_r \xi_r \xi_{r+1}\right], \quad \xi_{p+1} = 1, \qquad (20.2)$$

and

$$M_p = \text{g.l.b.}\left[\sum_{r=p+1}^{\infty} \xi_r^2 - 2\sum_{r=p+1}^{\infty} \alpha_r \xi_r \xi_{r+1}\right], \xi_{p+1} = 1. \quad [9.] \quad (20.3)$$

Here, the minimum of the form in (20.2) *is taken with respect to all real* ξ_r, *and in* (20.3), *the greatest lower bound is taken with respect to all rea* ξ_r *for which the infinite series converge.*

Proof. From the identity (16.10) we have

$$\sum_{r=1}^{p+1} \xi_r^2 - 2\sum_{r=1}^{p} \alpha_r \xi_r \xi_{r+1} = \sum_{r=1}^{p} (\sqrt{1 - m_{r-1}}\,\xi_r - \sqrt{m_r}\,\xi_{r+1})^2$$
$$+ (1 - m_p)\xi_{p+1}^2 \geq (1 - m_p)\xi_{p+1}^2.$$

By (19.5), $m_r = 0$ if $m_{r-1} = 1$. Hence, it follows that we may choose $\xi_1\ \xi_2, \cdots \xi_p$ such that the above form takes on the minimum value $1 - m_p$ for $\zeta_{p+1} = 1$. This proves (20.2).

Again, from (16.10), we find, for any $n \geq p + 1$,

$$\sum_{r=p+1}^{n+1} \xi_r^2 - 2\sum_{r=p+1}^{n} \alpha_r \xi_r \xi_{r+1} = M_p \xi_{p+1}^2$$
$$+ \sum_{r=p+1}^{n} (\sqrt{1 - M_{r-1}}\,\xi_r - \sqrt{M_r}\,\xi_{r+1})^2 + (1 - M_n)\xi_{n+1}^2$$
$$\geq M_p \xi_{p+1}^2.$$

Hence, the right-hand member of (20.3), which we shall denote by M_p', is greater than or equal to M_p. We must prove that $M_p' = M_p$, $p = 0, 1, 2, \cdots$.

Obviously, $M_p' \geq 0$. On putting $\xi_{p+1} = 1$, $\xi_r = 0$ for $r = p + 2$, $p + 3$, $p + 4$, $\cdots$, we see immediately that $M_p' \leq 1$.

From the way in which the maximal parameters were defined, it follows that $M_p, M_{p+1}, M_{p+2}, \cdots$ are the maximal parameters of the chain sequence $\alpha_{p+1}^2, \alpha_{p+2}^2, \alpha_{p+3}^2, \cdots$, for every p. By hypothesis, if $n \geq p + 1$,

$$\sum_{r=p+1}^{n+1} \xi_r^2 - 2\sum_{r=p+1}^{n} \alpha_r \xi_r \xi_{r+1} \geq M_p' \xi_{p+1}^2.$$

Inasmuch as $M_p'\xi_{p+1}^2 \geq 2\sqrt{M_p'}\xi_p\xi_{p+1} - \xi_p^2$, it therefore follows that

$$\sum_{r=p}^{n+1}\xi_r^2 - 2\sum_{r=p}^{n}\alpha_r'\xi_r\xi_{r+1} \geq 0, \quad n = p, p+1, p+2, \cdots,$$

where

$$\alpha_p' = \sqrt{M_p'}, \quad \alpha_r' = \alpha_r \quad \text{for} \quad r > p.$$

Therefore, $\alpha_p'^2, \alpha_{p+1}'^2, \alpha_{p+2}'^2, \cdots$ is a chain sequence. Let its minimal parameters be $m_0' = 0, m_1', m_2', \cdots$. Then,

$$\alpha_p'^2 = m_1' = M_p', \quad \alpha_{p+1}^2 = (1 - M_p')m_2',$$

$$\alpha_{p+2}^2 = (1 - m_2')m_3', \quad \cdots.$$

But $M_p', m_2', m_3', \cdots$ are in consequence parameters of the chain sequence $\alpha_{p+1}^2, \alpha_{p+2}^2, \alpha_{p+3}^2, \cdots$, and therefore $M_p' \leq M_p$, $m_2' \leq M_{p+1}$, $m_3' \leq M_{p+2}, \cdots$. The first of these inequalities, which holds for $p = 0, 1, 2, \cdots$, together with the inequality $M_p' \geq M_p$, establishes (20.3), and the proof of the theorem is complete.

We shall conclude this section by expressing the minimal parameters by means of certain determinants. We first state a known theorem of algebra concerning the general real quadratic form

$$F(\xi_1, \xi_2, \cdots, \xi_n) = \sum_{p,q=1}^{n} a_{pq}\xi_p\xi_q, \quad (a_{pq} = a_{qp}).$$

If $F > 0$ for all real ξ_p such that $\sum_{p=1}^{n}\xi_p^2 > 0$, then F is called *positive definite.*

THEOREM 20.3. *The form F is positive definite if, and only if, the determinants*

$$D_1 = a_{11}, \quad D_2 = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}, \cdots, \quad D_n = \begin{vmatrix} a_{11}, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22}, & \cdots, & a_{2n} \\ & \cdot & \cdot & \cdot \\ a_{n1}, & a_{n2}, & \cdots, & a_{nn} \end{vmatrix}$$

are all positive. If these determinants are different from zero, we have the decomposition

$$F \equiv D_1X_1^2 + \frac{D_2}{D_1}X_2^2 + \cdots + \frac{D_{n-1}}{D_{n-2}}X_{n-1}^2 + \frac{D_n}{D_{n-1}}\xi_n^2, \quad (20.4)$$

where

$$D_1X_1 = D_1\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 + \cdots + a_{1n}\xi_n,$$

$$D_2X_2 = D_2\xi_2 + \begin{vmatrix} a_{11}, & a_{13}\xi_3 + a_{14}\xi_4 + \cdots + a_{1n}\xi_n \\ a_{21}, & a_{23}\xi_3 + a_{24}\xi_4 + \cdots + a_{2n}\xi_n \end{vmatrix},$$

$$D_3X_3 = D_3\xi_3 + \begin{vmatrix} a_{11}, & a_{12}, & a_{14}\xi_4 + a_{15}\xi_5 + \cdots + a_{1n}\xi_n \\ a_{21}, & a_{22}, & a_{24}\xi_4 + a_{25}\xi_5 + \cdots + a_{2n}\xi_n \\ a_{31}, & a_{32}, & a_{34}\xi_4 + a_{35}\xi_5 + \cdots + a_{3n}\xi_n \end{vmatrix},$$

$$\cdots\cdot$$

The formula (20.4) shows that when the D_p are positive, then D_n/D_{n-1} is the minimum value of F under the condition $\xi_n = 1$. We shall apply this remark to the form

$$\sum_{r=1}^{p+1}(1+y)\xi_r^2 - 2\sum_{r=1}^{p}\alpha_r\xi_r\xi_{r+1}, \quad (20.5)$$

where $\{\alpha_r^2\}$ is a chain sequence. This form is positive definite for all $y > 0$. Hence, the determinants $D_r = D_r(y)$ formed from its coefficients, as in Theorem 20.3, are all positive. From the remark made above, it follows that $D_{p+1}(y)/D_p(y)$ is the minimum value of (20.5) under the condition $\xi_{p+1} = 1$. We shall prove that the minimal parameters of the chain sequence $\{\alpha_p^2\}$ are given by

$$m_p = 1 - \lim_{y\to 0}\frac{D_{p+1}(y)}{D_p(y)}, \quad p = 0, 1, 2, \cdots, \quad (D_0 = 1). \quad \textbf{[139.]} \quad (20.6)$$

(Cf. (20.2).) In fact, we have the recurrence formulas

$$D_{p+1}(y) = (1+y)D_p(y) - \alpha_p^2D_{p-1}(y), \quad p = 1, 2, 3, \cdots,$$

so that

$$\alpha_p^2 = \frac{D_p(y)}{D_{p-1}(y)}\left(1 + y - \frac{D_{p+1}(y)}{D_p(y)}\right). \quad (20.7)$$

Therefore,

$$0 < \frac{D_{p+1}(y)}{D_p(y)} < 1 + y.$$

Hence, the limit of the rational function of y,

$$\lim_{y=0} \frac{D_{p+1}(y)}{D_p(y)} = 1 - m_p',$$

exists and is not greater than 1 nor less than 0; and, by (20.7), $\alpha_p^2 = (1 - m_{p-1}')m_p'$, so that the m_p' are parameters of the chain sequence. Since $1 - m_n'$ is the limit for $y = 0$ of the minimum for $\xi_{n+1} = 1$ of the form (20.5), it is reasonable to expect, in view of (20.2), that $m_n' = m_n$. This can be easily proved by mathematical induction. We have, $m_0' = 0 = m_0$. Assuming that $m_k' = m_k$ for some index k, we see from the formula $(1 - m_k)m_{k+1} = (1 - m_k)m_{k+1}'$ that $m_{k+1}' = m_{k+1}$ provided $m_k < 1$. But, if $m_k = 1$, so that $\alpha_{k+1} = 0$, we have

$$\frac{D_{k+2}(y)}{D_{k+1}(y)} = 1 + y,$$

so that $m_{k+1}' = 0 = m_{k+1}$ (cf. (19.5)). This completes the proof of the formula (20.6).

Exercise 4

4.1. Let $\{\theta_p\}$ be a chain sequence with minimal and maximal parameters m_p and M_p, respectively. Show that parameters g_p may be chosen such that g_0 has any prescribed value between m_0 and M_0.

4.2. Show that if the polynomial

$$\sqrt{\beta_r(1 - g_{r-1})}Y_r(z) - \sqrt{\beta_{r+1}g_r}Y_{r+1}(z)$$

appearing in (17.19) does not vanish identically, and if $z_r = \zeta$, $r = 1, 2, 3, \cdots$, then this polynomial in ζ has all its zeros in the lower half-plane, $I(\zeta) \leq 0$. [**138.**]

4.3. Show that if the limit-point case holds for $z_r = \zeta = x + iy$, $y > 0$, then the limit-point case holds for $z_r = x + iY$ for every $Y > y$. [**138.**]

4.4. Show that the real poles of an approximant of a positive definite continued fraction in which $z_r = \zeta$ must be simple with positive residues.

4.5. If the coefficients a_p and b_p are real and $z_r = \zeta$, then all the poles of the approximants are real. [**109.**]

4.6.[8] Show that the linear transformation

$$z' = t(z) = \frac{-a}{b - z}, \quad a \neq 0,$$

carries the circular region $|z - 1| \leq 1$ into a part of itself if, and only if,

$$\Re\left(\frac{1 - b}{a}\right) \geq \frac{1}{2} + \left|\frac{1}{a}\right|.$$

Suggestion. Show that $t(z)$ maps the half-plane

$$\bar{a}(z - 1) + a(\bar{z} - 1) + 2a\bar{a}\left(\Re\left(\frac{1 - b}{a}\right) - \frac{1}{2}\right) \geq 0$$

upon $|z' - 1| \leq 1$; and this half-plane contains $|z - 1| \leq 1$ if, and only if, $\Re\left(\frac{1 - b}{a}\right) \geq \frac{1}{2} + \left|\frac{1}{a}\right|$.

4.7. If $k_p \neq 0$, $p = 0, 1, 2, \cdots$, the transformation

$$z' = t_n(z) = \frac{-a_n}{b_n - z}, \quad a_n \neq 0,$$

carries the circular region

$$\left|z - \frac{1}{k_n}\right| \leq \left|\frac{1}{k_n}\right|$$

into a part of the circular region

$$\left|z' - \frac{1}{k_{n-1}}\right| \leq \left|\frac{1}{k_{n-1}}\right|$$

if, and only if,

$$\Re\left(\frac{1 - k_n b_n}{k_n k_{n-1} a_n}\right) \geq \frac{1}{2} + \left|\frac{1}{k_n k_{n-1} a_n}\right|.$$

4.8. Put $a_n = 1/u_n$, $b_n = -c_n/u_n$, and let k_n be real and positive. Show that the last condition can be written

$$\Re(c_n) > 0, \quad |u_n| - \Re(u_n) \leq 2\Re(b_n)\Re(b_{n+1})(1 - g_{n-1})g_n,$$

where $0 < g_p < 1$. Apply this result to the continued fraction

$$\cfrac{1}{ic_1 - \cfrac{u_1}{ic_2 - \cfrac{u_2}{ic_3 - \cdots}}}$$

[8] Exercises 4.6, 4.7 and 4.8 were suggested to the author by R. E. Lane.

4.9. Let $g_1, (1 - g_1)g_2, (1 - g_2)g_3, \cdots$ be a chain sequence in which $0 < g_p < 1$, $p = 1, 2, 3, \cdots$, and suppose that the series

$$S = 1 + \Sigma \frac{g_1 g_2 \cdots g_p}{(1 - g_1)(1 - g_2) \cdots (1 - g_p)}$$

is convergent. Show that $Sg_1/(1 - S), (1 - g_1)g_2, (1 - g_2)g_3, \cdots$ is a chain sequence which determines its parameters uniquely.

Chapter V

SOME GENERAL CONVERGENCE THEOREMS

The starting point of the present chapter is the formula (17.19) for the radius $r_p(z)$ of the circle $K_p(z)$, which we have introduced in connection with a positive definite continued fraction. Let us recall that the continued fraction converges if $r_p(z) \to 0$ (limit point case). By inspection of the formula (17.19) for $r_p(z)$, we see that a *sufficient* condition for $r_p(z) \to 0$ is that the series $\Sigma y_p \mid Y_p(z) \mid^2$ be divergent. If, in particular, the y_p are bounded away from 0, then this series will be divergent if the series $\Sigma \mid Y_p(z) \mid^2$ is divergent. A glance at (17.7) and (17.9) will show immediately that the divergence of the series $\Sigma \mid X_p(z) \mid^2$ implies the divergence of the series $\Sigma \mid Y_p(z) \mid^2$. From these considerations it follows, then, that the positive definite continued fraction (16.1), in which the a_p are not zero, converges for any *particular* z_p such that $y_p = \mathfrak{J}(z_p) \geq \delta > 0$, $(p = 1, 2, 3, \cdots)$, provided that at least one of the series

$$\sum_{p=1}^{\infty} \mid X_p(z) \mid^2, \quad \sum_{p=1}^{\infty} \mid Y_p(z) \mid^2 \tag{a}$$

is divergent. These two series will play a fundamental role in the succeeding developments.

If our way of introducing these two series seems somewhat artificial, perhaps the reader will want to look for deeper-lying reasons in the following considerations. We are, of course, primarily interested in replacing the condition $r_p(z) \to 0$, involving the rather complicated formula (17.19), by some simpler condition. We are thus led in a natural way to use the series

$\Sigma| Y_p(z) |^2$. If we reflect upon the fact that the $X_p(z)$, as well as the $Y_p(z)$, are solutions of the system of equations

$$-a_{p-1}x_{p-1} + (b_p + z_p)x_p - a_p x_{p+1} = 0, \quad p = 1, 2, 3, \cdots, \tag{b}$$

and that the general solution of this system is $x_p = aX_p(z) + bY_p(z)$, we come naturally to consider the two series (a) together. The search for analogies between the system (b) and a second order linear differential equation enters into this train of ideas.

In the next section we prove Schwarz's inequality. Then, in § 22 we consider the question of dependence of the convergence of the series (a) upon the particular values of $z_1, z_2, \cdots$. We shall find that if these series converge for $z_p = z_p^*$, then they converge (uniformly) for all z_p such that $| z_p - z_p^* | < M$, where M is any finite constant. This result holds for continued fractions (16.1), subject to the *sole restriction* $a_p \neq 0$, $p = 1, 2, 3, \cdots$. In view of this "theorem of invariability," the question of convergence of the series (a) for any z_p such that $| z_p | < M$ is reduced to the question of convergence of these series for $z_p = 0$. We may therefore distinguish two cases, the **determinate case** and the **indeterminate case,** according as at least one of the series (a) diverges for $z_p = 0$, or both of these series converge for $z_p = 0$, respectively.[9]

In § 23 we consider questions of convergence of the general continued fraction (16.1) in the indeterminate case; § 24 contains some fundamental theorems on sequences of analytic functions; and § 25 deals with continued fractions (16.1) in which the z_p are equal to a common variable ζ ("J-fractions"). In § 26 and § 27 we treat J-fractions for which the a_p and b_p are bounded, and for which the a_p and b_p are real, respectively.

21. Schwarz's Inequality. In this and later chapters we shall need the following theorem.

THEOREM 21.1. (*Schwarz's inequality.*) *Let* $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ *be any complex numbers. Then*

$$\left| \sum_{p=1}^{n} a_p b_p \right|^2 \leq \sum_{p=1}^{n} | a_p |^2 \cdot \sum_{p=1}^{n} | b_p |^2, \tag{21.1}$$

[9] This classification was used by Hamburger [26] for continued fractions with real a_p and b_p and with $z_p = z$, $p = 1, 2, 3, \cdots$.

where equality holds if, and only if, either (a) $b_p = 0$, $p = 1, 2, 3, \cdots, n$, *or* (b) $a_p = c\bar{b}_p$, $p = 1, 2, 3, \cdots, n$, *where c is a complex constant.*

Proof. Let

$$a_p = c\bar{b}_p + e_p, \quad p = 1, 2, 3, \cdots, n, \tag{21.2}$$

where c and the e_p are to be determined. Since

$$a_p b_p = c| b_p |^2 + e_p b_p,$$

we see that

$$\sum_{p=1}^{n} a_p b_p = c \sum_{p=1}^{n} | b_p |^2, \tag{21.3}$$

if, and only if,

$$\sum_{p=1}^{n} e_p b_p = 0.$$

Since (21.1) obviously holds with equality in case the b_p are all 0, we shall assume that at least one $b_p \neq 0$. We may then determine c by (21.3), and the e_p are then determined by (21.2). Hence,

$$| a_p |^2 = (c\bar{b}_p + e_p)(\bar{c}b_p + \bar{e}_p) = | c |^2 | b_p |^2 + \bar{c}e_p b_p + c\bar{e}_p\bar{b}_p + | e_p |^2,$$

so that

$$\sum_{p=1}^{n} | a_p |^2 = | c |^2 \sum_{p=1}^{n} | b_p |^2 + \sum_{p=1}^{n} | e_p |^2,$$

or

$$\sum_{p=1}^{n} | a_p |^2 \geq | c |^2 \sum_{p=1}^{n} | b_p |^2, \tag{21.4}$$

where equality holds if, and only if, $e_p = 0$, $p = 1, 2, 3, \cdots$, i.e., if, and only if, $a_p = c\bar{b}_p$, $p = 1, 2, 3, \cdots, n$. Therefore, by (21.3) and (21.4),

$$\left| \sum_{p=1}^{n} a_p b_p \right|^2 = | c |^2 \sum_{p=1}^{n} | b_p |^2 \cdot \sum_{p=1}^{n} | b_p |^2 \leq \sum_{p=1}^{n} | a_p |^2 \cdot \sum_{p=1}^{n} | b_p |^2,$$

with equality if, and only if, $a_p = c\bar{b}_p$, $p = 1, 2, 3, \cdots, n$, as was to be proved.

This proof was given to the author by Ernst Hellinger.

22. The Theorem of Invariability. We shall now prove the following fundamental **theorem of invariability.**

THEOREM 22.1. *Let* (16.1) *be a continued fraction in which the coefficients* a_p *are different from zero. If the two series*

$$\Sigma \mid X_p(z) \mid^2, \quad \Sigma \mid Y_p(z) \mid^2, \tag{22.1}$$

formed for this continued fraction, converge for particular values z_p^* *of the variables* z_p, *then these series converge uniformly for* $\mid z_p - z_p^* \mid < M$, $p = 1, 2, 3, \cdots$, *where* M *is any constant independent of the* z_p.

Proof. The proof is accomplished by replacing the difference equation

$$L_p(z) \equiv -a_{p-1}x_{p-1} + (b_p + z_p)x_p - a_p x_{p+1} = 0,$$
$$p = 1, 2, 3, \cdots, \tag{22.2}$$

where we now take $a_0 = 1$, by a **Volterra sum equation** [10]

$$\zeta_p - \sum_{q=1}^{p} k_{pq}\zeta_q = h_p, \quad p = 1, 2, 3, \cdots. \tag{22.3}$$

(This is analogous to replacing a differential equation by an integral equation.) Denote by $L_p^*(x)$ the expression (22.2) in which z_p has been replaced by z_p^*. The solution of the system $L_p^*(x) = 0$ under the initial conditions $x_0 = -1$, $x_1 = 0$ is $x_p = X_p(z^*) = X_p^*$, and under the initial conditions $x_0 = 0$, $x_1 = 1$, the solution is $x_p = Y_p(z^*) = Y_p^*$. If x_p and x_p^* are arbitrary solutions of the systems $L_p(x) = 0$ and $L_p^*(x^*) = 0$, respectively, then we obtain immediately the identity (analogous to Green's formula)

$$\sum_{p=1}^{n} [x_p L_p^*(x^*) - x_p^* L_p(x)] = x_0 x_1^* - x_0^* x_1 - a_n(x_n x_{n+1}^* - x_n^* x_{n+1}) - \sum_{p=1}^{n}(z_p^* - z_p)x_p x_p^* = 0.$$

[10] This proof uses the idea which Weyl [141] applied in similar problems, namely, to express the relationship between solutions for two different parameter values as a Volterra integral or sum equation. This procedure, as well as the procedure used by Weyl [140] and by Hellinger [31] can be embraced in a single set-up using reciprocals of the J-matrix (cf. § 60).

On putting $x_p^* = X_p^*$ and then $x_p^* = Y_p^*$, we then get

$$x_1 - a_n(x_n X_{n+1}^* - x_{n+1}X_n^*) - \sum_{p=1}^{n}(z_p^* - z_p)x_p X_p^* = 0,$$

$$x_0 - a_n(x_n Y_{n+1}^* - Y_n^* x_{n+1}) - \sum_{p=1}^{n}(z_p^* - z_p)x_p Y_p^* = 0.$$

On multiplying the first of these equations by $-Y_n^*$, the second by X_n^*, and then adding, we obtain, if we use the determinant formula (17.11),

$$x_n - \sum_{p=1}^{n}(z_p - z_p^*)(X_p^* Y_n^* - X_n^* Y_p^*)x_p = x_1 Y_n^* - x_0 X_n^*.$$

Therefore, $\zeta_p = x_p$ is a solution of the equation (22.3) in which

$$k_{pq} = (z_q - z_q^*)(X_q^* Y_p^* - X_p^* Y_q^*),$$
$$h_p = x_1 Y_p^* - x_0 X_p^*.$$

The proof of the theorem will be complete if we show that the series $\Sigma|\zeta_p|^2$ converges uniformly for $|z_p - z_p^*| < M$, $p = 1, 2, 3, \cdots$, where ζ_p is *any* solution of this sum equation having bounded initial values ζ_0 and ζ_1.

Under the hypothesis that the two series $\Sigma|X_p^*|^2$ and $\Sigma|Y_p^*|^2$ are convergent, it readily follows by Schwarz's inequality that the double series $\Sigma|k_{pq}|^2$ is uniformly convergent for $|z_p - z_p^*| < M$. Hence, if ϵ is an arbitrarily chosen positive number, which we take less than unity, there exists an R, depending only upon ϵ, such that

$$\epsilon_r = \sum_{q=1}^{\infty}\sum_{p=r}^{\infty}|k_{pq}|^2 < \epsilon \quad \text{for} \quad r \geq R, \quad |z_p - z_p^*| < M.$$

Also, on taking R still larger if necessary, we have

$$C_r^2 = \sum_{p=r}^{\infty}|h_p|^2 < \epsilon^2, \quad \text{for} \quad r \geq R.$$

We now multiply the equation (22.3) by $\bar{\zeta}_p$ and sum over p from r to m, where $m > r$. This gives, if we apply Schwarz's inequality,

$$\sum_{p=r}^{m} |\zeta_p|^2 = \sum_{p=r}^{m} h_p\bar{\zeta}_p + \sum_{p=r}^{m}\sum_{q=1}^{p} k_{pq}\bar{\zeta}_p\zeta_q$$

$$\leq \sqrt{\sum_{p=r}^{m} |\zeta_p|^2}\left(C_r + \sqrt{\sum_{q=1}^{m} |\zeta_q|^2}\sqrt{\sum_{p=r}^{m}\sum_{q=1}^{m} |k_{pq}|^2}\right),$$

and therefore,

$$\sqrt{\sum_{p=r}^{m} |\zeta_p|^2} \leq C_r + \epsilon_r\sqrt{\sum_{q=1}^{m} |\zeta_q|^2}$$

$$\leq C_r + \epsilon_r\sqrt{\sum_{q=1}^{r-1} |\zeta_q|^2} + \epsilon_r\sqrt{\sum_{q=r}^{m} |\zeta_q^2|},$$

or

$$\sqrt{\sum_{p=r}^{m} |\zeta_p|^2}(1 - \epsilon) \leq \epsilon\left(1 + \sqrt{\sum_{q=1}^{r-1} |\zeta_q|^2}\right). \qquad (22.4)$$

Determine a constant K such that

$$\sum_{q=1}^{R-1} |\zeta_q|^2 < K \quad \text{for} \quad |z_p - z_p^*| < M.$$

The above inequality then shows that, for all m,

$$\sum_{p=1}^{m} |\zeta_p|^2 < K + \frac{\epsilon^2(1 + \sqrt{K}\,)^2}{(1 - \epsilon)^2} = H,$$

so that the series $\sum_{p=1}^{\infty} |\zeta_p|^2$ converges and its sum does not exceed H for $|z_p - z_p^*| < M$. Then, by (22.4), if $R \leq r < m$,

$$\sum_{p=r}^{m} |\zeta_p|^2 \leq \left[\frac{\epsilon(1 + \sqrt{H}\,)}{1 - \epsilon}\right]^2, \quad \text{for} \quad |z_p - z_p^*| < M.$$

This establishes the uniform convergence of the series $\Sigma|\zeta_p|^2$.

This theorem was proved by Hellinger [**31**] for the case where the a_p and b_p are real, and for the general case by Hellinger and Wall [**35**].

The theorem of invariability gives significance to the following definition.

DEFINITION 22.1.[11] *The determinate case or the indeterminate case is said to hold for the continued fraction* (16.1) *according as at least one of the following two series diverges or both these series converge, respectively:*

$$\Sigma | X_p(0) |^2, \quad \Sigma | Y_p(0) |^2. \tag{22.5}$$

23. The Indeterminate Case. In this section we consider **arbitrary** continued fractions (16.1) in which the a_p are not zero, and for which the **indeterminate case** holds (Definition 22.1). We define four polynomials as follows.

$$\begin{aligned} U_n(z) &= a_n[Y_n(0)X_{n+1}(z) - Y_{n+1}(0)X_n(z)], \\ V_n(z) &= a_n[Y_n(0)Y_{n+1}(z) - Y_{n+1}(0)Y_n(z)], \\ P_n(z) &= a_n[X_n(0)X_{n+1}(z) - X_{n+1}(0)X_n(z)], \\ Q_n(z) &= a_n[X_n(0)Y_{n+1}(z) - X_{n+1}(0)Y_n(z)], \end{aligned} \qquad n = 1, 2, 3, \cdots. \tag{23.1}$$

These polynomials satisfy the identity

$$P_n(z)V_n(z) - Q_n(z)U_n(z) \equiv 1. \tag{23.2}$$

In fact, we have, if we use the determinant formula (17.11),

$$\begin{aligned} &P_n(z)V_n(z) - Q_n(z)U_n(z) \\ &\quad = a_n[X_n(0)X_{n+1}(z) - X_{n+1}(0)X_n(z)]V_n(z) \\ &\qquad - a_n[X_n(0)Y_{n+1}(z) - X_{n+1}(0)Y_n(z)]U_n(z) \\ &\quad = a_nX_n(0)[X_{n+1}(z)V_n(z) - Y_{n+1}(z)U_n(z)] \\ &\qquad - a_nX_{n+1}(0)[X_n(z)V_n(z) - Y_n(z)U_n(z)] \\ &\quad = -a_nX_n(0)a_nY_{n+1}(0)[X_{n+1}(z)Y_n(z) - X_n(z)Y_{n+1}(z)] \\ &\qquad + a_nX_{n+1}(0)a_nY_n(0)[X_{n+1}(z)Y_n(z) - X_n(z)Y_{n+1}(z)] \\ &\quad = a_n[X_{n+1}(0)Y_n(0) - X_n(0)Y_{n+1}(0)] \equiv 1. \end{aligned}$$

Now, by (17.10),

$$a_{n+1}X_{n+2}(z) = (b_{n+1} + z_{n+1})X_{n+1}(z) - a_nX_n(z).$$

[11] Cf. footnote 9.

We multiply both members of this identity by $X_{n+1}(0)$, subtract $a_{n+1}X_{n+2}(0)X_{n+1}(z)$ from both members, and obtain

$$\begin{aligned} a_{n+1}[X_{n+1}(0)X_{n+2}(z) - X_{n+2}(0)X_{n+1}(z)] &= (b_{n+1} + z_{n+1})X_{n+1}(0)X_{n+1}(z) - a_nX_{n+1}(0)X_n(z) \\ &\quad - a_{n+1}X_{n+2}(0)X_{n+1}(z) \\ &= [b_{n+1}X_{n+1}(0) - a_{n+1}X_{n+2}(0)]X_{n+1}(z) + z_{n+1}X_{n+1}(0)X_{n+1}(z) \\ &\quad - a_nX_{n+1}(0)X_n(z) \\ &= a_n[X_n(0)X_{n+1}(z) - X_{n+1}(0)X_n(z)] + z_{n+1}X_{n+1}(0)X_{n+1}(z). \end{aligned}$$

Hence, by (23.1), we have the first of the following four relations.

$$\begin{aligned} P_{n+1}(z) &= P_n(z) + z_{n+1}X_{n+1}(0)X_{n+1}(z), \\ Q_{n+1}(z) &= Q_n(z) + z_{n+1}X_{n+1}(0)Y_{n+1}(z), \\ U_{n+1}(z) &= U_n(z) + z_{n+1}Y_{n+1}(0)X_{n+1}(z), \\ V_{n+1}(z) &= V_n(z) + z_{n+1}Y_{n+1}(0)Y_{n+1}(z). \end{aligned}$$

The others may be obtained in a similar way. From these relations we now obtain immediately the following formulas.

$$\begin{aligned} P_{n+1}(z) &= \sum_{p=2}^{n+1} z_pX_p(0)X_p(z), \\ Q_{n+1}(z) &= -1 + \sum_{p=2}^{n+1} z_pX_p(0)Y_p(z), \\ U_{n+1}(z) &= 1 + \sum_{p=2}^{n+1} z_pY_p(0)X_p(z), \\ V_{n+1}(z) &= z_1 + \sum_{p=2}^{n+1} z_pY_p(0)Y_p(z). \end{aligned} \tag{23.3}$$

We may now prove this theorem.

THEOREM 23.1.[12] *In the indeterminate case for the continued fraction* (16.1), *there exist four functions* $u(z)$, $v(z)$, $p(z)$, $q(z)$ *such that*

$$p(z)v(z) - q(z)u(z) = 1, \tag{23.4}$$

and such that

$$\lim_{n=\infty} U_n(z) = u(z), \quad \lim_{n=\infty} V_n(z) = v(z),$$
$$\lim_{n=\infty} P_n(z) = p(z), \quad \lim_{n=\infty} Q_n(z) = q(z), \tag{23.5}$$

uniformly for $|z_p| < M$, $p = 1, 2, 3, \cdots$, *where* M *is an arbitrarily large positive constant.* [137.]

Proof. By the theorem of invariability and Schwarz's inequality we get, for $|z_p| < M$,

$$\sum_{p=m}^{m+k} |z_p X_p(0) X_p(z)| \leq M \sqrt{\sum_{p=m}^{m+k} |X_p(0)|^2 \cdot \sum_{p=m}^{m+k} |X_p(z)|^2} < \epsilon,$$

where ϵ is any assigned positive number, provided that $m > N$, $(k = 1, 2, 3, \cdots)$, N being a sufficiently large number depending only upon ϵ and M. Therefore, we have, uniformly for $|z_p| < M$,

$$\lim_{n=\infty} P_{n+1}(z) = \sum_{p=2}^{\infty} z_p X_p(0) X_p(z).$$

The other limits in (23.5) can be established in the same way. Formula (23.4) follows at once by (23.2) and (23.5).

From (23.1) and the determinant formula we find that

$$X_{n+1}(z) = X_{n+1}(0) U_n(z) - Y_{n+1}(0) P_n(z),$$
$$Y_{n+1}(z) = X_{n+1}(0) V_n(z) - Y_{n+1}(0) Q_n(z). \tag{23.6}$$

Let

$$s_n = \frac{X_{n+1}(0)}{Y_{n+1}(0)}.$$

[12] This theorem, and the other theorems of this section, were proved by Hamburger [26] for the case $z_p = z$ and a_p, b_p real. Cf. also [35] and [9] for analogous theorems which hold in the (more restrictive) limit-circle case for positive definite J-fractions. Stieltjes [95] had the analogous theorem for his continued fraction.

If $\lim_{n=\infty} s_n = s$, a finite number, it then follows by Theorem 23.1 that

$$\lim_{n=\infty}\frac{X_{n+1}(z)}{X_{n+1}(0)} = su(z) - p(z), \quad \lim_{n=\infty}\frac{Y_{n+1}(z)}{X_{n+1}(0)} = sv(z) - q(z), \qquad (23.7)$$

uniformly for $|z_p| < M$, $p = 1, 2, 3, \cdots$. Since, by (23.4),

$$[sv(z) - q(z)]u(z) - [su(z) - p(z)]v(z) \equiv 1,$$

it follows that the limits (23.7) cannot both vanish. Therefore, for $|z_p| < M$, $p = 1, 2, 3, \cdots$, we have

$$\lim_{n=\infty}\frac{X_{n+1}(z)}{Y_{n+1}(z)} = \frac{su(z) - p(z)}{sv(z) - q(z)},$$

a finite number or ∞. If $\lim_{n=\infty} s_n = \infty$, then, for $|z_p| < M$, $p = 1, 2, 3, \cdots$,

$$\lim_{n=\infty}\frac{X_{n+1}(z)}{Y_{n+1}(z)} = \frac{u(z)}{v(z)},$$

a finite number or ∞. If the sequence $\{s_n\}$ has more than one limit-point, s' and s'', $s' \neq s''$, then, if s' and s'' are both finite, one infinite subsequence of approximants of the continued fraction has the limit

$$\frac{s'u(z) - p(z)}{s'v(z) - q(z)},$$

and another infinite subsequence of approximants has the limit

$$\frac{s''u(z) - p(z)}{s''v(z) - q(z)}.$$

These are unequal for all z_p inasmuch as

$$[s'u(z) - p(z)][s''v(z) - q(z)] - [s''u(z) - p(z)][s'v(z) - q(z)] = s' - s'' \neq 0.$$

Therefore, the continued fraction diverges by oscillation for all z_p, ($|z_p| < M$). The same evidently holds if one of the limit-points s' or s'' is ∞. From these considerations we conclude that the following theorem is true.

THEOREM 23.2. *Let the indeterminate case hold for the continued fraction* (16.1). *If the continued fraction or its reciprocal converges for a single set of values of the* z_p *in the domain* $|z_p| < M$, $p = 1, 2, 3, \cdots$, *(M an arbitrarily large constant), then, for every* z_p *in this domain, either the continued fraction or its reciprocal converges. If the continued fraction and its reciprocal both diverge for a particular set of values of the* z_p *with moduli less than M, then the continued fraction diverges by oscillation for all such* z_p.

In case the z_p are all equal to a common variable ζ, the continued fraction is called a **J-fraction.** This name is used because the related quadratic form $\Sigma(b_p + \zeta)x_p^2 - 2\Sigma a_p x_p x_{p+1}$ has long been called a **J-form.** For the J-fraction, the functions $u(\zeta)$, $v(\zeta)$, $p(\zeta)$, $q(\zeta)$ of Theorem 23.1 are entire functions of ζ. We may then state the following theorem.

THEOREM 23.3. *Let the indeterminate case hold for the J-fraction*

$$\cfrac{1}{b_1 + \zeta - \cfrac{a_1^2}{b_2 + \zeta - \cfrac{a_2^2}{b_3 + \zeta - \cdots}}} \tag{23.8}$$

Then, there exist four entire functions $u(\zeta)$, $v(\zeta)$, $p(\zeta)$, $q(\zeta)$ *such that*

$$p(\zeta)v(\zeta) - q(\zeta)u(\zeta) \equiv 1, \tag{23.9}$$

and such that

$$\lim \frac{X_{n+1}(\zeta)}{X_{n+1}(0)} = su(\zeta) - p(\zeta), \quad \lim \frac{Y_{n+1}(\zeta)}{X_{n+1}(0)} = sv(\zeta) - q(\zeta), \tag{23.10}$$

as n tends to ∞ *over a set of indices for which the sequence*

$$s_n = \frac{X_{n+1}(0)}{Y_{n+1}(0)}, \quad n = 1, 2, 3, \cdots,$$

has a finite limit s; and

$$\lim \frac{X_{n+1}(\zeta)}{s_n X_{n+1}(0)} = u(\zeta), \quad \lim \frac{Y_{n+1}(\zeta)}{s_n X_{n+1}(0)} = v(\zeta), \tag{23.11}$$

as n tends to ∞ *over a set of values for which the sequence* $\{s_n\}$ *has the limit* ∞. *In either case, the limits exist uniformly over every bounded domain of the* ζ*-plane. The continued fraction diverges by oscillation for every* *value of ζ, or converges to a meromorphic function of ζ, or else its reciprocal converges to the constant value* 0.

If (23.8) is positive definite (Definition 16.1), then

$$\left|\frac{X_p(\zeta)}{Y_p(\zeta)}\right| \leq \frac{1}{\delta}, \quad \text{for} \quad \Im(\zeta) \geq \delta > 0, \quad (p = 1, 2, 3, \cdots). \qquad (23.12)$$

Hence, it is impossible for the reciprocal of the continued fraction to have the constant value 0. We therefore have

THEOREM 23.4. *Let* (23.8) *be a positive definite J-fraction for which the indeterminate case holds. Then, the continued fraction diverges by oscillation for every value of ζ, or else it converges to a meromorphic function of ζ, for all values of ζ not poles of this function. If an infinite subsequence of approximants converges for a single value of ζ, then this subsequence converges to a meromorphic function of ζ which is analytic for* $\Im(\zeta) > 0$.

24. Convergence Continuation Theorem. Let $f_1(z), f_2(z), f_3(z), \cdots$ be an infinite sequence of functions which are analytic over a simply connected domain S of the complex z-plane. Suppose that we have established the convergence of the sequence over a subdomain of S, and that corresponding to every finite closed region S' entirely within S there exists a constant M depending only upon S', such that

$$|f_p(z)| < M, \quad \text{for} \quad p = 1, 2, 3, \cdots, z \quad \text{in} \quad S'.$$

That is, the sequence is **uniformly bounded** over S'. The **convergence continuation theorem** of Stieltjes [**95**] then asserts that the sequence converges uniformly over every finite closed region entirely within S to a function which is analytic over S.

This theorem is of fundamental importance in establishing the convergence of continued fractions and the character of the functions represented by them. In particular, we shall need this theorem in § 25 for our investigation of positive definite J-fractions in the determinate case.

The convergence continuation theorem has been improved upon by other mathematicians. Osgood [**64**] and Arzelà [**2**] showed that the conclusion of Stieltjes' theorem holds if the sequence is uniformly bounded, as above, and converges over a set of points everywhere dense along a closed rectifiable contour within S. Vitali [**114**] and Porter [**71, 72**] replaced the contour by an infinite set of points having at least one limit-point interior to S. Blaschke [**4**] showed that this limit-point may be on the boundary of S provided only that there is a sequence of points of the given set which approaches the limit-point "sufficiently slowly." Montel [**60**] extended and developed this theory by means of his notion of **normal families** of functions.

We shall begin by proving

THEOREM 24.1. *Let $\{f_p(z)\}$ be a sequence of functions, analytic in a simply connected open domain S, which is uniformly bounded over every closed domain entirely within S. Then, there exists an infinite subsequence of these functions which is uniformly convergent over every finite closed domain entirely within S, to a limit-function which is analytic in S.*

Proof. Let E be a sequence of points $e_1, e_2, e_3, \cdots$ of S which is everywhere dense in S. Then, since the given sequence of functions $f_p(z)$ is uniformly bounded over every finite closed region within S, there exist constants M_n such that

$$|f_p(e_n)| < M_n, \quad p = 1, 2, 3, \cdots, \quad n = 1, 2, 3, \ldots,$$

where M_n is independent of p. Therefore, by the Bolzano-Weierstrass theorem, we may select from the sequence $\{f_p(z)\}$ an infinite subsequence $\{f_{n_p}(z)\}$, where $n_1 < n_2 < n_3 < \cdots$, which converges for $z = e_1$. From the sequence $\{f_{n_p}(z)\}$ we may then select an infinite subsequence $\{f_{n_p'}(z)\}$, where $n_1 < n_1' < n_2' < n_3' < \cdots$, which converges for $z = e_2$. From the last chosen sequence we may select an infinite subsequence $\{f_{n_p''}(z)\}$, where $n_1' < n_1'' < n_2'' < n_3'' < \cdots$, which converges for $z = e_3$, and so on. The "diagonal sequence"

$$f_{n_1}(z), \quad f_{n_2'}(z), \quad f_{n_3''}(z), \cdots \tag{24.1}$$

is then obviously an infinite subsequence of the given sequence $\{f_p(z)\}$ which is convergent for $z = e_n$, $n = 1, 2, 3, \cdots$. We shall, for the sake of simplicity, denote the pth member of the sequence (24.1) by $F_p(z)$.

Let S' be an arbitrarily assigned bounded closed domain entirely within S. We shall prove that the sequence $\{F_p(z)\}$ converges uniformly over S'. It will follow from a theorem of Weierstrass that the limit of this sequence is an analytic function of z in S.

Let S'' be a bounded closed region lying entirely within S, containing S' on its interior, and having a rectifiable boundary C of length L at a minimum distance $\delta > 0$ from S'. We may obtain S'', for instance, by covering S' with a finite number of circles within S, using for this purpose the Heine-Borel theorem. The boundary C of S'' then consists of a finite number of arcs of circles.

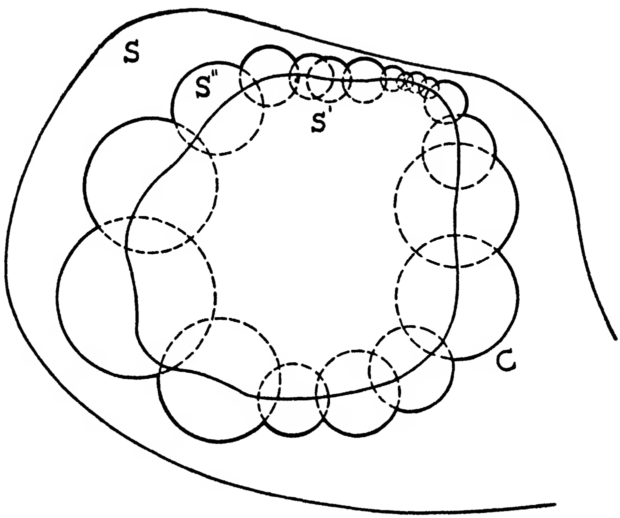

FIG. 4.

If z is in S', we then have, by Cauchy's integral formula,

$$F_p(z) = \frac{1}{2\pi i}\int_C \frac{F_p(t)dt}{t - z}.$$

Hence, for any two points z' and z'' of S',

$$| F_p(z') - F_p(z'') | = \frac{1}{2\pi}\left| \int_C \frac{z' - z''}{(t - z')(t - z'')} F_p(t)dt \right|$$

$$< \frac{ML}{2\pi\delta^2} | z' - z'' |,$$

where, in accordance with the hypothesis, M is so chosen that $|F_p(z)| < M, p = 1, 2, 3, \cdots$, for all z in S', M being independent of z and of p. Consequently, for every $\epsilon > 0$, there exists an $\eta(\epsilon) > 0$, depending only upon ϵ, such that

$$|F_p(z') - F_p(z'')| < \epsilon \quad \text{if} \quad |z' - z''| < \eta(\epsilon),$$
$$z', z'' \quad \text{in} \quad S'. \tag{24.2}$$

That is, the sequence $\{F_p(z)\}$ is **equi-continuous** over S'.

Let z be any point of S'. Since the set E is everywhere dense in S, we can find in S' a point e_k of E in the circle with center z and radius $\eta(\epsilon)$. Then,

$$|F_r(z) - F_s(z)| \leq |F_r(z) - F_r(e_k)| + |F_r(e_k) - F_s(e_k)| + |F_s(e_k) - F_s(z)|.$$

The first and last of the quantities on the right are less than ϵ by virtue of (24.2). The second will be less than ϵ if $r, s > m$, where m is sufficiently large, since $\{F_p\}$ converges over E. Therefore $|F_r(z) - F_s(z)| < 3\epsilon$ if $r, s > m$. Here m depends upon ϵ and upon z. Thus, the sequence converges for every z in S'. Denote the limit-function by $F(z)$. Then,

$$|F(z) - F_p(z)| < \epsilon \quad \text{if} \quad p > n(\epsilon, z), \tag{24.3}$$

where $n(\epsilon, z)$ is a sufficiently large number depending upon ϵ and z. We may evidently agree to take $n(\epsilon, z)$ to be the least positive integer for which (24.3) holds. In order to prove uniform convergence, it is required to prove that $n(\epsilon, z)$ is a bounded function of z for each $\epsilon > 0$.

For a fixed $\epsilon > 0$, let $N \leq \infty$ be the least upper bound of $n(\epsilon, z)$ as z ranges over S'. Choose a sequence $z_1, z_2, z_3, \cdots$ of points in S' such that

$$\lim_{p=\infty} n(\epsilon, z_p) = N.$$

Let ζ be a limit-point of the sequence $\{z_p\}$, and z any point of S' in the circle of radius $\eta(\epsilon/5)$ and center ζ. Then

$$|F_p(z) - F(z)| \leq |F(z) - F(\zeta)| + |F(\zeta) - F_p(\zeta)| + |F_p(\zeta) - F_p(z)|.$$

The last term on the right does not exceed $\epsilon/5$ by (24.2). The second term is less than $\epsilon/5$ if $p > n(\epsilon/5, \zeta)$. Finally, the first term is not greater than

$$| F(z) - F_h(z) | + | F_h(z) - F_h(\zeta) | + | F_h(\zeta) - F(\zeta) | < \frac{3\epsilon}{5}$$

for all sufficiently large values of h. Therefore,

$$| F_p(z) - F(z) | < \epsilon \quad \text{if} \quad p > n\left(\frac{\epsilon}{5}, \zeta\right), \quad | z - \zeta | < \eta\left(\frac{\epsilon}{5}\right).$$

Inasmuch as $| z_p - \zeta | < \eta(\epsilon/5)$ for all sufficiently large values of p, we therefore conclude that $n(\epsilon, z_p) \leq \eta(\epsilon/5, \zeta)$ for all sufficiently large values of p, and hence $N \leq n(\epsilon/5, \zeta)$. Thus, N is finite, and $| F(z) - F_p(z) | < \epsilon$ if $p > N$, for all z in S'. That is, the sequence $\{F_p(z)\}$ converges uniformly over S'.

This completes the proof of Theorem 24.1.

From Theorem 24.1 we may readily obtain the **convergence continuation theorem:**

THEOREM 24.2. *Let $\{f_p(z)\}$ be an infinite sequence of functions, analytic over a simply connected open domain S, which is uniformly bounded over every finite closed domain S' entirely within S. Let the sequence converge over an infinite set of points having at least one limit-point interior to S. Then, the sequence converges uniformly over every finite closed domain entirely within S to a function of z which is analytic in S.*

Proof. Let $z_1, z_2, z_3, \cdots$ be an infinite sequence of points of S having the limit z_0, interior to S, over which the given sequence of functions is convergent. In accordance with Theorem 24.1, we may select an infinite subsequence $\{F_p(z)\}$ of the given sequence, which converges uniformly over every finite closed domain entirely within S, to a function $F(z)$ which is analytic in S. Then, we evidently have

$$\lim_{p=\infty} f_p(z_n) = F(z_n), \quad n = 1, 2, 3, \cdots.$$

Let S' be a finite closed domain entirely within S which contains the point z_0 on the interior. Let

$$\delta_p = \text{l.u.b.} \quad |f_p(z) - F(z)|, \quad z \text{ in } S', \quad p = 1, 2, 3, \cdots,$$

and let

$$\delta = \limsup \delta_p.$$

We shall prove that $\delta = 0$, and thereby establish the uniform convergence over S' of the sequence $\{f_p(z)\}$. Let $p_1 < p_2 < p_3 < \cdots$ be a sequence of indices such that

$$\delta = \lim_{k=\infty} \delta_{p_k}.$$

From the sequence $f_{p_k}(z)$, $k = 1, 2, 3, \cdots$, we may select an infinite subsequence uniformly convergent over S' to a function $F^*(z)$. Inasmuch as $F^*(z) = F(z)$ for all $z = z_r$ which are in S', it follows that $F^*(z) \equiv F(z)$. Hence we conclude immediately that $\delta = 0$, as was to be proved.

25. The Determinate Case. We suppose now that the determinate case holds for the continued fraction (16.1), that is, at least one of the two series (22.5) is divergent. There are two simple *sufficient* conditions for the determinate case. In fact, from the determinant formula and Schwarz's inequality, we find that

$$\sum_{p=1}^{n} \left| \frac{1}{a_p} \right| = \sum_{p=1}^{n} | X_{p+1}(0)Y_p(0) - X_p(0)Y_{p+1}(0) |$$
$$\leq 2 \sqrt{\sum_{p=1}^{n+1} | X_p(0) |^2 \cdot \sum_{p=1}^{n+1} | Y_p(0) |^2},$$

and therefore we have [9]

THEOREM 25.1.[13] *If the series*

$$\sum_{p=1}^{\infty} \left| \frac{1}{a_p} \right| \tag{25.1}$$

diverges, then the determinate case holds for the continued fraction (16.1).

One may easily verify that

$$X_{p+2}(0)Y_p(0) - X_p(0)Y_{p+2}(0) = \frac{b_{p+1}}{a_p a_{p+1}},$$

and conclude, as above, that the following theorem is true [9].

THEOREM 25.2. *If the series*

$$\sum_{p=1}^{\infty} \left| \frac{b_{p+1}}{a_p a_{p+1}} \right| \tag{25.2}$$

diverges, then the determinate case holds for the continued fraction (16.1).

[13] Hellinger [31] showed that the determinate case holds for a real J-fraction if $\liminf | a_p |$ is finite. Carleman [6] had Theorem 25.1 for the case of real J-fractions.

We can establish the convergence of the continued fraction (16.1) in the determinate case provided the coefficients a_p and b_p are suitably restricted. If we require that the a_p and b_p satisfy the conditions imposed in Definition 16.1, i.e., that the continued fraction is positive definite, we have this theorem.

THEOREM 25.3. *Let* (16.1) *be a positive definite continued fraction for which the determinate case holds. Then, the continued fraction converges if* $\mathfrak{J}(z_p) \geq \delta$, $|z_p| < M$, $p = 1, 2, 3, \cdots$, *where* δ *and* M *are arbitrarily chosen positive constants.*

Proof. It is only necessary to recall the discussion given at the beginning of this chapter, and to apply Theorem 22.1.

For positive definite J-fractions, we have the following theorem.

THEOREM 25.4. *Let* (23.8) *be a positive definite J-fraction for which the determinate case holds. Then, the J-fraction converges uniformly over every bounded closed region in the upper half-plane* $I(\zeta) > 0$, *and its value is an analytic function of* ζ *for* $\mathfrak{J}(\zeta) > 0$. [**138.**]

Proof. The uniform convergence over every finite closed domain in $\mathfrak{J}(\zeta) > 0$, and the fact that the value is an analytic function of ζ, follow immediately from Theorem 25.3, (23.12), and Theorem 24.2.

On combining Theorems 23.4 and 25.4, we obtain at once

THEOREM 25.5. *If a positive definite J-fraction* (23.8), *or an infinite subsequence of its approximants, converges for a particular value of* ζ *such that* $\mathfrak{J}(\zeta) > 0$, *then it converges uniformly over every closed finite region in the half-plane* $\mathfrak{J}(\zeta) > 0$, *and its value is an analytic function of* ζ *in this half-plane.* [**9.**]

In later chapters we shall investigate the class of functions represented by positive definite J-fractions. We shall show, in particular, that these functions can be expressed by means of certain definite integrals.

26. Bounded J-fractions. The J-fraction (23.8), *where we now permit the partial numerators to be zero,* is called **bounded,** if there exists a number M such that

$$|a_p| \leq \frac{M}{3}, \quad |b_p| \leq \frac{M}{3}, \quad p = 1, 2, 3, \cdots, \tag{26.1}$$

the number M being independent of p. If the J-fraction is bounded, then the least number M which can be used in (26.1) is called the **bound** of the J-fraction.

The condition (26.1) can be formulated in terms of J-forms as follows.

THEOREM 26.1. *The J-fraction* (23.8) *is bounded if, and only if, there exists a number N such that*

$$\left|\sum_{p=1}^{n} b_p u_p v_p - \sum_{p=1}^{n-1} a_p(u_p v_{p+1} + u_{p+1} v_p)\right| \leq N\sqrt{\sum_{p=1}^{n} |u_p|^2 \cdot \sum_{p=1}^{n} |v_p|^2}, \quad (26.2)$$

for all values of the variables u_p and v_p, the constant N being independent of the variables and of n.

Proof. If (26.1) holds, then we find by Schwarz's inequality that the left-hand member of (26.2) does not exceed

$$\frac{M}{3}\sqrt{\sum_{p=1}^{n} |u_p|^2 \cdot \sum_{p=1}^{n} |v_p|^2} + \frac{2M}{3}\sqrt{\sum_{p=1}^{n} |u_p|^2 \cdot \sum_{p=1}^{n} |v_p|^2},$$

so that (26.2) holds with $N = M$. Conversely, if (26.2) holds, then we find, on putting $u_p = v_p = 1$, $u_r = v_r = 0$ for $r \neq p$, that $|b_p| \leq N$; and, on putting $u_p = v_{p+1} = 1$, $u_r = v_s = 0$ for $r \neq p$, $s \neq p+1$, that $|a_p| \leq N$. Thus, (26.1) holds with $M = 3N$.

When (26.2) holds, the J-form $\Sigma b_p u_p v_p - \Sigma a_p(u_p v_{p+1} + u_{p+1} v_p)$ is said to be **bounded**, and the least value of N which can be used in (26.2) is called the **norm** of the J-form. We shall also call this number the **norm** of the J-fraction.

The above proof shows that the norm of a J-fraction does not exceed its bound. That the norm may be *less* than the bound is shown by the example

$$\cfrac{1}{\zeta - \cfrac{(\frac{1}{4})}{\zeta - \cfrac{(\frac{1}{4})}{\zeta - \cdots}}}$$

Here the norm is $N = 1$, whereas the bound is $M = \frac{3}{2}$.

We shall now prove

THEOREM 26.2. *A bounded J-fraction with bound M is uniformly convergent for* $|\zeta| \geq M$.

Proof. Let (23.8) be a given J-fraction with bound M. If the J-fraction is transformed by means of an equivalence transformation so that all the partial denominators are equal to unity, then the nth partial numerator is

$$\frac{-a_{n-1}^2}{(b_{n-1} + \zeta)(b_n + \zeta)}.$$

If $|\zeta| \geq M$ and (26.1) holds, this has modulus not greater than $\frac{1}{4}$. Hence, by Theorem 10.1, the J-fraction converges uniformly for $|\zeta| \geq M$.

We shall now undertake to improve this result by showing that there exists a region K in the circle $|\zeta| = N$, where N is the norm of the J-fraction, such that the J-fraction converges for all ζ not in K. To that end, let $a = e^{i\theta}$ be a complex number with modulus unity, and consider the J-fraction

$$\cfrac{a}{b_1 a + \lambda - \cfrac{(a_1 a)^2}{b_2 a + \lambda - \cfrac{(a_2 a)^2}{b_3 a + \lambda - \cdots}}}, \qquad \lambda = a\zeta, \tag{26.3}$$

which is obtained from (23.8) by means of an equivalence transformation. It is easy to see that (23.8) and (26.3) have one and the same norm.

We now put

$$\alpha_p(\theta) = \Im(a_p a), \quad \beta_p(\theta) = \Im(b_p a), \quad p = 1, 2, 3, \cdots.$$

Then, if (23.8) is bounded, there exists a finite constant $Y(\theta)$, depending upon θ, such that

$$\sum_{p=1}^{n} [\beta_p(\theta) + Y(\theta)]x_p^2 - 2\sum_{p=1}^{n-1} \alpha_p(\theta)x_p x_{p+1} \geq 0, \quad n = 1, 2, 3, \cdots, \tag{26.4}$$

for all real values of $x_1, x_2, x_3, \cdots$. We may evidently assume that

$$|Y(\theta)| \leq N, \quad 0 \leq \theta < 2\pi,$$

where N is the norm of the J-fraction. For each θ, we shall let $-Y_0(\theta)$ denote the greatest lower bound of the values of

$$\sum_{p=1}^{n} \beta_p(\theta) x_p^2 - 2\sum_{p=1}^{n-1} \alpha_p(\theta) x_p x_{p+1}$$

for $n = 1, 2, 3, \cdots$, and for $\sum_{p=1}^{n} x_p^2 = 1$. Then, $|Y_0(\theta)| \leq N$, and (26.4) holds with $Y(\theta) = Y_0(\theta)$.

We now make the change of variable $\lambda = iY(\theta) + \xi$ in (26.3). It follows from (26.4) that (26.3) is then a positive definite J-fraction in the variable ξ. Therefore, if δ is a positive constant, it results from (23.12) that the nth approximant of (26.3), which is the same as the nth approximant of (23.8), satisfies the inequality

$$\left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| \leq \frac{1}{\delta},$$

provided that $\mathfrak{J}(\xi) \geq \delta$, i.e., provided that

$$x \sin \theta + y \cos \theta \geq Y(\theta) + \delta, \quad \text{where} \quad \zeta = x + iy.$$

This result has the following geometrical interpretation. Let K denote the set of points $\zeta = x + iy$ such that

$$x \sin \theta + y \cos \theta \leq Y(\theta) \quad \text{for} \quad 0 \leq \theta < 2\pi.$$

If $\zeta_1 = x_1 + iy_1, \zeta_2 = x_2 + iy_2$ are any two points of K, then every $\zeta = t\zeta_1 + (1 - t)\zeta_2 = [tx_1 + (1 - t)x_2] + i[ty_1 + (1 - t)y_2]$, $0 < t < 1$, a point on the line segment joining ζ_1 and ζ_2, is in K. Thus, K is a **convex** set. The zeros of all the denominators $B_n(\zeta)$ are in K. Moreover, the approximants of the J-fraction are uniformly bounded over any region at a positive distance from K. We shall let K_0 denote the convex set of points obtained in this way corresponding to the function $Y_0(\theta)$ introduced before.

By Theorem 26.2, the J-fraction converges for all ζ with sufficiently large moduli. Hence, we may apply the convergence continuation theorem (Theorem 24.2) and obtain

THOEREM 26.3. *A bounded J-fraction* (23.8) *converges uniformly over every finite closed region whose distance from the convex set K_0 defined above is positive. In particular, the J-fraction converges if* $|\zeta| > N$, *where N is its norm.* [**137a**].

We note the following special cases. If the coefficients a_p and b_p are all real, then $Y_0(0) = Y_0(\pi) = 0$, so that K_0 reduces to a subinterval of the interval $-N \leq x \leq +N$. If the a_p are pure imaginary and the b_p are real and nonnegative, then the set K_0 is contained in the left half-plane, $\mathfrak{R}(\zeta) \leq 0$.

27. Real J-fractions. The J-fraction (23.8) is called **real** if the coefficients a_p and b_p are all real. A real J-fraction is obviously positive definite. If ζ is replaced by $-\zeta$ in (23.8), we see by means of an equivalence transformation that the effect is to multiply the J-fraction by -1 and to replace b_p by $-b_p$. If the J-fraction is real, it therefore follows that the role of the upper and lower half-planes may be interchanged. In particular, we have the following theorems.

THEOREM 27.1. *The zeros of the denominators of a real J-fraction are all real* [**95, 109**].

THEOREM 27.2. *If a real J-fraction converges for a single nonreal value of ζ, then it converges uniformly over every finite closed region whose distance from the real axis is positive, and its value in each of the half-planes* $\mathfrak{I}(\zeta) < 0$ *and* $\mathfrak{I}(\zeta) > 0$ *is an analytic function of ζ in that half-plane.* [**26.**]

For real J-fractions we have, instead of the inequality (23.12), the inequality

$$\left|\frac{A_p(\zeta)}{B_p(\zeta)}\right| \leq \frac{1}{|y|} \quad \text{for} \quad y = \mathfrak{I}(\zeta) \neq 0, \quad p = 1, 2, 3, \cdots. \tag{27.1}$$

Inasmuch as

$$\mathfrak{I}\left(\frac{A_p(\zeta)}{B_p(\zeta)}\right) < 0 \quad \text{for} \quad \mathfrak{I}(\zeta) > 0, \quad p = 1, 2, 3, \cdots, \tag{27.2}$$

it follows that the poles of $A_p(\zeta)/B_p(\zeta)$, which are all real by Theorem 27.1, are all *simple* and have *positive residues.* In fact, if this were not true for one of the poles, then we could choose a path of ζ in the upper half-plane approaching this pole in such a way that (27.2) would fail to hold.

Let the poles of the pth approximant be denoted by $x_1, x_2, \cdots, x_p$. Then $x_r \neq x_s$ for $r \neq s$, and we have a partial fraction development of the form

$$\frac{A_p(\zeta)}{B_p(\zeta)} = \sum_{r=1}^{p} \frac{L_r}{\zeta - x_r}, \quad \text{where} \quad L_r > 0. \tag{27.3}$$

On expanding both members of (27.3) in descending powers of ζ and comparing coefficients of $1/\zeta$ on either side, we obtain

$$\sum_{r=1}^{p} L_r = 1. \tag{27.4}$$

From results found in § 26, we have

THEOREM 27.3. *A real bounded J-fraction of norm N converges uniformly over every finite closed region whose distance from the real interval* $-N \leq x \leq +N$ *is positive.*

We shall now prove the following theorem concerning real bounded J-fractions (23.8) in which $b_p = 0$, $p = 1, 2, 3, \cdots$.

THEOREM 27.4. *A real bounded J-fraction of the form*

$$\cfrac{1}{\zeta - \cfrac{a_1^2}{\zeta - \cfrac{a_2^2}{\zeta - \cdots}}} \tag{27.5}$$

has norm less than or equal to unity if, and only if, the sequence $\{a_p^2\}$ *is a chain sequence* (cf. § 19). **[139.]**

Proof. If the norm of (27.5) does not exceed unity, then

$$-\sum_{p=1}^{n} x_p^2 \leq -2\sum_{p=1}^{n-1} a_p x_p x_{p+1} \leq \sum_{p=1}^{n} x_p^2, \quad n = 2, 3, 4, \cdots, \tag{27.6}$$

for all real values of the x_p. Therefore,

$$\sum_{p=1}^{n} x_p^2 - 2\sum_{p=1}^{n-1} a_p x_p x_{p+1} \geq 0, \tag{27.7}$$

so that, by Theorem 20.1, the sequence $\{a_p^2\}$ is a chain sequence. Conversely, if $\{a_p^2\}$ is a chain sequence, so that (27.7) holds, then the first inequality in (27.6) holds inasmuch as it is the same as (27.7), while the second follows from (27.7) if we there replace x_p by $(-1)^p x_p$. Hence the J-fraction has norm not greater than unity.

By means of an equivalence transformation, we may throw (27.5) into the form

$$\cfrac{1/\zeta}{1 - \cfrac{a_1^2/\zeta^2}{1 - \cfrac{a_2^2/\zeta^2}{1 - \cdots}}}$$

Hence, if we drop the factor $1/\zeta$ and then put $z = -1/\zeta^2$, we obtain

$$\cfrac{1}{1 + \cfrac{a_1^2 z}{1 + \cfrac{a_2^2 z}{1 + \cdots}}}$$

The real ζ-interval $-1 \leq x \leq +1$ is carried by the above substitution into the real z-interval $-\infty \leq x \leq -1$. On combining Theorems 27.3 and 27.4 we then have

THEOREM 27.5. *If* $0 \leq g_p \leq 1$, $p = 0, 1, 2, \cdots$, *then the continued fraction*

$$\cfrac{1}{1 + \cfrac{(1-g_0)g_1 z}{1 + \cfrac{(1-g_1)g_2 z}{1 + \cfrac{(1-g_2)g_3 z}{1 + \cdots}}}} \tag{27.8}$$

converges uniformly over every finite closed region whose distance from the real interval $-\infty < x \leq -1$ *is positive.*

Exercise 5

5.1. Let $Y_p^{(n)}(z)$ denote the polynomial obtained from $Y_p(z)$ by advancing the subscripts of all the a_k and b_k by n. Show that the determinate case holds if the series

$$\sum_{p=1}^{\infty} \left| \frac{Y_k^{(p)}(0)}{a_p} \right|$$

is divergent **[9]**.

5.2. Show that the determinate case holds for a bounded J-fraction.

5.3. Show that if the limit-point case holds for a positive definite J-fraction when $\zeta = \zeta_0$, $\mathfrak{J}(\zeta_0) > 0$, then the limit-point case holds for all ζ in the half-plane $\mathfrak{J}(\zeta) > 0$. **[138, 9.]**

5.4. Show that the continued fraction

$$\cfrac{1}{1 + \zeta - \cfrac{1}{1 + \zeta - \cfrac{1}{1 + \zeta - \ddots}}}$$

may diverge if $|\zeta|$ is less than the bound.

5.5. Let $f_p(z)$, $p = 1, 2, 3, \cdots$, be a sequence of functions which are analytic for $|z| < 1$, such that $|f_p(z)| < M$, $p = 1, 2, 3, \cdots$, where M is a constant independent of p and of z, $|z| < 1$. Let $z_1, z_2, z_3, \cdots$ be an infinite sequence of points with moduli less than unity such that the series $\Sigma(1 - |z_p|)$ diverges, and suppose that the sequence of functions converges for $z = z_p$, $p = 1, 2, 3, \cdots$. Then the sequence converges uniformly for $|z| \leq r$ for every r in the interval $0 < r < 1$. **[4.]**

5.6. A function which is analytic and has modulus not exceeding a constant M for every point in the interior of the half-plane $R(z) > 0$, and which vanishes at the integral points $1, 2, 3, \cdots$, vanishes identically. (Use 5.5 and the mapping $z = 1 - [2/(1 + w)]$.

Chapter VI

STIELTJES TYPE CONTINUED FRACTIONS

In his celebrated memoir of 1894, Stieltjes [**95**] developed a theory of the continued fraction (28.1) in which the k_p are real and positive. We shall connect this continued fraction with a certain real J-fraction, and shall be able to discover some of its properties by means of known properties of the J-fraction. More generally, we shall investigate the continued fraction (28.2), and shall obtain a general convergence theorem of Scott and Wall [**88a**], which includes theorems of Van Vleck [**107**], Hamburger [**26**], and Mall [**59**].

28. Convergence and Divergence of the Continued Fraction of Stieltjes. The continued fraction preferred by Stieltjes has the form

$$\cfrac{1}{k_1 z + \cfrac{1}{k_2 + \cfrac{1}{k_3 z + \cfrac{1}{k_4 + \ddots}}}}, \tag{28.1}$$

where the k_p are real and positive and z is a complex variable.

It will sometimes be convenient to consider a more general continued fraction, namely,

$$\cfrac{1}{k_1 z_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 z_2 + \cfrac{1}{k_4 + \ddots}}}}, \tag{28.2}$$

where the k_p are complex constants and $z_1, z_2, z_3, \cdots$ are independent complex variables. We shall refer to (28.2) as a **Stieltjes type** continued fraction.

If the k_p are all different from zero, then the even part (cf. § 4) of (28.2) is a continued fraction of the form (16.1), namely,

$$\cfrac{1/k_1}{c_1 + z_1 - \cfrac{c_1 c_2}{c_2 + c_3 + z_2 - \cfrac{c_3 c_4}{c_4 + c_5 + z_3 - \cdots}}}, \tag{28.3}$$

where we have put

$$c_p = \frac{1}{k_p k_{p+1}}, \quad p = 1, 2, 3, \cdots. \tag{28.4}$$

It is easy to verify that the approximants of (28.2) can be expressed in terms of the transformations

$$T_p(w) = \cfrac{1/k_1}{c_1 + z_1 - \cfrac{c_1 c_2}{c_2 + c_3 + z_2 - \cdots - \cfrac{c_{2p-1} c_{2p}}{w}}}, \tag{28.5}$$

by means of the formulas

$$\begin{aligned} T_p(z_{p+1} + c_{2p} + c_{2p+1}) &= \frac{A_{2p+2}}{B_{2p+2}}, \\ T_p(z_{p+1} + c_{2p}) &= \frac{A_{2p+1}}{B_{2p+1}}. \end{aligned} \tag{28.6}$$

From (28.6) we see that the approximants of the continued fraction (28.1), in which $k_p > 0$, may be regarded as approximants of *real* J-fractions. From this remark it follows (cf. § 27) that the poles of the approximants are all real, simple, and have positive residues. Inasmuch as the denominators are obviously polynomials in z with positive coefficients, it follows that these poles all lie on the negative half of the real axis. On applying (27.3) and (27.4), we then conclude, by (28.6), that the approximants A_{2p+2}/B_{2p+2} and A_{2p+1}/B_{2p+1} of (28.1) have partial fraction developments of the form

$$\sum_{r=1}^{p} \frac{L_r}{z + x_r}, \tag{28.7}$$

where $L_r > 0$, $\Sigma L_r = 1/k_1$, and $0 \leq x_1 < x_2 < \cdots < x_p$. It follows immediately that if z is at a distance not less than $\delta > 0$ from the negative half of the real axis, then the approximants of (28.1) satisfy the inequality

$$\left| \frac{A_p(z)}{B_p(z)} \right| \leq \frac{1}{k_1 \delta}, \quad p = 1, 2, 3, \cdots. \tag{28.8}$$

We are now in a position to prove the following fundamental convergence theorem of Stieltjes [**95**].

Theorem 28.1. *The continued fraction* (28.1), *in which the k_p are positive constants and the series Σk_p diverges, is uniformly convergent over every finite closed domain of z whose distance from the negative half of the real axis is positive, and its value is an analytic function of z for all z not on the negative half of the real axis.*

Proof. Let z be real and positive. Then it follows at once from the **parabola theorem** (Theorem 14.2) that the continued fraction (28.1) is convergent if the series Σk_p diverges. Now, if G is any finite closed region whose distance from the negative half of the real axis is positive, it follows from (28.8) that the approximants of (28.1) are uniformly bounded over G. Consequently, by the **convergence continuation theorem** (Theorem 24.2), we conclude that Theorem 28.1 is true.

With the aid of Theorem 6.1 and the second remark after that theorem, we readily obtain this theorem.

Theorem 28.2. *The continued fraction* (28.1), *in which the k_p are positive constants such that the series Σk_p converges, is divergent by oscillation for every value of z. The sequences of even and odd approximants converge uniformly over every finite closed region whose distance from the negative half of the real axis is positive, to distinct meromorphic limit-functions $p(z)/q(z)$ and $p_1(z)/q_1(z)$, respectively, whose poles are all on the negative half of the real axis. Here, $p(z)$, $q(z)$, $p_1(z)$, and $q_1(z)$ are entire functions of z which satisfy the identity*

$$p_1(z)q(z) - p(z)q_1(z) \equiv 1,$$

and are the limits, uniformly over every finite domain of z, *of the sequences* $\{A_{2p}(z)\}$, $\{B_{2p}(z)\}$, $\{A_{2p+1}(z)\}$, *and* $\{B_{2p+1}(z)\}$ *of the even and odd numerators and denominators of* (28.1). **[95.]**

We have seen that the even part of the continued fraction of Stieltjes is a real J-fraction. Stieltjes employed the following criterion for determining when, conversely, a given real J-fraction is the even part of a continued fraction of the form (28.1) in which the k_p are positive.

THEOREM 28.3. *A real J-fraction*

$$\cfrac{\lambda_0}{b_1 + z - \cfrac{\lambda_1}{b_2 + z - \cfrac{\lambda_2}{b_3 + z - \cdots}}}, \tag{28.9}$$

in which $\lambda_{p-1} > 0$, $b_p > 0$, $p = 1, 2, 3, \cdots$, *is the even part of a continued fraction*

$$\cfrac{c_0}{z + \cfrac{c_1}{1 + \cfrac{c_2}{z + \cfrac{c_3}{1 + \cdots}}}}, \tag{28.10}$$

in which $c_p > 0$, $p = 0, 1, 2, \cdots$, *provided there exist numbers* $P_1 = 0$, $P_n > 0$, $n = 2, 3, 4, \cdots$, *such that*

$$\left.\begin{aligned} P_n &< b_n, \\ \frac{\lambda_n}{b_n - P_n} &\leq P_{n+1}, \end{aligned}\right\} \quad n = 1, 2, 3, \cdots. \quad \textbf{[95.]} \tag{28.11}$$

Proof. It is required to show that when (28.11) holds, then the system of equations

$$\begin{aligned} c_0 &= \lambda_0, & c_1 &= b_1, \\ c_1c_2 &= \lambda_1, & c_2 + c_3 &= b_2, \\ c_3c_4 &= \lambda_2, & c_4 + c_5 &= b_3, \end{aligned}$$

$$\cdots$$

can be solved for $c_0, c_1, c_2, \cdots$, and that these numbers are positive. We have

$$c_2 = \frac{\lambda_1}{b_1} > 0, \quad c_2 \leq \frac{\lambda_1}{b_1 - P_1} \leq P_2.$$

If we assume that $0 < c_{2n-1} \leq P_n$, then we conclude from the equation

$$c_{2n} = \frac{\lambda_n}{b_n - c_{2n-2}}$$

that

$$0 < c_{2n} < \frac{\lambda_n}{b_n - P_n} \leq P_{n+1}.$$

Therefore, $0 < c_{2n} \leq P_{n+1}$, $n = 1, 2, 3, \cdots$. Since $c_{2n-1} = \lambda_n/c_{2n}$, then $c_{2n-1} > 0$, $n = 1, 2, 3, \cdots$.

29. The Condition (H). We have seen in § 6 that a *necessary* condition for the continued fraction

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}} \tag{29.1}$$

to be convergent is that the infinite series $\Sigma |b_p|$ be divergent. More generally, we showed that it is necessary for the convergence of (29.1) that the sequence $b_1, b_2, b_3, \cdots$ satisfy a certain condition, which we shall now call **condition (H)**, and which is defined as follows.

Definition 29.1. *A sequence $b_1, b_2, b_3, \cdots$ of complex numbers is said to satisfy* **condition (H)** *if at least one of the following three statements holds.*

(a) *The series* $\Sigma |b_{2p+1}|$ *diverges.*
(b) *The series* $\Sigma |b_{2p+1}(b_2 + b_4 + \cdots + b_{2p})^2|$ *diverges.*
(c) $\lim_{p=\infty} |b_2 + b_4 + \cdots + b_{2p}| = \infty$.

In § 7, we found one important case where the condition (H) is sufficient for the convergence of (29.1). We note that if

$k_p > 0, p = 1, 2, 3, \cdots$, then the series Σk_p diverges if, and only if, the sequence $k_1, k_2, k_3, \cdots$ satisfies condition (H). Hence we see by Theorems 28.1 and 28.2 that the continued fraction of Stieltjes converges if, and only if, the sequence of coefficients $k_1, k_2, k_3, \cdots$ satisfies condition (H).

The condition (H) was first used by Hamburger [**26**] in the case of the continued fraction (28.1) in which the k_p are real, $k_{2p+1} > 0$ and $k_{2p} \neq 0$, $(p = 1, 2, 3, \cdots)$. He showed that the continued fraction converges for z nonreal if, and only if, the k_p satisfy condition (H). (Cf. § 30.)

The theorem which we shall now give, due to Scott and Wall [**88a**], includes this result of Hamburger as well as an extension of Hamburger's theorem due to Mall [**59**]. It also includes an important convergence theorem of Van Vleck [**107**].

THEOREM 29.1. *Let $k_1, k_2, k_3, \cdots$ be constants such that*

$$k_1 > 0, \quad k_{2p+1} \geq 0, \quad \Re(k_{2p}) \geq 0, \quad p = 1, 2, 3, \cdots, \tag{29.2}$$

and let $z_1, z_2, z_3, \cdots$ be complex variables. The continued fraction (28.2) *converges for*

$$\Re(z_p) \geq \delta, \quad |z_p| < M, \quad p = 1, 2, 3, \cdots, \tag{29.3}$$

where δ and M are any positive constants, if, and only if, the sequence $k_1, k_2, k_3, \cdots$ satisfies condition (H).

Remark 1. By means of an equivalence transformation, we may write (28.2) in the form

$$\cfrac{i}{k_1 i z_1 + \cfrac{1}{-ik_2 + \cfrac{1}{k_3 i z_2 + \cfrac{1}{-ik_4 + \cdots}}}}$$

On replacing iz_p by z_p, $-ik_{2p}$ by k_{2p}, and dropping the factor i, we see that this resumes the form (28.2) with the inequalities (29.2) and (29.3) replaced by the inequalities

$$k_1 > 0, \quad k_{2p+1} \geq 0, \quad \Im(k_{2p}) \leq 0, \quad p = 1, 2, 3, \cdots, \tag{29.4}$$

and

$$\mathfrak{J}(z_p) \geq \delta, \quad |z_p| < M, \quad p = 1, 2, 3, \cdots, \tag{29.5}$$

respectively. The statements (a), (b), (c) of the condition (H) remain unchanged. Conversely, if (29.4) and (29.5) hold, then the continued fraction is equivalent to another of the same form which satisfies (29.2) and (29.3).

Remark 2. Instead of the condition (29.2) we may use the more general condition

$$k_{2p-1} \geq 0, \quad \mathfrak{R}(k_{2p}) \geq 0, \quad p = 1, 2, 3, \cdots, \tag{29.6}$$

with the proviso that $k_{2p-1} > 0$ for at least one value of p. (Cf. the remark after Theorem 7.1.)

Proof of Theorem 29.1. Since $|z_p| < M$, $p = 1, 2, 3, \cdots$, it follows by Theorem 6.2 that it is necessary for the convergence of the continued fraction that the k_p satisfy condition (H). In case the series

$$\Sigma k_{2p+1} \tag{29.7}$$

and

$$\Sigma k_{2p+1} | k_2 + k_4 + \cdots + k_{2p} |^2 \tag{29.8}$$

both converge, and

$$\lim_{p=\infty} | k_2 + k_4 + \cdots + k_{2p} | = \infty, \tag{29.9}$$

then the continued fraction converges by Theorem 7.1. Hence, there is to be considered here the case where at least one of the two series (29.7) or (29.8) is divergent.

As in the proof of Theorem 7.1, we find that the pth approximant of the continued fraction satisfies the inequality

$$\left| \frac{A_p}{B_p} \right| \leq \frac{1}{k_1 \delta}, \quad p = 1, 2, 3, \cdots, \tag{29.10}$$

and therefore $B_p \neq 0$.

We have to consider two cases (A) and (B) according as $k_p \neq 0$ for all values of p, or $k_p = 0$ for one or more values of p.

(A) *Suppose that* $k_p \neq 0$ *for* $p = 1, 2, 3, \cdots$. If we make the substitution (28.4) and suppose that (29.4) holds (cf. Remark 1), then

$$\Im(c_p) = \gamma_p \geq 0, \quad p = 1, 2, 3, \cdots.$$

Let $\gamma_0 = 0$,

$$\beta_p = \gamma_{2p-2} + \gamma_{2p-1},$$

$$g_0 = 0,$$

$$g_p = \begin{cases} 1 & \text{if} \quad \gamma_{2p+1} = 0, \\ \dfrac{\gamma_{2p}}{\gamma_{2p} + \gamma_{2p+1}} & \text{if} \quad \gamma_{2p+1} > 0, \end{cases} \qquad p = 1, 2, 3, \cdots.$$

$$\alpha_p = \Im\left(\sqrt{c_{2p-1}c_{2p}}\right),$$

Then we find that $0 \leq g_{p-1} \leq 1$, and, since $\arg c_{2p-1} = \arg c_{2p}$,

$$\alpha_p^2 = \gamma_{2p-1}\gamma_{2p} = \beta_p\beta_{p+1}(1 - g_{p-1})g_p, \quad p = 1, 2, 3, \cdots.$$

Therefore, (28.3) *is in this case a positive definite continued fraction.*

Following § 17, we now define the circular region K_p as the image of the half-plane

$$\Im(w) \geq \beta_{p+1}g_p = \gamma_{2p} = \Im\left(\frac{1}{k_{2p}k_{2p+1}}\right) \tag{29.11}$$

under the transformation $t = T_p(w)$ of (28.5). Inasmuch as $\Im(z_{p+1} + c_{2p} + c_{2p+1}) > \gamma_{2p}$, $\Im(z_{p+1} + c_{2p}) > \gamma_{2p}$, it follows by (28.6) that A_{2p+2}/B_{2p+2} and A_{2p+1}/B_{2p+1} both have their values in K_p. To prove that the continued fraction (28.2) converges, it therefore suffices to prove that the radius r_p of K_p has the limit 0 for $p = \infty$, i.e., that the limit-point case holds for the positive definite continued fraction (28.3).

By an argument used before (cf. the beginning of Chapter V) and the theorem of invariability (§ 22), we know that $r_p \to 0$ if $\Im(z_p) \geq \delta > 0$, $|z_p| < M$, $p = 1, 2, 3, \cdots$, provided that at least one of the infinite series

$$\Sigma |X_p(0)|^2, \quad \Sigma |Y_p(0)|^2 \tag{29.12}$$

is divergent, where the X_p and Y_p are the polynomials constructed by means of (17.10) for the continued fraction (28.3).

Our theorem will then be established for the case where $k_p \neq 0$, $p = 1, 2, 3, \cdots$, if we show that the series (29.12) are the same as the series (29.7) and (29.8). But this can easily be done by mathematical induction with the aid of the recurrence formulas

$$Y_0 = 0, \quad Y_1 = 1, \quad X_0 = -1, \quad X_1 = 0,$$

$$-\sqrt{c_{2p-3}c_{2p-2}}Y_{p-1}(0) + (c_{2p-2} + c_{2p-1})Y_p(0) - \sqrt{c_{2p-1}c_{2p}}Y_{p+1}(0) = 0,$$

$$-\sqrt{c_{2p-3}c_{2p-2}}X_{p-1}(0) + (c_{2p-2} + c_{2p-1})X_p(0) - \sqrt{c_{2p-1}c_{2p}}X_{p+1}(0) = 0,$$

$$p = 1, 2, 3, \cdots.$$

In fact, we find that

$$\begin{aligned} Y_{p+1}{}^2(0) &= k_{2p+1}, \\ X_{p+1}{}^2(0) &= k_{2p+1}(k_2 + k_4 + \cdots + k_{2p})^2, \end{aligned} \qquad p = 1, 2, 3, \cdots. \tag{29.13}$$

(B) *Suppose that $k_p = 0$ for one or more values of p.* We shall reduce this case to the preceding case, or else to the consideration of the convergence of what is essentially a terminating continued fraction of the form (28.2). The ideas involved are intuitively simple, but the detailed treatment is somewhat tedious.

We shall associate with the continued fraction (28.2) another continued fraction obtained from it by a certain contraction process, namely:

$$\cfrac{1}{h_1\zeta_1 + \cfrac{1}{h_2 + \cfrac{1}{h_3\zeta_2 + \cfrac{1}{h_4 + \cdots}}}}, \tag{29.14}$$

where, for certain indices n_p specified below,

$$
\begin{aligned}
h_1 &= k_1 + k_3 + \cdots + k_{2n_1-1}, \\
h_2 &= k_{2n_1} + k_{2n_1+2} + \cdots + k_{2n_2}, \\
h_3 &= k_{2n_2+1} + k_{2n_2+3} + \cdots + k_{2n_3-1}, \\
h_4 &= k_{2n_3} + k_{2n_3+2} + \cdots + k_{2n_4}, \\
h_5 &= k_{2n_4+1} + k_{2n_4+3} + \cdots + k_{2n_5-1}, \\
& \quad \cdot \quad \cdot \quad \cdot ; \\
h_1\zeta_1 &= k_1 z_1 + k_3 z_2 + \cdots + k_{2n_1-1} z_{n_1}, \\
h_3\zeta_2 &= k_{2n_2+1} z_{n_2+1} + k_{2n_2+3} z_{n_2+2} + \cdots + k_{2n_3-1} z_{n_3}, \\
h_5\zeta_3 &= k_{2n_4+1} z_{n_4+1} + k_{2n_4+3} z_{n_4+2} + \cdots + k_{2n_5-1} z_{n_5}, \\
& \quad \cdot \quad \cdot \quad \cdot .
\end{aligned}
\tag{29.15}
$$

The indices $n_0 = 0, n_1, n_2, n_3, \cdots$ satisfy the following conditions.

$$0 = n_0 < n_1 \leq n_2 < n_3 \leq n_4 < n_5 \leq \cdots; \tag{i}$$

$$k_{2n_p+1} > 0, \quad p = 0, 2, 4, 6, \cdots; \tag{ii}$$

$$k_{2p+1} = 0 \quad \text{for} \quad n_1 \leq p < n_2, \quad n_3 \leq p < n_4, \quad n_5 \leq p < n_6, \quad \cdots; \tag{iii}$$

$$
\begin{aligned}
& k_2 + k_4 + \cdots + k_{2p-2} \neq 0 \quad \text{for} \quad 1 < p < n_1 \\
& \qquad\qquad\qquad \text{implies } k_{2p-1} = 0, \\
& k_2 + k_4 + \cdots + k_{2n_1-2} = 0, \\
& k_{2n_2+2} + k_{n_2+4} + \cdots + k_{2p-2} \neq 0 \quad \text{for} \quad n_2 < p < n_3 \\
& \qquad\qquad\qquad \text{implies } k_{2p-1} = 0, \\
& k_{2n_2+2} + k_{2n_2+4} + \cdots + k_{2n_3-2} = 0, \\
& k_{2n_4+2} + k_{2n_4+4} + \cdots + k_{2p-2} \neq 0 \quad \text{for} \quad n_4 < p < n_5 \\
& \qquad\qquad\qquad \text{implies } k_{2p-1} = 0, \\
& k_{2n_4+2} + k_{2n_4+4} + \cdots + k_{2n_5-2} = 0, \\
& \quad \cdot \quad \cdot \quad \cdot ;
\end{aligned}
\tag{iv}
$$

$$
\begin{aligned}
& k_{2n_1} + k_{2n_1+2} + \cdots + k_{2p} \neq 0 \quad \text{for} \quad n_1 \leq p \leq n_2, \\
& k_{2n_3} + k_{2n_3+2} + \cdots + k_{2p} \neq 0 \quad \text{for} \quad n_3 \leq p \leq n_4, \\
& k_{2n_5} + k_{2n_5+2} + \cdots + k_{2p} \neq 0 \quad \text{for} \quad n_5 \leq p \leq n_6,
\end{aligned}
\tag{v}
$$

It is easy to verify that all the approximants of (29.14) are approximants of (28.2).

If, by exception, the sequence of indices n_p satisfying the conditions (i), $\cdots$, (v) fails to exist, then (28.2) has an infinite subsequence of approximants of the form

$$f(w_r) = \cfrac{1}{h_1\zeta_1 + \cfrac{1}{h_2 + \cfrac{1}{h_3\zeta_2 + \cfrac{1}{h_4 + \cdots + \cfrac{1}{h_{2m} + \cfrac{1}{w_r}}}}}}, \tag{29.16}$$

where

$$w_r = \sum_{p=n_{2m}}^{n_{2m}+r} k_{2p+1}z_{p+1}.$$

Here the h_p are defined as in (29.15) with $n_0, n_1, \cdots, n_{2m}$ a *finite* sequence of indices satisfying the conditions (i), $\cdots$, (v), and the further condition

$$k_{2n_{2m}+1} > 0,$$

$$k_{2n_{2m}+2} + k_{2n_{2m}+4} + \cdots + k_{2p-2} \neq 0 \quad \text{for} \quad p > n_{2m} \tag{vi}$$
$$\text{implies } k_{2p-1} = 0.$$

Inasmuch as at least one of the two series (29.7) and (29.8) diverges, by hypothesis, it follows that $k_{2p-1} > 0$ for infinitely many values of p.

The case where (29.14) exists will be referred to as *Case* 1, and the exceptional case will be referred to as *Case* 2. We shall prove that when (29.4) and (29.5) hold, and at least one of the series (29.7) or (29.8) diverges, then (29.14) converges in Case 1, and the sequence (29.16) converges in Case 2. It will then remain to be shown that the sequence of approximants of (28.2) which are not approximants of (29.14) converges to the value of (29.14) in Case 1, and the analogous fact must be established in Case 2.

We shall first dispose of Case 2. From condition (vi) we get, for $n > n_{2m} = \sigma$,

$$\sum_{p=1}^{n} k_{2p+1}| k_2 + k_4 + \cdots + k_{2p} |^2$$
$$= \sum_{p=1}^{\sigma} k_{2p+1}| k_2 + k_4 + \cdots + k_{2p} |^2$$
$$+ \sum_{p=\sigma+1}^{n} k_{2p+1}| k_2 + k_4 + \cdots + k_{2\sigma} + k_{2\sigma+2} + \cdots + k_{2p} |^2$$
$$= \sum_{p=1}^{\sigma} k_{2p+1}| k_2 + k_4 + \cdots + k_{2p} |^2$$
$$+ | k_2 + k_4 + \cdots + k_{2\sigma} |^2 \sum_{p=\sigma+1}^{n} k_{2p+1},$$

and consequently the series (29.7) is divergent. Therefore, the sequence (29.16) converges to the finite (cf. (29.10)) limit $f(\infty)$. All except at most a finite number of those approximants of (28.2) not included in the sequence (29.16) are equal to $f(\infty)$ or are of the form $f(w_r + v)$, where

$$v = \frac{1}{k_{2\sigma+2r+2} + k_{2\sigma+2r+4} + \cdots + k_{2p}}, \quad p \geq \sigma + r + 1.$$

Since $\mathfrak{I}(v) \geq 0$, then $\mathfrak{I}(w_r + v) \geq \mathfrak{I}(w_r)$, and therefore

$$\lim_{r=\infty} f(w_r + v) = f(\infty).$$

Thus, (28.2) converges to $f(\infty)$.

Turning now to Case 1, we first prove that the continued fraction (29.14) converges. To that end, it suffices to prove that it satisfies all the conditions used in (A). There will then be associated with (29.14) a nest of circles K_1', K_2', K_3', $\cdots$, each contained in the preceding, whose radii have the limit 0. We shall show then that, for each n, there exists an index $\mu = \mu(n)$ such that all the approximants of the original continued fraction (28.2) from and after the nth have their values in the circle K_μ'; and $\lim_{n=\infty} \mu(n) = \infty$. It will then follow that (28.2) converges, and the proof of the theorem will be complete.

From the formulas (29.15), the conditions (i), $\cdots$, (v), and (29.4), (29.5), we find that

$$h_{2p-1} > 0, \quad h_{2p} \neq 0, \quad \Im(h_{2p}) \leq 0, \quad \Im(\zeta_p) \geq \delta, \quad |\zeta_p| < M.$$

Let $S_p = h_2 + h_4 + \cdots + h_{2p}$. Then, by (29.15) and (iv), $S_p = s_{n_{2p}}$, where $s_n = k_2 + k_4 + \cdots + k_{2n}$. It is not difficult to show that $\Sigma h_{2p+1}|S_p|^2 = \Sigma k_{2p+1}|s_p|^2$. Therefore, if at least one of the series (29.7) or (29.8) diverges, then at least one of the series Σh_{2p+1} or $\Sigma h_{2p+1}|S_p|^2$ diverges. Therefore, by (A), the continued fraction (29.14) converges.

Let A_p'/B_p' denote the pth approximant of (29.14). The sequence of these approximants is a subsequence of the sequence of approximants of (28.2). We shall prove that those approximants of (28.2) which fall between A_{2p}'/B_{2p}' and A_{2p+2}'/B_{2p+2}' in the sequence have their values in the circle K_{p-1}' associated with (29.14). This will imply the existence of the index $\mu(n)$ having the required properties.

The approximants of (28.2) in question all have the form

$$\cfrac{1}{h_1\zeta_1 + \cfrac{1}{h_2 + \cfrac{1}{h_3\zeta_2 + \cfrac{1}{h_4 + \cdots + \cfrac{1}{h_{2p} + \cfrac{1}{w}}}}}}, \tag{29.17}$$

where w has one of the possible forms

$$w = k_{2n_{2p}+1}z_{n_{2p}+1} + k_{2n_{2p}+3}z_{n_{2p}+2} + \cdots + k_{2r-1},$$

$$n_{2p} < r < n_{2p+1},$$

or

$$w = k_{2n_{2p}+1}z_{n_{2p}+1} + k_{2n_{2p}+3}z_{n_{2p}+2} + \cdots + k_{2r-1}z_r + \frac{1}{k_{2r} + k_{2r+2} + \cdots + k_{2s}},$$

$$n_{2p} < r \leq n_{2p+1}, \quad r \leq s < n_{2p+2}.$$

We note that when (29.4) and (29.5) hold, then $I(w) \geq 0$.

Since (29.17) is the $2p$th approximant of (29.14) in which h_{2p} has been replaced by $h_{2p} + 1/w$, it is given by the expression

$$T_{p-1}'\left(\zeta_p + \frac{1}{h_{2p-2}h_{2p-1}} + \frac{1}{h_{2p-1}\left(h_{2p} + \frac{1}{w}\right)}\right),$$

where T_{p-1}' is the transformation related to (29.14) in the same way that T_{p-1} is related to (28.2) (cf. the first formula (28.6)). Inasmuch as

$$\mathfrak{I}\left(\zeta_p + \frac{1}{h_{2p-2}h_{2p-1}} + \frac{1}{h_{2p-1}\left(h_{2p} + \frac{1}{w}\right)}\right) \geq \mathfrak{I}\left(\frac{1}{h_{2p-2}h_{2p-1}}\right),$$

we conclude immediately that the value of (29.17) is in the circle K_{p-1}'. This is what we wished to prove, and Theorem 29.1 is therefore established.

In case the z_p of (28.2) are all equal to a single complex variable z, we have, by (29.10) and the convergence continuation theorem, this corollary to Theorem 29.1.

THEOREM 29.2. *If $k_1, k_2, k_3, \cdots$ are numbers satisfying the conditions* (29.2), *then the continued fraction* (28.1) *converges uniformly over every finite closed region in the half-plane $\mathfrak{R}(z) > 0$, and its value is analytic in this half-plane, provided the k_p satisfy the condition* (*H*). *If condition* (*H*) *is not satisfied, then the continued fraction diverges for every value of z.*

30. Three Convergence Theorems. The first of the three theorems which we shall consider in this section is the following theorem of Van Vleck [**107**].

THEOREM 30.1.[14] *If $\mathfrak{R}(b_1) > 0$ and $|\mathfrak{I}(b_p)| \leq c\mathfrak{R}(b_p)$ for $p = 1, 2, 3, \cdots$, where c is a positive constant, then the continued fraction*

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}} \tag{30.1}$$

converges if, and only if, the infinite series $\Sigma|b_p|$ diverges.

[14] Van Vleck used his theorem to obtain Theorem 28.1 of Stieltjes. Besides Van Vleck's original proof, and the proof given here (cf. [**88a**]), other proofs are to be found in [**40**] (cf. [**69**, § 54]) and in [**35**].

Proof. In the continued fraction of Theorem 29.1, put

$$k_{2p} = b_{2p}, \quad k_{2p-1} = |\, b_{2p-1} \,|, \quad k_{2p-1}z_p = b_{2p-1}$$

where

$$z_p = 1 \quad \text{if} \quad b_{2p-1} = 0.$$

Then,

$$\Re(k_{2p}) \geq \frac{|\, \Im(b_{2p}) \,|}{c} \geq 0,$$

$$k_1 = |\, b_1 \,| > 0, \quad k_{2p+1} = |\, b_{2p+1} \,| \geq 0,$$

$$|\, z_p \,| = 1 < M, \quad \Re(z_p) \geq \frac{1}{\sqrt{1 + c^2}} = \delta.$$

If the series $\Sigma |\, b_p \,|$ diverges, then either (a) of condition (H) holds (cf. Definition 29.1), or else $\Sigma |\, b_{2p} \,|$ diverges, in which case (c) of condition (H) holds, inasmuch as $|\, b_{2p} \,| \leq \sqrt{1 + c^2}\Re(b_{2p})$. We then conclude by Theorem 29.1 that the continued fraction (30.1) converges if $\Sigma |\, b_p \,|$ diverges. If, on the other hand, this series converges, then the continued fraction diverges by Theorem 6.1.

If we require that $|\, \Im(b_p) \,| \leq c\Re(b_p)$ only for *odd* values of p, we obtain the following theorem of Mall [**59**].

THEOREM 30.2. *If* $\Re(b_1) > 0$, $|\, \Im(b_{2p-1}) \,| \leq c\Re(b_{2p-1})$, *where* c *is a positive constant, and* $\Re(b_{2p}) \geq 0$, $(p = 1, 2, 3, \cdots)$, *then the continued fraction* (30.1) *converges if, and only if,* $b_1, b_2, b_3, \cdots$ *satisfies condition* (*H*).

Proof. As in the proof of Theorem 30.1, we put

$$k_{2p-1} = |\, b_{2p-1} \,|, \quad k_{2p} = b_{2p}, \quad k_{2p-1}z_p = b_{2p-1}$$

where

$$z_p = 1 \quad \text{if} \quad b_{2p-1} = 0, \quad p = 1, 2, 3, \cdots.$$

Then, $k_1 = |\, b_1 \,| > 0$, $k_{2p+1} = |\, b_{2p+1} \,| \geq 0$, $\Re(k_{2p}) = \Re(b_{2p}) \geq 0$, $|\, z_p \,| = 1 < M$, and $\Re(z_p) \geq 1/(\sqrt{1 + c^2}\,) = \delta$. Hence, by Theorem 29.1, the continued fraction converges if, and only if, the b_p satisfy condition (H).

The third theorem is the following theorem of Hamburger [**26**].

THEOREM 30.3. *If k_p is real, $k_1 > 0$, $k_{2p+1} \geq 0$, $p = 1, 2, 3, \cdots$, then the continued fraction* (28.1) *converges for all nonreal z if, and only if, the k_p satisfy condition (H). When condition (H) holds, then the continued fraction converges uniformly over every finite closed domain whose distance from the real axis is positive.*

Proof. By means of an equivalence transformation, we may throw (28.1) into the form

$$\cfrac{i}{k_1(iz) + \cfrac{1}{-ik_2 + \cfrac{1}{k_3(iz) + \cfrac{1}{-ik_4 + \cdots}}}}$$

Then, $k_1 > 0$, $k_{2p+1} \geq 0$, $\Re(-ik_{2p}) = 0$. By Theorem 29.1, the continued fraction converges for $\Re(iz) = -\Im(z) \geq \delta > 0$ if, and only if, $k_1, -ik_2, k_3 - ik_4, k_5, \cdots$ satisfies condition (H). Therefore, the continued fraction (28.1) converges for $\Im(z) \leq -\delta$ if, and only if, $k_1, k_2, k_3, \cdots$ satisfies condition (H). Again, by means of an equivalence transformation, we may throw (28.1) into the form

$$\cfrac{-i}{k_1(-iz) + \cfrac{1}{ik_2 + \cfrac{1}{k_3(-iz) + \cfrac{1}{ik_4 + \cdots}}}},$$

and conclude, as before, that (28.1) converges for $\Re(-iz) = \Im(z) \geq \delta > 0$, if, and only if, $k_1, k_2, k_3, \cdots$ satisfies condition (H).

Since, by (29.10), the approximants of (28.1) are uniformly bounded for $\Im(z) \geq \delta > 0$ and for $\Im(z) \leq -\delta$, we conclude by the convergence continuation theorem that, when condition (H) is satisfied, then the continued fraction converges uniformly over every finite closed domain at a positive distance from the real axis.

Remark 1. In connection with all three of these theorems, cf. Remark 2 following Theorem 29.1.

Remark 2. Both Theorems 30.2 and 30.3 are here proved in a somewhat more general form than is to be found in the papers of Mall and of Hamburger, respectively (loc. cit.).

Exercise 6

6.1. Show that Stieltjes' theorem (Theorem 28.1) holds under the more general hypothesis $k_1 > 0$, $k_p \geq 0$, $p = 2, 3, 4, \cdots$.

6.2. In the continued fraction (30.1), put $b_p = |b_p| e^{i\theta_p}$. Show that if there exists a real number c and a positive number a such that

$$\begin{aligned} c \leq \theta_{2p-1} \leq c + \pi - a, \\ -c - \pi + a \leq \theta_{2p} \leq \pi - c, \end{aligned} \qquad p = 1, 2, 3, \cdots,$$

then the continued fraction converges if, and only if, the series $\Sigma |b_p|$ diverges [**107**].

6.3. Let $\delta_1, \delta_2, \delta_3, \cdots$ be any numbers such that

$$|\delta_p| - \Re(\delta_p) \leq \tfrac{1}{2}, \quad \delta_{2p}\delta_{2p+1} > 0, \quad \Im(\delta_{2p-1} + \delta_{2p}) \geq 0, \quad p = 1, 2, 3, \cdots.$$

Then, the continued fraction

$$\cfrac{1}{\delta_1 + \delta_2 + z - \cfrac{\delta_2\delta_3}{\delta_3 + \delta_4 + z - \cfrac{\delta_4\delta_5}{\delta_5 + \delta_6 + z - \cdots}}}$$

converges if z is outside the region bounded by the lower half of the parabola $|z| + \Re(z) = 2$.

6.4. Show that the continued fraction (28.9) in which

$$\begin{aligned} \lambda_0 &= 1, \\ \lambda_n &= (2n - 1)(2n)^2(2n + 1)k^2, \\ b_n &= (2n - 1)^2(1 + k^2), \\ k &> 0, \end{aligned}$$

is the even part of a continued fraction (28.10) in which $c_p > 0$. [**95.**]

Chapter VII

EXTENSIONS OF THE PARABOLA THEOREM

The theorems of this chapter may be regarded as extensions or refinements of the theorems of Chapter III, which result from the general theory of positive definite continued fractions.

31. A Family of Parabolic Domains. We specialize the continued fraction (16.1) by there taking

$$b_p = -\sin\phi_p(1+\delta\sec\phi_p) + i\cos\phi_p,$$
$$-\frac{\pi}{2} < \phi_p < \frac{\pi}{2}, \quad z_p = i\delta, \quad \delta > 0. \tag{31.1}$$

Then $b_p + z_p = ie^{i\phi_p}(1+\delta\sec\phi_p)$. We require the continued fraction to be positive definite, i.e., we require that

$$|a_p{}^2| - \Re(a_p{}^2) \leq 2\cos\phi_p\cos\phi_{p+1}(1-g_{p-1})g_p,$$
$$0 \leq g_{p-1} \leq 1, \quad p = 1, 2, 3, \ldots. \tag{31.2}$$

We do not require the a_p to be different from zero.

By means of an equivalence transformation, the continued fraction can be written as $1/ie^{i\phi_1}(1+\delta\sec\phi_1)$ times

$$\cfrac{1}{1+\cfrac{c_1}{1+\cfrac{c_2}{1+\cdots}}}, \tag{31.3}$$

where

$$c_p = \frac{a_p{}^2 e^{-i(\phi_p+\phi_{p+1})}}{(1+\delta\sec\phi_p)(1+\delta\sec\phi_{p+1})}. \tag{31.4}$$

On substituting the value of a_p^2 from (31.4) into (31.2), we get the following inequalities for the c_p.

$$|c_p| - \Re(c_p e^{i(\phi_p+\phi_{p+1})}) \leq \frac{2\cos\phi_p \cos\phi_{p+1}(1-g_{p-1})g_p}{(1+\delta\sec\phi_p)(1+\delta\sec\phi_{p+1})}, \quad p = 1, 2, 3, \cdots. \tag{31.5}$$

Conversely, if (31.5) holds, then (31.2) holds.

If A_p/B_p is the pth approximant of the continued fraction (31.3), and A_p'/B_p' is the pth approximant of (16.1), with b_p, z_p and a_p determined by (31.1) and (31.4), then

$$\frac{A_p'}{B_p'} = \frac{1}{ie^{i\phi_1}(1+\delta\sec\phi_1)}\frac{A_p}{B_p}.$$

Therefore, if (31.5) holds, it follows from (17.3) that

$$\left|\frac{A_p}{B_p} - \frac{e^{i\phi_1}(1+\delta\sec\phi_1)}{2(g_0\cos\phi_1+\delta)}\right| \leq \frac{1+\delta\sec\phi_1}{2(g_0\cos\phi_1+\delta)}, \quad p = 1, 2, 3, \cdots, \tag{31.6}$$

provided $g_0 > 0$ or $\delta > 0$.

We have thus proved the following theorem.

THEOREM 31.1. *Let the partial numerators c_p of the continued fraction* (31.3) *satisfy the inequalities* (31.5), *where $\delta \geq 0$, and*

$$-\frac{\pi}{2} < \phi_p < \frac{\pi}{2}, \quad 0 \leq g_{p-1} \leq 1, \quad p = 1, 2, 3, \cdots.$$

If $g_0 > 0$ or $\delta > 0$, then the approximants of the continued fraction satisfy the inequalities (31.6). [**139.**]

This is a generalization of Theorem 14.3. In fact, we see that this theorem is the special case $\phi_p = 0$, $g_{p-1} = \frac{1}{2}$, $p = 1, 2, 3, \cdots$, $\delta = 0$, of Theorem 31.1.

Geometrically, the inequality (31.5) means that, for each p, c_p has its value in a certain parabolic domain depending upon p. In contrast with the parabolic domains of Chapter III, the axes of these domains do not necessarily coincide with the real axis.

From Theorems 5.1, 16.1 and 17.1, we have immediately

THEOREM 31.2. *If the inequalities* (31.5) *hold with* $\delta > 0$ *or with* $g_p > 0$, $p = 0, 1, 2, \cdots$, *then the denominators of the continued fraction* (31.3) *are all different from zero. Hence, the continued fraction converges in this case when some partial numerator* c_p *vanishes* [**9**].

If $\delta > 0$ and the determinate case holds for the continued fraction (16.1) equivalent to (31.3), then the latter converges. By Theorem 25.1 and (31.4) we therefore have

THEOREM 31.3. *If the inequalities* (31.5) *hold with* $\delta > 0$, *and if the series*

$$\Sigma \frac{1}{\sqrt{|c_p|(1 + \delta \sec \phi_p)(1 + \delta \sec \phi_{p+1})}} \tag{31.7}$$

diverges, then the continued fraction (31.3) *converges.*

32. "Convergence Neighborhoods" of a Point (2). In § 15 we found that every point c with real part greater than $-\frac{1}{4}$ is the center of a circular region which is a convergence region for the continued fraction (31.3). Perron [**69**] showed that this holds if c is not real and less than or equal to $-\frac{1}{4}$. Paydon and Wall [**68**] proved the following more general theorem.

THEOREM 32.1. *Let* W *be a finite closed region containing* c. *Then* W *is a convergence neighborhood of* c *for the continued fraction* (31.3) *provided there exists a parabola with focus at the origin, not containing the point* $-\frac{1}{4}$ *in its interior, which contains* W *in its interior.*

Proof. Let $|z| - \Re(ze^{2i\phi}) = h$, $h > 0$, be a parabola with focus at the origin, passing through $-\frac{1}{4}$, which contains W in its interior. Then we must have

$$-\frac{\pi}{2} < \phi < \frac{\pi}{2}, \quad h = \frac{\cos^2 \phi}{2}.$$

Thus, if c_p is in W, then

$$|c_p| - \Re(c_p e^{2i\phi}) \leq \frac{\cos^2 \phi}{2(1 + \delta \sec \phi)^2},$$

provided δ is a sufficiently small positive number. That is, (31.5) is satisfied with $\phi_p = \phi$, $g_{p-1} = \frac{1}{2}$, $\delta > 0$. Since W is a finite region, we conclude from Theorems 31.2 and 31.3 that the continued fraction converges when the c_p are in W, i.e., W is a convergence set for the continued fraction.

From Theorem 31.1 we have at once

COROLLARY 32.1. *The approximants of the continued fraction* (31.3) *do not exceed* $(2 + 2\delta \sec \phi)/(\cos \phi + 2\delta)$ *in absolute value, provided the* c_p *are in the domain* W.

33. A Theorem of Van Vleck. We shall now prove the following important convergence theorem of E. B. Van Vleck [**110**].

THEOREM 33.1. *Let* h_p, $p = 1, 2, 3, \cdots$, *be a sequence of numbers having a finite limit* h. *In case* $h \neq 0$, *let* L *denote the rectilinear cut from*

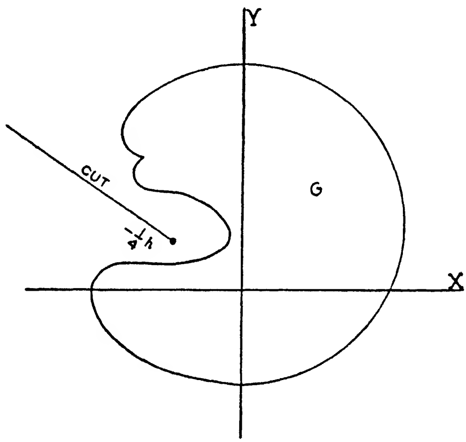

FIG. 5.

$-1/4h$ *to* ∞ *in the direction of the vector from* 0 *to* $-1/4h$. *Let* G *denote an arbitrary finite closed region whose distance from* L *is positive, or, in case* $h = 0$, *an entirely arbitrary finite closed region. There exists an index* N, *depending only upon* G, *such that the continued fraction*

$$\cfrac{1}{1 + \cfrac{h_n z}{1 + \cfrac{h_{n+1} z}{1 + \cfrac{h_{n+2} z}{1 + \cdots}}}} \tag{33.1}$$

converges uniformly over G *for* $n > N$.

Proof. If $h = 0$, it is only necessary to choose N so large that $|h_p z| \leq \frac{1}{4}$, for $p > N$, and for z in G, and then to apply Theorem 10.1.

If $h \neq 0$ and z' is in G, then hz' is not on the real interval $-\infty < x \leq -\frac{1}{4}$. By Theorem 32.1 and Corollary 32.1, together with the convergence continuation theorem, we therefore conclude that there exist an index $N(z')$ and a circle $C(z')$ with center z', such that (33.1) converges uniformly for z in $C(z')$, provided $n > N(z')$. Since G is a finite closed region, we may select from the family of circles $C(z')$ a finite set of these circles such that every point of G is interior to at least one of them. Let N denote the larger of the finite set of indices $N(z')$ which correspond to the circles selected. Then (33.1) is obviously uniformly convergent over G if $n > N$.

The above theorem can be generalized in the following way. Suppose that there are k finite numbers $\sigma_1, \sigma_2, \cdots, \sigma_k$, such that

$$\lim_{n=\infty} h_{nk+p} = \sigma_p, \quad \text{for} \quad p = 1, 2, 3, \cdots, k.$$

For the sake of simplicity, let $k = 2$. For any n, the even part of (33.1) can be thrown into a form where the partial denominators are unity, and the partial numerators are

$$\frac{-h_{n+2p}h_{n+2p+1}z^2}{(1 + h_{n+2p-1}z + h_{n+2p}z)(1 + h_{2+2p+1}z + h_{n+2p+2}z)}.$$

For $p = \infty$, this becomes

$$T(z) = \frac{-\sigma_1\sigma_2 z^2}{(1 + \sigma_1 z + \sigma_2 z)^2}.$$

The odd part gives exactly the same expression. One can now conclude, by the argument used in proving Theorem 33.1, that if G is any finite closed region at a positive distance from the curve along which $T(z)$ has its values in the real interval $-\infty < x \leq -\frac{1}{4}$, then there exists an index N such that the even and odd parts of (33.1) converge uniformly over G if $n > N$. Since the region G can always be chosen so that it contains the origin on its interior, and since, by Theorem 10.1, (33.1) converges in a sufficiently small neighborhood of the origin, we conclude that the even and odd parts must have a common value. Therefore, if $n > N$, the continued fraction (33.1) converges uniformly over G. In this case the cut is the curve into which the real interval $-\infty \leq x \leq -\frac{1}{4}$ is transformed by the transformation $t = T(x)$. For further details, cf. [121].

34. The Cardioid Theorem. We now specialize (16.1) by taking $b_p = ih$, $h \geq 0$, $z_p = z$, $(p = 1, 2, 3, \cdots)$. The condition for positive definiteness then becomes

$$|a_p^2| - \Re(a_p^2) \leq 2h^2(1 - g_{p-1})g_p, \quad 0 \leq g_{p-1} \leq 1, \qquad p = 1, 2, 3, \cdots. \tag{34.1}$$

Instead of (31.4) we now have

$$c_p = \frac{-a_p^2}{(ih + z)^2} = a_p^2 t, \tag{34.2}$$

where

$$t = \frac{-1}{(ih + z)^2}, \quad z = i\left(\frac{1}{\sqrt{t}} - h\right). \tag{34.3}$$

Since $z = i$ for $t = 1/(1 + h)^2$, we must take that branch of $\sqrt{t}$ which is real and positive for t real and positive. In the t-plane we make a cut from 0 to $-\infty$ along the negative half of the real axis, thus:

$$t = |t|e^{-2i\phi}, \quad |t| > 0, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \tag{34.4}$$

Then, by (34.3),

$$z = i\left(\frac{1}{\sqrt{|t|}} e^{i\phi} - h\right),$$

so that

$$\Im(z) = \frac{1}{\sqrt{|t|}} \cos\phi - h.$$

If $h = 0$, then $\Im(z) > 0$ for all t in the domain (34.4).

If $h > 0$, then $\Im(z) > 0$ if, and only if,

$$|t| < \frac{1}{2h^2}(1 + \cos 2\phi), \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}, \quad \arg t = --2\phi. \tag{34.5}$$

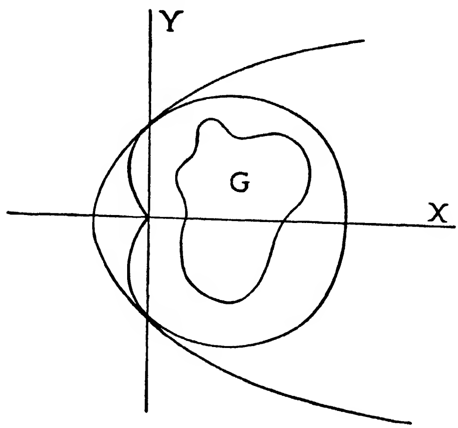

FIG. 6.

Let G be any finite closed region of the t-plane in the domain (34.4) if $h = 0$, or in the cardioid domain (34.5) if $h > 0$. Then, if σ is a sufficiently small positive constant, we shall have $\mathfrak{J}(z) \geq \sigma$ for all t in G, and consequently the approximants of the continued fraction

$$\cfrac{1}{1 + \cfrac{a_1^2 t}{1 + \cfrac{a_2^2 t}{1 + \cdots}}} \tag{34.6}$$

are uniformly bounded for t in G. Now, if t is real and

$$0 \leq t \leq \frac{1}{4h^2}, \tag{34.7}$$

then it follows from (34.1) that $|a_p^2 t| - \mathfrak{R}(a_p^2 t) \leq \frac{1}{2}, p = 1, 2, 3, \cdots$. Hence, for these values of t, we may apply the parabola theorem to (34.6), and conclude that it converges for t in the interval (34.7) provided that (a) some a_p vanishes, or (b) $a_p \neq 0$, $p = 1, 2, 3, \cdots$, and the series $\Sigma |k_p|$ diverges, where

$$k_1 = 1, \quad a_p^2 = \frac{1}{k_p k_{p+1}}, \quad p = 1, 2, 3, \cdots. \tag{34.8}$$

By the convergence continuation theorem, we therefore conclude that (34.6) converges uniformly over G if (a) or (b) holds. Similarly, if the series $\Sigma | k_p |$ converges, then we find that the even and odd parts of (34.6) converge uniformly over G to distinct limit-functions.

We have thus proved the **cardioid theorem** [**68, 9**]:

THEOREM 34.1. *Let the partial numerators a_p^2 of the continued fraction* (34.6) *satisfy the inequalities* (34.1). *Let G be any finite closed region in the t-plane which is in the domain* (34.4) *if $h = 0$, or in the cardioid domain* (34.5) *if $h > 0$. The continued fraction* (34.6) *converges uniformly over G if* (a) *some a_p vanishes, or* (b) *$a_p \neq 0$, $p = 1, 2, 3, \cdots$, and the series $\Sigma | k_p |$ diverges, where the k_p are given by* (34.8). *If this series converges, then the even and odd parts of* (34.6) *converge uniformly over G to separate limits, and the continued fraction diverges by oscillation for all t in G.*

Remark. If $h = 0$, this theorem is essentially Stieltjes' theorem, Theorem 28.1.

An arbitrary value t in the interior of the cardioid domain (34.5) can be expressed in the form

$$t = \frac{r \cos^2 \phi}{h^2} e^{-2i\phi}, \quad 0 < r < 1, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}, \quad h > 0.$$

If we combine (34.1) and (34.2), we then obtain the inequality

$$| c_p | - \Re(c_p e^{2i\phi}) \leq 2r \cos^2 \phi (1 - g_{p-1}) g_p, \qquad p = 1, 2, 3, \cdots. \tag{34.9}$$

Therefore, by Theorem 34.1, we have immediately [**68, 9**]

THEOREM 34.2. *If the partial numerators c_p of the continued fraction*

$$\cfrac{1}{1 + \cfrac{c_1}{1 + \cfrac{c_2}{1 + \cdots}}}$$

satisfy the inequalities (34.9) *in which* $-(\pi/2) < \phi < (\pi/2)$, $0 < r < 1$, $0 \leq g_p \leq 1$, $p = 0, 1, 2, \cdots$, *then the continued fraction converges if,* *and only if,* (a) *some* c_p *vanishes, or* (b) $c_p \neq 0$, $p = 1, 2, 3, \cdots$, *and the series* $\Sigma|k_p|$ *diverges, where* $k_1 = 1$, $c_p = 1/k_p k_{p+1}$, $p = 1, 2, 3, \cdots$.

If $\phi = 0$, $g_p = \frac{1}{2}$, $p = 0, 1, 2, \cdots$, then, if it were allowable to have $r = 1$, this theorem would reduce to the parabola theorem. It is reasonable to conjecture that Theorem 34.2 holds even for $r = 1$. However, our method of proof does not permit this value of r. Having $r = 1$ is equivalent to allowing the region G of the cardioid theorem to have boundary points upon the boundary of the cardioid domain.

35. An Extension of a Theorem of Szász. Szász showed that the continued fraction

$$\cfrac{1}{1+\cfrac{c_1}{1+\cfrac{c_2}{1+\cdots}}} \tag{35.1}$$

converges if the c_p are not zero, provided the series $\Sigma|c_p|$ converges and provided $\Sigma(|c_p| - \Re(c_p)) \leq 2$. This may be regarded as an extension of von Koch's theorem, Theorem 12.1. In the following theorem, we drop the requirement that the c_p be different from zero and the requirement that the series $\Sigma|c_p|$ be convergent, and employ an extended form of the condition $\Sigma(|c_p| - \Re(c_p)) \leq 2$.

THEOREM 35.1. *If*

$$\sum_{p=1}^{n} [|c_p| - \Re(c_p e^{2i\phi})] < 2\cos^2\phi, \quad n = 1, 2, 3, \cdots, \tag{35.2}$$

where $-(\pi/2) < \phi < (\pi/2)$, *then the continued fraction* (35.1) *converges if, and only if,* (a) *some* c_p *vanishes, or* (b) $c_p \neq 0$, $p = 1, 2, 3, \cdots$, *and the series* $\Sigma|k_p|$ *diverges, where* $k_1 = 1$, $c_p = 1/k_p k_{p+1}$, $p = 1, 2, 3, \cdots$. **[139, 9.]**

Proof. Let

$$C_p = \frac{|c_p| - \Re(c_p e^{2i\phi})}{2\cos^2\phi}.$$

Then, $C_p \geq 0$, and, by (35.2), $\sum_{p=1}^{n} C_p < 1$, $n = 1, 2, 3, \cdots$.

Therefore, by Lemma 12.1, the sequence $\{C_p\}$ is a chain sequence whose minimal parameters m_p satisfy the inequalities $0 \leq m_p < 1$, $p = 0, 1, 2, \cdots$. Moreover, by Theorem 12.1, the continued fractions

$$\cfrac{1}{1 - \cfrac{C_{p+1}}{1 - \cfrac{C_{p+2}}{1 - \cdots}}}, \qquad p = 0, 1, 2, \cdots,$$

are convergent. Therefore, by Theorem 19.4, the maximal parameters M_p of the chain sequence $\{C_p\}$ are all positive. We may then write $C_p = (1 - M_{p-1})M_p$, or

$$|c_p| - \Re(c_p e^{2i\phi}) = 2\cos^2\phi(1 - M_{p-1})M_p, \quad 0 < M_{p-1} \leq 1,$$
$$p = 1, 2, 3, \cdots.$$

If we now put $c_p e^{-2i\varphi} = a_p{}^2$, then (35.1) can be written as $ie^{-i\phi}$ times the continued fraction

$$\cfrac{1}{ie^{-i\phi} - \cfrac{a_1{}^2}{ie^{-i\phi} - \cfrac{a_2{}^2}{ie^{-i\phi} - \cdots}}}.$$

This is a positive definite continued fraction (16.1) in which

$$b_p = ie^{-i\phi}, \quad \beta_p = \cos\phi > 0, \quad z_p = 0.$$

On applying Theorem 17.1, we then conclude that its denominators are all different from zero, so that the denominators of (35.1) are all different from zero. Therefore, (35.1) converges in case some c_p vanishes.

Suppose that $c_p \neq 0$, $p = 1, 2, 3, \cdots$. By (35.2) we may choose n such that

$$|c_p| - \Re(c_p e^{2i\phi}) \leq 2r\cos^2\phi(1 - \tfrac{1}{2})\tfrac{1}{2}, \quad \text{for} \quad p \geq n,$$

where $0 < r < 1$. Therefore, by Theorem 34.2, the continued fraction

$$\cfrac{1}{1+\cfrac{c_n}{1+\cfrac{c_{n+1}}{1+\cdot_{\cdot_{\cdot}}}}}$$

converges when the series $\Sigma|\, k_p \,|$ diverges. If we denote its value by v, then (35.1) must converge to the value

$$\frac{A_{n-1} + c_{n-1}vA_{n-2}}{B_{n-1} + c_{n-1}vB_{n-2}}, \tag{35.3}$$

where A_p/B_p is the pth approximant of (35.1), provided the denominator of this expression is not zero. Now, by the determinant formula,

$$(A_{n-1} + c_{n-1}vA_{n-2})B_{n-2} - (B_{n-1} + c_{n-1}vB_{n-2})A_{n-2} \neq 0,$$

so that the numerator and denominator of (35.3) cannot vanish together. Inasmuch as the approximants of (35.1) have moduli not greater than $1/M_0 \cos \phi$, it follows that the denominator of (35.3) is not zero, so that the continued fraction converges.

If the series $\Sigma|\, k_p \,|$ converges, then the continued fraction diverges by Theorem 6.1.

The proof of Theorem 35.1 is now complete.

Exercise 7

7.1. Show that if t is in the domain G of the cardioid theorem, then the approximants of the continued fraction satisfy the inequality

$$\left|\frac{A_p}{B_p} - \frac{e^{i\phi}}{2\cos\phi - h(1-g_0)\sqrt{|\,t\,|}}\right| \leq \frac{1}{2\cos\phi - h(1-g_0)\sqrt{|\,t\,|}},$$

$$p = 1, 2, 3, \cdots.$$

7.2. Show that the approximants of the continued fraction of Theorem 34.2 satisfy the inequality

$$\left|\frac{A_p}{B_p} - \frac{e^{i\phi}}{2\cos\phi[1-(1-g_0)\sqrt{r}]}\right| \leq \frac{1}{2\cos\phi[1-(1-g_0)\sqrt{r}]}.$$

Chapter VIII

THE VALUE REGION PROBLEM

By the **value region problem** for a continued fraction, we shall understand the problem of determining a region V of the complex plane in which the values of the approximants lie when the elements of the continued fraction are restricted to have their values in a preassigned region E. We shall restrict our attention to continued fractions of the form

$$\cfrac{1}{1+\cfrac{c_1}{1+\cfrac{c_2}{1+\ddots}}}, \tag{a}$$

and shall call the pair of regions (E, V), having the above described property, a **solution** of the value region problem. A solution (E, V) is called **minimal** if V is closed and if, moreover, the closure of V_1 contains V whenever (E, V_1) is a solution. Thus, if (E, V) is a minimal solution, then the region V contains all the limit-points of the set of values of the approximants of the continued fraction whose elements c_p are in E, and is the smallest closed region having this property.

In the preceding chapters, we have found two solutions of the value region problem for the continued fraction (a), namely,

$$E: |z| - \Re(z) \le \tfrac{1}{2}, \quad V: |z-1| \le 1,$$

(Theorem 14.3); and

$$E: |z| \le \tfrac{1}{4}, \quad V: |z - \tfrac{4}{3}| \le \tfrac{2}{3},$$

(Theorem 10.1). Both these solutions (E, V) are minimal solutions.

In the present chapter we propose to treat the value region problem in some detail, with a view toward correlating a number of results, including the two cited above.

36. A Sufficient Condition. In the investigation of the value region problem, the following theorem is often useful [**87**].

THEOREM 36.1. *Let*

$$s = s(z; t) = \frac{1}{1 + zt}$$

be a linear transformation of the variable t into the variable s, the transformation depending upon the parameter z. Let E and V be two regions having the following two properties.

(a) *The region V contains the point* 1.

(b) *The transformation $s = s(z; t)$ transforms V into all or a part of itself for every value of the parameter z in E:*

$$s(z; V) \subset V, \quad \text{for all} \quad z \quad \text{in} \quad E.$$

Under these conditions, (E, V) is a solution of the value region problem for the continued fraction

$$\cfrac{1}{1 + \cfrac{c_1}{1 + \cfrac{c_2}{1 + \cdots}}} \tag{36.1}$$

Proof. By (a), the first approximant $A_1/B_1 = 1$ of (36.1) is in V. By (a), (b),

$$s(c_n; 1) = \frac{1}{1 + c_n} = u_n$$

is in V. Hence, by (b),

$$s(c_{n-1}; u_n) = \cfrac{1}{1 + \cfrac{c_{n-1}}{1 + c_n}} = u_{n-1}$$

is in V. Therefore, again by (b),

$$s(c_{n-2}; u_{n-1}) = \cfrac{1}{1 + \cfrac{c_{n-2}}{1 + \cfrac{c_{n-1}}{1 + c_n}}} = u_{n-2}$$

is in V. Continuing in this way we conclude, finally, that

$$s(c_1; u_2) = \cfrac{1}{1 + \cfrac{c_1}{1 + \cfrac{c_2}{1 + \cdots + \cfrac{c_n}{1}}}} = \frac{A_{n+1}}{B_{n+1}}$$

is in V, and the theorem is proved.

37. The Two-Circle Theorem. We now consider the problem of determining conditions upon the parameter z in order that the transformation $s = s(z; t)$ carry a given circular region H into all or a part of another given circular region K. We have the following theorem of Lane [**47**], which will be used later on to obtain solutions of the value region problem.

THEOREM 37.1. *Let H and K be arbitrary circular regions of the form*

$$H: |w - c| \leq p|c|, \quad K: |w - d| \leq q|d|, \quad c, d \neq 0, \quad p > 0, \quad q > 0,$$

in the complex w-plane. Let $s = s(z; w) = 1/(1 + zw)$ be a linear transformation of the variable w into the variable s, the transformation depending upon the parameter z. This transformation carries H into all or a part of K if, and only if,

$$p|z| \leq \begin{cases} \dfrac{q}{|cd|(1-q^2)} - \left| z + \dfrac{1}{c}\left(1 - \dfrac{1}{d(1-q^2)}\right)\right|, & \text{if } q < 1, \\[2ex] \dfrac{cdz + \overline{cdz} + d + \bar{d} - 1}{2|cd|}, & \text{if } q = 1, \\[2ex] \dfrac{q}{|cd|(1-q^2)} + \left| z + \dfrac{1}{c}\left(1 - \dfrac{1}{d(1-q^2)}\right)\right|, & \text{if } q > 1. \end{cases} \tag{37.1}$$

Remark. It may of course happen that there is no value of z for which (37.1) holds. It can be shown that the domain of z is nonvacuous if, and only if, one of the following three conditions holds:

(a) *K contains the point* 1;

(b) *K contains the origin and H does not contain the origin*;

(c) *Neither H nor K contains the origin and*

$$| d - 1 |^2 \leq q^2 | d |^2 + \frac{q^2(1 - p^2)}{p^2(1 - q^2)}.$$

Inasmuch as we do not make use of this criterion, we shall omit the proof (cf. [**47**]).

Proof of Theorem 37.1. One may verify directly that $z = 0$ satisfies (37.1) if, and only if, K contains the point 1, i.e., if, and only if, $s(0; H) = 1 \subset K$. We therefore need consider only values of z which are different from zero. It is required to show that such a value of z satisfies (37.1) if, and only if, $s(z; H) \subset K$, i.e., if, and only if,

$$| w - c | \leq p | c | \quad \text{implies} \quad \left| \frac{1}{1 + zw} - d \right| \leq q | d |. \tag{37.2}$$

We introduce a new variable ζ by means of the equation $\zeta = zw/c$ or $w = c\zeta/z$. Then (37.2) holds, if, and only if,

$$| \zeta - z | \leq p | z | \quad \text{implies} \quad \left| \zeta + \frac{1}{c}\left(1 - \frac{1}{d}\right) \right| \leq q \left| \zeta + \frac{1}{c} \right|. \tag{37.3}$$

Let A and B denote the ζ-regions (depending upon z), defined by the inequalities in (37.3). It is required to find a necessary and sufficient condition on z, $z \neq 0$, such that $A \subset B$. This condition is obviously as follows. *The center z of the circular region A lies within B at a distance at least $p| z |$ from the nearest point of the boundary of B.* There are three cases to be considered, according as $q < 1$, $q = 1$, or $q > 1$.

Case 1. Suppose that $q < 1$. In this case, B is the region

$$\left| \zeta + \frac{1}{c}\left(1 - \frac{1}{d(1 - q^2)}\right) \right| \leq \frac{q}{| cd |(1 - q^2)}.$$

Hence, it follows immediately from the above stated condition on the center z of A, that $A \subset B$ if, and only if, the first of the inequalities (37.1) holds.

Case 2. *Suppose that* $q = 1$. In this case, B is the region

$$cd\zeta + \overline{cd\zeta} + d + \bar{d} - 1 \geq 0.$$

We then conclude by the same reasoning which was used in Case 1 that $A \subset B$ if, and only if, the second inequality (37.1) holds.

Case 3. *Suppose that* $q > 1$. In this case, B is the region

$$\left| \zeta + \frac{1}{c}\left(1 - \frac{1}{d(1 - q^2)}\right) \right| \geq \frac{q}{|cd|(q^2 - 1)}.$$

Hence, as before, $A \subset B$ if, and only if, the last inequality (37.1) holds.

38. Circular Element Regions with Centers at the Origin. Continued fractions of the form (36.1) whose partial numerators have their values in the neighborhood of the origin are of considerable importance in analysis. (Cf., for instance, some of the examples in Chapter XVIII.) The following theorem furnishes estimates for the values of such continued fractions [**68**].

THEOREM 38.1. *A minimal solution of the value region problem for the continued fraction* (36.1) *is given by*

$$E: |z| \leq r(1 - r), \quad V: \left| z - \frac{1}{1 - r^2} \right| \leq \frac{r}{1 - r^2}, \quad 0 < r \leq \tfrac{1}{2}. \quad (38.1)$$

Proof. In the two-circle theorem (Theorem 37.1), take

$$c = d = \frac{1}{1 - r^2}, \quad p = q = r.$$

Since $0 < q < 1$, (37.1) gives for z the condition $|z| \leq r(1 - r)$; and the regions H and K of Theorem 37.1 are one and the same region, namely, the region V of (38.1); and since V contains the point 1, and since $s(z; V) \subset V$ for every z in E, it therefore follows by Theorem 36.1 that (E, V) is a solution of the value region problem for the continued fraction (36.1).

To prove that this is a minimal solution, consider the continued fraction

$$\cfrac{1}{1+\cfrac{c_1}{1+\cfrac{r(r-1)}{1+\cfrac{r(r-1)}{1+\cdots}}}} \tag{38.2}$$

If $|c_1| \leq r(1-r)$, this continued fraction converges by Theorem 10.1. Since it is periodic from and after the third partial quotient, one may readily find that its value is

$$v = \frac{1-r}{1-r+c_1}.$$

As c_1 ranges over the region E of (38.1), v ranges over V.

This completes the proof of Theorem 38.1.

Since the continued fraction (36.1) converges if the c_p are in the region E of (38.1), and since E contains 0, we may speak of the values of the continued fraction, and these values include all the values of the approximants. We shall now prove that the values of the continued fraction are, in general, on the **interior** of the region V. In fact, we have this theorem.

THEOREM 38.2. *The only continued fraction* (36.1), *with partial numerators c_p in the domain E of* (38.1), *which takes on values upon the boundary of the domain V of* (38.1), *is the continued fraction* (38.2) *with* $|c_1| = r(1-r)$. [68.]

Proof. We have seen that the continued fraction (38.2) takes on every value v on the boundary of V, as c_1 ranges over the boundary of E. We write such a value in the form

$$v = \frac{1+re^{i\phi}}{1-r^2}, \quad 0 \leq \phi < 2\pi,$$

and consider any continued fraction (36.1) with partial numerators c_p in E, whose value is v. Such a continued fraction is equal to

$$v = \frac{1}{1 + c_1 w},$$

where w is the value of another continued fraction of the form (36.1) with elements in E. We then have

$$c_1 w = \frac{1 - v}{v} = \frac{-r[2r + (1 + r^2)\cos\phi + i(1 - r^2)\sin\phi]}{1 + 2r\cos\phi + r^2}, \qquad (38.3)$$

so that

$$|c_1 w| = r, \quad |c_1| = \frac{r}{|w|} = r(1 - r) \cdot \frac{1}{(1 - r)|w|}.$$

Inasmuch as w is in V, we have

$$|w| \leq \frac{1}{1 - r}, \quad \frac{1}{(1 - r)|w|} \geq 1,$$

and consequently $|c_1| \geq r(1 - r)$. On the other hand, $|c_1| \leq r(1 - r)$ since c_1 is in E. Therefore,

$$|c_1| = r(1 - r), \quad |w| = \frac{1}{1 - r}, \quad w = \frac{1}{1 - r}.$$

On putting this value of w in (38.3), we find that the value of c_1 is thereby uniquely determined.

Starting now with $w = 1/(1 - r)$ as the value upon the boundary of V to be attained, we find in the same way that c_2 must be given by the expression in the right-hand member of (38.3), multiplied by $1 - r$, but with ϕ now set equal to zero. This gives $c_2 = r(r - 1)$; and on repeating this argument we find that $c_3, c_4, c_5, \cdots$ all have this same value. On referring to (38.2) we now see that the proof of the theorem is complete.

39. A Family of Parabolic Element Regions. We turn now to the case where the element region E is the region bounded by a parabola with focus at the origin,

$$|z| - \Re(ze^{i\phi}) \leq 2\cos^2\frac{\phi}{2}\, h, \qquad (39.1)$$

where $-\pi < \phi < +\pi$, $0 < h \leq \frac{1}{4}$. On putting $\phi_p = \phi/2$, $\delta = 0$, $(1 - g_{p-1})g_p = h$ (cf. Theorem 19.1), in (31.5), we see by

Theorem 31.1 that a corresponding value region V is the region

$$\left| z - \frac{e^{i\phi/2} \sec(\phi/2)}{1+\sqrt{1-4h}} \right| \leq \frac{\sec(\phi/2)}{1+\sqrt{1-4h}}. \tag{39.2}$$

We have taken $g_0 = (1+\sqrt{1-4h})/2$, in accordance with (19.13). When $\phi = 0$, $h = \frac{1}{4}$, we know that this is a minimal solution of the value region problem (cf. Theorem 14.3).

In the present section we shall improve the above result in the case where $h < \frac{1}{4}$.

If we put $2\cos^2(\phi/2)\cdot h = t/2$ in (39.1), and write $a = e^{i\phi}$, then the inequality defining the region E takes the form

$$\begin{gathered} E\colon az + \overline{az} + t \geq 2|z|, \quad a = e^{i\phi}, \\ -\pi < \phi < \pi, \quad 0 < t \leq \cos^2\frac{\phi}{2}. \end{gathered} \tag{39.3}$$

To obtain a corresponding value region, we start with two circular regions of the form

$$H\colon |z-c| \leq |c|, \quad K\colon |z-d| \leq |d|, \quad c \neq 0, \quad d \neq 0,$$

and apply the two-circle theorem. We find at once that $s(z; H) \subset K$ if, and only if,

$$cdz + \overline{cdz} + d + \bar{d} - 1 \geq 2|cdz|; \tag{39.4}$$

and that $s(z; K) \subset H$ if, and only if,

$$cdz + \overline{cdz} + c + \bar{c} - 1 \geq 2|cdz|. \tag{39.5}$$

The domain E of (39.3) will be contained in the intersection of the domains (39.4) and (39.5) if, and only if,

$$\frac{cd}{|cd|} = a, \quad \frac{d+\bar{d}-1}{|cd|} \geq t, \quad \frac{c+\bar{c}-1}{|cd|} \geq t. \tag{39.6}$$

If c and d satisfy these conditions, we shall say that (c, d) is a *solution* of (39.6). We shall write $H = H(c)$ and $K = K(d)$.

In (39.4) and (39.5), let c, d be such that (c, d) is a solution of (39.6). If z is any point of E, so that z is in both (39.4) and (39.5), then $s(z; H(c)) \subset K(d)$ and $s(z; K(d)) \subset H(c)$. Therefore, $s(z; V_{cd}) \subset V_{cd}$, where V_{cd} is the intersection of $H(c)$ and $K(d)$.

From (39.6) it readily follows that V_{cd} contains the point 1. By Theorem 36.1 we therefore conclude that (E, V_{cd}) is a solution of the value region problem for the continued fraction (36.1). Let V denote the intersection of all the regions V_{cd} as (c, d) ranges over all solutions of (39.6). Then $s(z; V) \subset V$ for all z in E, and V contains the point 1, so that (E, V) is a solution of the value region problem.

We shall now prove the following theorem, which will facilitate the construction of the region V by successive approximation.

THEOREM 39.1. *Let W denote the set of all points c such that for some $k > 0$,*

$$\left| c - \frac{1}{kt} \right| \leq \frac{\sqrt{1 - kt}}{kt}, \quad \left| c - \frac{a}{t} \right| \leq \frac{\sqrt{1 - (t/k)}}{t}. \tag{39.7}$$

Then the value region V just defined, corresponding to the element region E of (39.3), *is the intersection of all the regions $|z - c| \leq |c|$ as c ranges over W* [47].

Proof. The totality of solutions (c, d) of the system (39.6) is identical with the totality of pairs $(c, ka\bar{c})$, where c and $k > 0$ satisfy (39.7). In fact, if (c, d) is a solution of (39.6), then from the last of those relations we get

$$\frac{c + \bar{c} - 1}{c\bar{c}} \geq kt, \quad \text{where} \quad k = \left| \frac{d}{c} \right|.$$

Also, since $a|cd| = cd$, we have

$$\bar{a}c + a\bar{c} - tc\bar{c} = \frac{\bar{c}c\bar{d}}{|cd|} + \frac{c\bar{c}d}{|cd|} - tc\bar{c}$$

$$= \left| \frac{c}{d} \right| [(d + \bar{d} - 1) + 1 - t|cd|] \geq \left| \frac{c}{d} \right|.$$

Consequently,

$$\frac{c + \bar{c} - 1}{c\bar{c}} \geq kt \geq \frac{t}{\bar{a}c + a\bar{c} - tc\bar{c}} > 0, \tag{39.8}$$

which is equivalent to (39.7). Moreover,

$$d = \left| \frac{d}{c} \right| \cdot a\bar{c} = ka\bar{c}.$$

Conversely, let c and k satisfy (39.7), i.e., (39.8). Then one may readily verify that c and $d = ka\bar{c}$ satisfy (39.6).

If (c, d) satisfies (39.6), then c is in W and, by symmetry, d is in W. Let V_1 denote the intersection of all the circular regions $|z - c| \leq |c|$ for c in W. We are to show that $V_1 = V$, where V is the region defined above as the intersection of all the regions V_{cd}. If z is in V, then z is in every circle $|z - c| = |c|$ for c in W, so that $V \subset V_1$; and if z is in V_1, then z is in every region V_{cd}, so that $V_1 \subset V$. Hence, $V = V_1$, and the proof of the theorem is complete.

Since E is nonvacuous, it is clear that V is nonvacuous. Let us consider the range of $k > 0$ such that (39.7) has a solution c. A necessary and sufficient condition for (39.7) to have a solution c is that

$$\left|\frac{1}{kt} - \frac{a}{t}\right| \leq \frac{\sqrt{1 - kt}}{kt} + \frac{\sqrt{1 - (t/k)}}{t}, \quad t \leq k \leq \frac{1}{t},$$

i.e., that

$$k(t - \cos\phi) \leq \sqrt{k(1 - kt)(k - t)}, \quad t \leq k \leq \frac{1}{t}, \quad (a = e^{i\phi}).$$

Therefore, (39.7) has a solution c if, and only if,

$$0 \leq t \leq \cos\phi, \quad t \leq k \leq \frac{1}{t}, \tag{39.9}$$

or else,

$$t > \cos\phi, \quad \frac{1}{k} + k \leq 2\cos\phi + \frac{\sin^2\phi}{t}, \quad t \leq k \leq \frac{1}{t}. \tag{39.10}$$

The minimum of $(1/k) + k$ for $k > 0$ is 2. Then $2\cos\phi + (\sin^2\phi)/t \geq 2$ if, and only if, $t \leq \cos^2(\phi/2)$, which is the condition imposed upon t in (39.3). Since $\cos\phi \leq \cos^2(\phi/2)$, this condition upon t is necessary also under (39.9). We have proved

THEOREM 39.2. *The values of k for which the simultaneous inequalities* (39.7) *have solutions c are given by* (39.9) *and* (39.10). *In particular, they have a solution c if $k = 1$.*

If $t = \cos^2(\phi/2)$, then (39.7) has a solution *only* when $k = 1$, whereupon c is uniquely determined, and W has but one point.

If $t < \cos^2(\phi/2)$, then W contains more than one point. *Therefore, the boundary of V is a circle if, and only if,* $t = \cos^2(\phi/2)$. We find that the equation of this circle is

$$\left| z - e^{i\phi/2}\sec\left(\frac{\phi}{2}\right) \right| \leq \sec\left(\frac{\phi}{2}\right),$$

which is the same as the region (39.2), for $h = \frac{1}{4}$.

If $\phi = 0$, so that $0 < t \leq 1$, $k = 1$, the boundary of V may be obtained as follows. The inequalities (39.7) now coincide, so that V is the set of all points w such that

$$| w - c | \leq | c | \tag{39.11}$$

for all c in the circular region

$$\left| c - \frac{1}{t} \right| \leq \frac{\sqrt{1-t}}{t}. \tag{39.12}$$

If w is any fixed value such that (39.11) holds, then c lies in the half of the u-plane defined by $\bar{w}u + w\bar{u} - w\bar{w} \geq 0$. If this is to hold for all c satisfying (39.12), then $1/t$ must lie in this half-plane, at a distance at least $(1/t)\sqrt{1-t}$ from the boundary line, i.e.,

$$\frac{\bar{w}t^{-1} + wt^{-1} - w\bar{w}}{2|w|} \geq \frac{\sqrt{1-t}}{t}.$$

Conversely, if w satisfies this inequality, then w is in V. Thus the region V is the region bounded by the inner loop of the limaçon

$$z + \bar{z} - tz\bar{z} = 2\sqrt{1-t}\,|z|. \tag{39.13}$$

One can readily show that this solution of the value region problem is *minimal* [**55**]. In fact, as z ranges over the parabola $z + \bar{z} + t \geq t|z|$ bounding E, the value w of the continued fraction of Exercise 2.1, which satisfies the equation $w\bar{w}z + w - 1 = 0$, ranges over the boundary of V, namely, the inner loop of the limaçon (39.13).

Exercise 8

8.1. Show that a solution of the value region problem is given by

$$E: r|z| \leq \frac{r}{1-r^2} - \left| z - \frac{r^2}{1-r^2} \right|, \quad V: |z-1| \leq r,$$

where $0 < r < 1$. The boundary of E is a natural generalization of a conic section. It is the locus of the point P whose distance from the circle

$$\left| z - \frac{r^2}{1-r^2} \right| = \frac{r}{1-r^2}, \quad \text{(directrix)},$$

is r (eccentricity) times its distance from the origin (focus) [**47**].

8.2. If $1 \leq c \leq 2$, then

$$E: |z| - \Re(z) \leq \frac{2-c}{2c}, \quad V: \left| z - \frac{1}{c} \right| \leq \frac{1}{c}$$

is a solution of the value region problem.

8.3. If a value region V is a region whose boundary is a circle C passing through the origin and containing 1 on the interior, if E is a corresponding element region, and if there is a continued fraction (36.1) with its elements in E whose value is on C, then there is only one such continued fraction [**68**].

Part II

FUNCTION THEORY

Chapter IX

J-FRACTION EXPANSIONS FOR RATIONAL FUNCTIONS

In this chapter we consider the problem of expanding a rational function of a variable z into a J-fraction, with particular emphasis upon the development of an algorithm suited to numerical computation. These expansions have applications in the theory of equations (cf. Chapter X), and are useful in certain problems in physics, for instance, in the theory of electrical networks.

40. The Expansion Algorithm. For the purposes of the present chapter, it will be convenient to write the J-fraction in the form

$$\cfrac{1}{r_1 z + s_1 + \cfrac{1}{r_2 z + s_2 + \cfrac{1}{r_3 z + s_3 + \cdots}}} \tag{40.1}$$

The r_p and s_p are complex numbers of which $r_p \neq 0$, $(p = 1, 2, 3, \cdots)$. The pth approximant is a rational fraction whose numerator and denominator are of degrees $p - 1$ and p, respectively, whose coefficients are certain polynomials in the r_n and s_n.

Let

$$\begin{aligned} f_0 &= a_{00}z^n + a_{01}z^{n-1} + \cdots + a_{0n}, \\ f_1 &= a_{11}z^{n-1} + a_{12}z^{n-2} + \cdots + a_{1n}, \end{aligned} \tag{40.2}$$

be two polynomials of degree n and $n - 1$, respectively. The problem of determining numbers $r_p \neq 0$ and s_p, $p = 1, 2, 3, \cdots, n$, such that

$$\frac{f_1}{f_0} = \cfrac{1}{r_1 z + s_1 + \cfrac{1}{r_2 z + s_2 + \cdot \, _{\cdot} \, _{\cdot} + \cfrac{1}{r_n z + s_n}}} \tag{40.3}$$

is equivalent to the problem of determining polynomials f_p of degree $n - p$, $p = 2, 3, 4, \cdots, n - 1$, which are connected with f_0 and f_1 by the recurrence relations

$$f_{p-1} = (r_p z + s_p) f_p + f_{p+1}, \quad p = 1, 2, 3, \cdots, n, \tag{40.4}$$

where $f_{n+1} = 0$ and f_n is a constant different from zero. For, if (40.4) holds, then

$$\frac{f_p}{f_{p-1}} = \cfrac{1}{r_p z + s_p + \cfrac{f_{p+1}}{f_p}}, \quad p = 1, 2, 3, \cdots, n,$$

so that (40.3) holds; and if (40.3) holds, we may compute the f_p step by step by means of (40.4), starting with $p = n, f_{n+1} = 0$, f_n equal to a constant not zero, $f_n = c$. This determines the f_p up to a constant factor not zero. This factor may be made equal to unity by suitably adjusting the value of c.

It is easy to show that when the expansion (40.3) exists, then the r_p and s_p are unique. In fact, let (40.4) hold with $r_p \neq 0$, and suppose that $f_0 = f_0', f_1 = f_1'$,

$$f_{p-1}' = (r_p' z + s_p') f_p' + f_{p+1}', \quad p = 1, 2, 3, \cdots,$$

where $f_{n+1}' = 0$ and f_n' is a constant not zero. Suppose, moreover, that f_p' is of degree $n - p$. Then, if $p = 1$ we get

$$0 = f_0 - f_0' = [(r_1 - r_1')z + (s_1 - s_1')]f_1 + (f_2 - f_2').$$

Since the degree of $f_2 - f_2'$ is at most $n - 2$, and since f_1 is of degree $n - 1$, it follows that $r_1 = r_1'$, $s_1 = s_1'$, $f_2 = f_2'$. On taking $p = 2$, we then find in the same way that $r_2 = r_2'$, $s_2 = s_2'$, $f_3 = f_3'$, and so on.

We shall now obtain recurrence formulas for computing the constants r_p and s_p. The division process involved in forming the first of the equations (40.4) may be expressed by means of the following equations.

$$\frac{a_{00}z^n + a_{01}z^{n-1} + \cdots + a_{0n}}{a_{11}z^{n-1} + a_{12}z^{n-2} + \cdots + a_{1n}} = r_1 z + \frac{b_{11}z^{n-1} + b_{12}z^{n-2} + \cdots + b_{1n}}{a_{11}z^{n-1} + a_{12}z^{n-2} + \cdots + a_{1n}},$$

$$r_1 = \frac{a_{00}}{a_{11}}, \quad b_{11} = a_{01} - r_1 a_{12},$$

$$b_{12} = a_{02} - r_1 a_{13}, \quad \cdots, \quad b_{1n} = a_{0n}, \tag{40.5}$$

$$\frac{b_{11}z^{n-1} + b_{12}z^{n-2} + \cdots + b_{1n}}{a_{11}z^{n-1} + a_{12}z^{n-2} + \cdots + a_{1n}} = s_1 + \frac{a_{22}z^{n-2} + a_{23}z^{n-3} + \cdots + a_{2n}}{a_{11}z^{n-1} + a_{12}z^{n-2} + \cdots + a_{1n}},$$

$$s_1 = \frac{b_{11}}{a_{11}}, \quad a_{22} = b_{12} - s_1 a_{12},$$

$$a_{23} = b_{13} - s_1 a_{13}, \quad \cdots, \quad a_{2n} = b_{2n} - s_1 a_{1n}.$$

These equations determine r_1, s_1 and $f_2 = a_{22}z^{n-2} + a_{23}z^{n-3} + \cdots + a_{2n}$. The equations for determining r_2, s_2 and f_3 may be obtained from the preceding by advancing all subscripts by unity and decreasing all exponents by unity. Then r_3, s_3 and f_4 may may be obtained by repeating these operations, and so on.

We observe that the process can be continued as long as the numbers $a_{00}, a_{11}, a_{22}, \cdots$ are different from zero. *The expansion* (40.3) *exists if, and only if,* $a_{00}, a_{11}, \cdots, a_{nn}$ *are different from zero.*

The computation involved in (40.5) can be conveniently arranged in the following table.

$$\begin{array}{lllll}
 & a_{00}, & a_{01}, & a_{02}, & \cdots \\
 & a_{11}, & a_{12}, & a_{13}, & \cdots \\
r_1 = \dfrac{a_{00}}{a_{11}}, & b_{11} = a_{01} - r_1 a_{12}, & b_{12} = a_{02} - r_1 a_{13}, & b_{13} = a_{03} - r_1 a_{14}, & \cdots \\
s_1 = \dfrac{b_{11}}{a_{11}}, & a_{22} = b_{12} - s_1 a_{12}, & a_{23} = b_{13} - s_1 a_{13}, & a_{24} = b_{14} - s_1 a_{14}, & \cdots, \\
r_2 = \dfrac{a_{11}}{a_{22}}, & b_{22} = a_{12} - r_2 a_{23}, & b_{23} = a_{13} - r_2 a_{24}, & b_{24} = a_{14} - r_2 a_{25}, & \cdots, \\
s_2 = \dfrac{b_{22}}{a_{22}}, & a_{33} = b_{23} - s_2 a_{23}, & a_{34} = b_{24} - s_2 a_{24}, & a_{35} = b_{25} - s_2 a_{25}, & \cdots, \\
r_3 = \dfrac{a_{22}}{a_{33}}, & b_{33} = a_{23} - r_3 a_{34}, & b_{34} = a_{24} - r_3 a_{35}, & b_{35} = a_{25} - r_3 a_{36}, & \cdots, \\
 & & \cdot \quad \cdot \quad \cdot, & &
\end{array} \tag{40.6}$$

$$(a_{pq} = b_{pq} = 0 \quad \text{for} \quad q > n).$$

These are the recurrence formulas we wished to obtain. They are well-suited for the purpose of numerical computation.

Example. Let

$$f_0 = z^3 + (2+i)z^2 + (3+i)z + (2i+2),$$
$$f_1 = 2z^2 + iz + 2. \tag{40.7}$$

In this case, the table (40.6) is as follows.

$$\begin{array}{llllll}
 & 1, & 2+i, & 3+i, & 2i+2 \\
 & 2, & i, & 2, & \\
r_1 = \frac{1}{2}, & 2+\frac{i}{2}, & 2+i, & 2i+2, & \\
s_1 = 1+\frac{i}{4}, & \frac{9}{4}, & \frac{3i}{2}, & & \\
r_2 = \frac{8}{9}, & -\frac{i}{3}, & 2, & & \\
s_2 = -\frac{4i}{27}, & \frac{16}{9}, & & & \\
r_3 = \frac{81}{64}, & \frac{3i}{2} & & & \\
s_3 = \frac{27i}{32}. & & & &
\end{array}$$

Then, we have

$$\frac{f_1}{f_0} = \cfrac{1}{\frac{1}{2}z + 1 + \frac{i}{4} + \cfrac{1}{\frac{8}{9}z - \frac{4i}{27} + \cfrac{1}{\frac{81}{64}z + \frac{27i}{32}}}}. \tag{40.8}$$

41. Conditions Involving Determinants. We shall now formulate the condition for the J-fraction expansion (40.3) to hold, in terms of certain determinants involving the coefficients of the polynomials f_0 and f_1.

THEOREM 41.1. *The quotient* f_1/f_0 *of two polynomials*

$$f_1 = a_{11}z^{n-1} + a_{12}z^{n-2} + \cdots + a_{1n},$$

$$f_0 = a_{00}z^n + a_{01}z^{n-1} + \cdots + a_{0n}$$

has a J-fraction expansion (40.3) *in which* $r_p \neq 0,\ p = 1, 2, 3, \cdots, n,$ *if, and only if,*

$$D_p \neq 0, \quad p = 0, 1, 2, \cdots, \quad n, \tag{41.1}$$

where $D_0 = a_{00}$, *and* $D_1, D_2, \cdots, D_n$ *are the first, third,* $\cdots$, $(2n-1)$th *principal minors (blocked off with lines) in the array*

$$\begin{array}{ccccccccc}
a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15}, & a_{16}, & a_{17}, & a_{18}, & \cdots \\
a_{00}, & a_{01}, & a_{02}, & a_{03}, & a_{04}, & a_{05}, & a_{06}, & a_{07}, & \cdots \\
0, & a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15}, & a_{16}, & a_{17}, & \cdots \\
0, & a_{00}, & a_{01}, & a_{02}, & a_{03}, & a_{04}, & a_{05}, & a_{06}, & \cdots \\
0, & 0, & a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15}, & a_{16}, & \cdots \\
0, & 0, & a_{00}, & a_{01}, & a_{02}, & a_{03}, & a_{04}, & a_{05}, & \cdots \\
0, & 0, & 0, & a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15}, & \cdots \\
0, & 0, & 0, & a_{00}, & a_{01}, & a_{02}, & a_{03}, & a_{04}, & \cdots \\
 & & & & \cdots & & & &
\end{array} \tag{41.2}$$

$(a_{0p} = a_{1p} = 0 \quad \text{for} \quad p > n)$. [15.]

Proof. We suppose first that the expansion (40.3) exists, so that the numbers $a_{00}, a_{11}, \cdots, a_{nn}$ in (40.6) are different from zero. Thus $D_0 = a_{00} \neq 0$, $D_1 = a_{11} \neq 0$. Let us write $D_p = D_p(f_0, f_1)$, to indicate the dependence of the determinant upon the polynomials f_0 and f_1. We shall prove that

$$D_p(f_0, f_1) = (-1)^{p-1}a_{11}{}^2 D_{p-1}(f_1, f_2), \quad p = 2, 3, \cdots, \quad n. \tag{41.3}$$

In fact, on subtracting r_1 times the $(2k-1)$th row from the $2k$th row for $k = 1, 2, \cdots, p-1$ in $D_p(f_0, f_1)$, $2 \leq p \leq n$, and making use of (40.6), we obtain

$$D_p(f_0, f_1) = a_{11} \begin{vmatrix}
b_{11}, & b_{12}, & b_{13}, & \cdots \\
a_{11}, & a_{12}, & a_{13}, & \cdots \\
0, & b_{11}, & b_{12}, & \cdots \\
0, & a_{11}, & a_{12}, & \cdots \\
0, & 0, & b_{11}, & \cdots \\
 & \cdot & \cdot & \cdot
\end{vmatrix},$$

where the new determinant is of order $2p - 2$. If in the latter we subtract s_1 times the $2k$th row from the $(2k - 1)$th row for $k = 1, 2, 3, \cdots, p - 1$, and again make use of (40.6), we readily obtain (41.3). For $p = 2$, (41.3) gives $D_2(f_0, f_1) = -a_{11}^2 a_{22}$. On applying the formula for $p = 2$ to the polynomials f_1 and f_2 we get $D_2(f_1, f_2) = -a_{22}^2 D_1(f_2, f_3) = -a_{22}^2 a_{33}$, and consequently $D_3(f_0, f_1) = a_{11}^2 D_2(f_1, f_2) = -a_{11}^2 a_{22}^2 a_{33}$. In this way we obtain the formula

$$D_p = (-1)^{\frac{p(p-1)}{2}} a_{11}^2 a_{22}^2 \cdots a_{p-1,p-1}^2 a_{pp}, \qquad p = 2, 3, \cdots, n. \tag{41.4}$$

Therefore, if the expansion (40.3) exists, then $D_p \neq 0$, $p = 0, 1, \cdots, n$.

If, conversely, these determinants are different from zero, then a_{00}, a_{11} are different from zero since, by definition, $D_0 = a_{00}$, $D_1 = a_{11}$. Since $a_{11} \neq 0$, then (41.4) holds for $p = 2$, so that $D_2 = -a_{11}^2 a_{22} \neq 0$, or $a_{22} \neq 0$. This guarantees that (41.4) holds for $p = 3$, so that $D_3 = -a_{11}^2 a_{22}^2 a_{33} \neq 0$, or $a_{33} \neq 0$. On continuing this argument, we finally arrive at $a_{nn} \neq 0$, and therefore the expansion (40.3) exists when $D_p \neq 0$, $p = 0, 1, \cdots, n$.

This completes the proof of Theorem 41.1.

42. Relationship Between the J-fraction and the Power Series for f_1/f_0. We now make an equivalence transformation in (40.3), and put

$$a_0 = \frac{1}{r_1}, \quad a_p = \frac{-1}{r_p r_{p+1}}, \quad p = 1, 2, 3, \cdots, n - 1, \tag{42.1}$$

$$b_p = \frac{s_p}{r_p}, \quad p = 1, 2, 3, \cdots, n. \tag{42.2}$$

The J-fraction then takes the form

$$\frac{f_1}{f_0} = \cfrac{a_0}{b_1 + z - \cfrac{a_1}{b_2 + z - \cdot \;\; \cdot - \cfrac{a_{n-1}}{b_n + z}}}. \tag{42.3}$$

By (40.6), we have

$$a_0 = \frac{a_{11}}{a_{00}}, \quad a_p = \frac{-a_{p+1,p+1}}{a_{p-1,p-1}}, \quad p = 1, 2, 3, \cdots, n-1, \tag{42.4}$$

and

$$b_p = \frac{b_{p,p}}{a_{p-1,p-1}}, \quad p = 1, 2, 3, \cdots, n. \tag{42.5}$$

If we assume that $a_{00} = 1$, which we may do with no loss in generality, then we obtain from (41.4) and (42.4) the formulas

$$D_{p+1} = a_0 a_1 \cdots a_p D_p, \quad p = 0, 1, 2, \cdots, n-1, \tag{42.6}$$

and

$$a_0 = \frac{D_1}{D_0}, \quad a_p = \frac{D_{p-1}D_{p+1}}{D_p^2}, \quad p = 1, 2, 3, \cdots, n-1. \tag{42.7}$$

We now call attention to a relationship which exists between the pth approximant of the J-fraction (42.3) and the expansion in descending powers of z of the rational function f_1/f_0. Let

$$\frac{f_1}{f_0} = \sum_{p=0}^{\infty} \frac{c_p}{z^{p+1}}. \tag{42.8}$$

From the determinant formula (1.5) it follows that

$$\frac{A_{p+1}(z)}{B_{p+1}(z)} - \frac{A_p(z)}{B_p(z)} = \frac{a_0 a_1 \cdots a_p}{B_{p+1}(z)B_p(z)}, \quad p = 0, 1, \cdots, n-1. \tag{42.9}$$

Inasmuch as $B_{p+1}(z)B_p(z)$ is a polynomial in z of degree $2p + 1$, this shows that the power series expansion in descending powers of z of the difference in the left-hand member begins with the term in $1/z^{2n+1}$. In other words,

$$\frac{A_p(z)}{B_p(z)} = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_{2p-1}}{z^{2p}} + \frac{c_{2p}{}^{(p)}}{z^{2p+1}} + \frac{c_{2p+1}{}^{(p)}}{z^{2p+2}} + \cdots, \tag{42.10}$$

$p = 1, 2, 3, \cdots, n-1$. Thus, the pth approximant $(p < n)$ *approximates* f_1/f_0 in the sense that its expansion in descending powers of z agrees term by term with the expansion in descending powers of z of f_1/f_0 for just the first $2p$ terms.

43. Rational Fractions with Simple Poles and Positive Residues. We shall now prove the following theorem.

THEOREM 43.1. *A necessary and sufficient condition for a rational fraction to have the form*

$$\frac{f_1}{f_0} = \sum_{p=1}^{n} \frac{L_p}{z - x_p}, \tag{43.1}$$

where the x_p are real and distinct and the L_p positive, is that f_1/f_0 have a J-fraction expansion (42.3) *in which the b_p are real and the a_p are positive* [23].

Proof. The sufficiency of the condition follows from the remarks in § 27. (Cf. (27.3).)

To prove the necessity, let us suppose that (43.1) holds, and write f_1/f_0 in the form (42.8). The coefficients c_p in that power series expansion are evidently given by

$$c_p = \sum_{k=1}^{n} x_k{}^p L_k, \quad p = 0, 1, 2, \cdots.$$

Inasmuch as the x_k are distinct, a polynomial in u of degree $m < n$, not identically zero, say, $X_0 + X_1 u + \cdots + X_m u^m$, cannot vanish for all the values $u = x_k$, $k = 1, 2, 3, \cdots, n$. Therefore, the quadratic form

$$\sum_{p,q=0}^{m} c_{p+q} X_p X_q = \sum_{u=x_1}^{u=x_n} (X_0 + X_1 u + \cdots + X_m u^m)^2 L(u),$$

$$L(x_k) = L_k,$$

is positive definite for $m < n$, and consequently, by Theorem 20.3, the determinants

$$\Delta_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix} > 0 \quad \text{for} \quad p = 0, 1, \cdots, n-1. \tag{43.2}$$

We consider now the rational fraction

$$\frac{c_0 z^{2n} + c_1 z^{2n-1} + \cdots + c_{2n-1} z + t}{z^{2n+1}} = \sum_{p=0}^{2n-1} \frac{c_p}{z^{p+1}} + \frac{t}{z^{2n+1}}, \tag{43.3}$$

where t is a constant to be determined. In this case, the determinants D_p of Theorem 41.1 are

$$D_0 = 1, \quad D_1 = c_0 = \Delta_0, \quad D_2 = \Delta_1, \quad \cdots, \quad D_n = \Delta_{n-1},$$

$$D_{n+1} = D_{n+1}(t) = t\Delta_{n-1} + D_{n+1}(0).$$

By (43.2), $D_p > 0$, $p = 0, 1, 2, \cdots, n$, and, if t is a sufficiently large positive number, then $D_{n+1} > 0$. We may therefore construct the first $n + 1$ partial quotients of the J-fraction for the rational function (43.3), namely,

$$\cfrac{a_0}{b_1 + z - \cfrac{a_1}{b_2 + z - \cdot_{\displaystyle\cdot}\cdot - \cfrac{a_{n-1}}{b_n + z - \cfrac{a_n}{b_{n+1} + z}}}},$$

where the b_p are real and, by (42.7), the a_p are positive.

By (42.10) we have, for the nth approximant of this continued fraction,

$$\frac{A_n(z)}{B_n(z)} = \sum_{p=0}^{2n-1} \frac{c_p}{z^{p+1}} + \frac{c_{2n}^{(n)}}{z^{2n+1}} + \cdots,$$

so that

$$f_1(z)B_n(z) - A_n(z)f_0(z) = \left(\frac{1}{z}\right),$$

where $(1/z)$ denotes a power series in descending powers of z beginning with the term in $1/z$. Inasmuch as the left-hand member of this identity contains no negative powers of z, it follows that $f_1(z)B_n(z) - A_n(z)f_0(z) \equiv 0$, and therefore

$$\frac{A_n(z)}{B_n(z)} \equiv \frac{f_1}{f_0}, \tag{43.4}$$

i.e., f_1/f_0 has a J-fraction expansion (42.3) in which the b_p are real and the a_p are positive.

This completes the proof of Theorem 43.1.

44. Expansion of Rational Functions into Stieltjes Type Continued Fractions. If we replace z by $-z$ in (40.3) and make an equivalence transformation, we obtain

$$\frac{f_1(-z)}{f_0(-z)} = \cfrac{-1}{r_1 z - s_1 + \cfrac{1}{r_2 z - s_2 + \cdots + \cfrac{1}{r_n z - s_n}}}.$$

Consequently, if $f_1(z)/f_0(z)$ is an odd function of z, it follows from the uniqueness of the J-fraction expansion that the s_p are all equal to zero. Conversely, if the s_p are all equal to zero, then $f_1(z)/f_0(z)$ is evidently an odd function of z.

In view of this remark, if the function $zf_1(z^2)/f_0(z^2)$ has a J-fraction expansion, then, since this is an odd function of z, the s_p must all be equal to zero in that expansion. Therefore, if we change the notation and write k_p for r_p,

$$\frac{zf_1(z^2)}{f_0(z^2)} = \cfrac{1}{k_1 z + \cfrac{1}{k_2 z + \cdots + \cfrac{1}{k_m z}}}, \tag{44.1}$$

where $m = 2n - 1$ or $2n$ according as $f_0(z)$, supposed of degree n, does or does not vanish at $z = 0$, respectively. On making an equivalence transformation and then replacing z^2 by z, after removing a factor z, we then obtain

$$\frac{f_1(z)}{f_0(z)} = \cfrac{1}{k_1 z + \cfrac{1}{k_2 + \cfrac{1}{k_3 z + \cdots + \cfrac{1}{T}}}}, \qquad T = \begin{cases} k_{2n-1}z & \text{if } f_0(0) = 0, \\ k_{2n} & \text{if } f_0(0) \neq 0. \end{cases} \tag{44.2}$$

If, conversely, the expansion (44.2) exists, then the expansion (44.1) exists and the k_p are determined uniquely by the polynomials f_0 and f_1.

To compute the continued fraction (44.2), it is only necessary to insert a 0 between every two successive numbers in the first two rows of the table (40.6). Every second entry in all the rows of the table will then be equal to 0. If we change the notation, the table may then be written more compactly as follows.

$$
\begin{array}{lllll}
 & a_{00}, & a_{01}, & a_{02}, & \cdots \\
 & a_{11}, & a_{12}, & a_{13}, & \cdots \\
k_1 = \dfrac{a_{00}}{a_{11}}, & a_{22} = a_{01} - k_1 a_{12}, & a_{23} = a_{02} - k_1 a_{13}, & a_{24} = a_{03} - k_1 a_{14}, & \cdots \\
k_2 = \dfrac{a_{11}}{a_{22}}, & a_{33} = a_{12} - k_2 a_{23}, & a_{34} = a_{13} - k_2 a_{24}, & a_{35} = a_{14} - k_2 a_{25}, & \cdots, \\
 & & \cdot \quad \cdot \quad \cdot & &
\end{array}
\tag{44.3}
$$

By Theorem 41.1, the continued fraction expansion (44.2) exists if, and only if, the determinants of that theorem formed for the fraction $zf_1(z^2)/f_0(z^2)$ are different from zero. We find that these determinants, which we shall denote by D_p', can be factored into products of the determinants $D_p = D_p(f_0, f_1)$ times other determinants W_q. The latter are the principal minors of *even* order in the array (41.2). Thus,

$$D_0' = a_{00} = D_0 W_0, \quad W_0 = 1,$$

$$D_1' = a_{11} = D_1 W_0,$$

$$D_2' = -D_1 W_1,$$

$$D_3' = -D_2 W_1,$$

$$D_4' = D_2 W_2, \cdots,$$

with the general formulas

$$D_{2p}' = (-1)^p D_p W_p, \quad D_{2p+1}' = (-1)^p D_{p+1} W_p. \tag{44.4}$$

Hence, we have the following theorem.

THEOREM 44.1. *The quotient f_1/f_0 of two polynomials* (40.2) *can be expanded into a continued fraction of the form* (44.2) *in which the k_p are different from zero if, and only if, the condition* (41.1) *holds and, in addition, the principal minors of the array* (41 2) *of even order are different from zero up to and including the one of order $2n - 2$ or $2n$ according as $f_0(0) = 0$ or $f_0(0) \neq 0$, respectively.*

If we put

$$a_0 = \frac{1}{k_1}, \quad a_p = \frac{-1}{k_p k_{p+1}}, \quad p = 1, 2, 3, \cdots, \tag{44.5}$$

then the continued fraction (44.2) takes the form

$$\cfrac{a_0}{z - \cfrac{a_1}{1 - \cfrac{a_2}{z - \cfrac{a_3}{1 - \cdot_{\cdot_{\cdot}}}}}} \tag{44.6}$$

By (42.7) and (44.4), we then obtain the formulas

$$a_0 = \frac{D_1}{D_0}, \quad a_{2p-1} = -\frac{D_{p-1}W_p}{D_p W_{p-1}}, \quad a_{2p} = -\frac{W_{p-1}D_{p+1}}{D_p W_p}. \tag{44.7}$$

Exercise 9

9.1. Find the J-fraction expansion for f_1/f_0 where

$$f_0 = z^4 - (\tfrac{9}{4} + t)z^3 + z^2 - \tfrac{9}{4}z + \tfrac{5}{2},$$
$$f_1 = -(\tfrac{9}{4} + t)z^3 - \tfrac{9}{4}z.$$

9.2. Find the J-fraction expansion for $P'(z)/P(z)$, where

$$P(z) = 64z^3 + 80z^2 + 24z + 1,$$

and $P'(z) = dP(z)/dz$.

9.3. The roots of $P(z)$ are all real if, and only if, $P'(z)/P(z)$ has a J-fraction expansion (42.3) in which the b_p are real and the a_p are positive.

Chapter X

THEORY OF EQUATIONS

One of the main problems in the theory of equations is the location of the roots of polynomials. There are two main aspects of this problem. One is the problem in the large, of determining regions in which the roots lie, and of determining the number of roots in a given region. The other is the problem in the small, of computing the roots to any desired degree of accuracy. For some purposes, one needs to know only information on the problem in the large. The solution of the problem in the small depends in general upon a preliminary investigation of the problem in the large.

Continued fractions have proved to be useful in both aspects of this problem of the theory of equations. Given a polynomial $P(z)$ of degree n, we seek to find another polynomial $N(z)$ of degree $n - 1$ such that properties of $P(z)$ can be determined from the J-fraction expansion of the quotient $N(z)/P(z)$. For example, if we take $N(z) = dP(z)/dz$, then the roots of $P(z)$ are all real if, and only if, the J-fraction for $N(z)/P(z)$ is of a certain kind (cf. Exercise 9.3). There is one choice for $N(z)$ which is particularly convenient, namely, we take $N(z)$ to be a certain polynomial $Q(z)$, called the **alternant** of $P(z)$ (cf. § 45 following). The J-fraction for $Q(z)/P(z)$ is called the **test-fraction** for $P(z)$. By means of this test-fraction, we are able to find polygonal regions in which the roots of $P(z)$ lie, to determine the number of roots of $P(z)$ in each of the half-planes $\Re(z) < 0$ and $\Re(z) > 0$, and to obtain a method, similar to Horner's method, for computing the roots (real or complex) by successive approximations.

45. The Test-Fraction. Let

$$P(z) = z^n + \delta_1 z^{n-1} + \delta_2 z^{n-2} + \cdots + \delta_n$$

be a polynomial of degree $n > 0$ with complex coefficients δ_k. We shall put $\Re(\delta_k) = p_k$, $\Im(\delta_k) = q_k$, so that $\delta_k = p_k + iq_k$. The polynomial

$$Q(z) = p_1 z^{n-1} + iq_2 z^{n-2} + p_3 z^{n-3} + iq_4 z^{n-4} + \cdots$$

is called the **alternant** of $P(z)$. The quotient $Q(z)/P(z)$ has, in general, a J-fraction expansion of the form

$$\frac{Q(z)}{P(z)} = \cfrac{1}{c_1 z + 1 + k_1 + \cfrac{1}{c_2 z + k_2 + \cfrac{1}{c_3 z + k_3 + \cdots + \cfrac{1}{c_n z + k_n}}}}, \tag{45.1}$$

called the **test-fraction** of $P(z)$.

In (45.1), the c_p are real and different from zero, and the k_p are pure imaginary or zero. In fact, from the identity

$$\frac{Q}{P - Q} = \frac{1}{\dfrac{P}{Q} - 1},$$

it follows that $Q(z)/P(z)$ has a J-fraction expansion if, and only if, $Q(z)/(P(z) - Q(z))$ has a J-fraction expansion; and when (45.1) holds, then

$$\frac{Q(z)}{P(z) - Q(z)} = \cfrac{1}{c_1 z + k_1 + \cfrac{1}{c_2 z + k_2 + \cdots + \cfrac{1}{c_n + k_n}}},$$

and vice versa. Now, the table (40.6) for obtaining the latter expansion is as follows.

$$\begin{array}{llll}
a_{00} = 1, & a_{01} = iq_1, & a_{02} = p_2, & \cdots \\
a_{11} = p_1, & a_{12} = iq_2, & a_{13} = p_3, & \cdots
\end{array}$$

$$\begin{array}{lllll}
c_1 = \dfrac{a_{00}}{a_{11}}, & b_{11} = a_{01} - c_1a_{12}, & b_{12} = a_{02} - c_1a_{13}, & b_{13} = a_{03} - c_1a_{14}, & \cdots \\
k_1 = \dfrac{b_{11}}{a_{11}}, & a_{22} = b_{12} - k_1a_{12}, & a_{23} = b_{13} - k_1a_{13}, & a_{24} = b_{14} - k_1a_{14}, & \cdots \\
c_2 = \dfrac{a_{11}}{a_{22}}, & b_{22} = a_{12} - c_2a_{23}, & b_{23} = a_{13} - c_2a_{24}, & b_{24} = a_{14} - c_2a_{25}, & \cdots \\
k_2 = \dfrac{b_{22}}{a_{22}}, & a_{33} = b_{23} - k_2a_{23}, & a_{34} = b_{24} - k_2a_{24}, & a_{35} = b_{25} - k_2a_{25}, & \cdots \\
c_3 = \dfrac{a_{22}}{a_{33}}, & b_{33} = a_{23} - c_3a_{34}, & b_{34} = a_{24} - c_3a_{35}, & b_{35} = a_{25} - c_3a_{36}, & \cdots \\
 & & \cdots\cdot & &
\end{array} \tag{45.2}$$

Hence, the entries in the table are alternately real and pure imaginary. We verify immediately that the c_p are real and the k_p are pure imaginary or zero. Since all the entries in the table (45.2) are pure imaginary or real, the choice of $Q(z)/P(z)$ as our "test-fraction" is a particularly fortunate one for numerical computation.

In exceptional cases the test-fraction for $P(z)$ may fail to exist. Ways of circumventing this difficulty are illustrated in Exercise 10.

There is some theoretical interest in expressing the condition for the test-fraction to exist by means of the determinants D_p of Theorem 41.1. We take

$$f_0 = P(z) - Q(z) = z^n + iq_1z^{n-1} + p_2z^{n-2} + iq_3z^{n-3} + \cdots,$$
$$f_1 = Q(z) = p_1z^{n-1} + iq_2z^{n-2} + p_3z^{n-3} + iq_4z^{n-4} + \cdots.$$

We then find by simple transformations that

$$D_k = (-1)^{\frac{k(k-1)}{2}} \cdot F_k,$$

where $F_1 = p_1$, and

$$F_k = \begin{vmatrix}
p_1, & p_3, & p_5, & \cdots, & p_{2k-1}, & -q_2, & -q_4, & -q_6, & \cdots, & -q_{2k-2} \\
1, & p_2, & p_4, & \cdots, & p_{2k-2}, & -q_1, & -q_3, & -q_5, & \cdots, & -q_{2k-3} \\
0, & p_1, & p_3, & \cdots, & p_{2k-3}, & 0, & -q_2, & -q_4, & \cdots, & -q_{2k-4} \\
 & \cdot & \cdot & \cdot & & & & \cdot\ \cdot\ \cdot & & \\
0, & & \cdots, & & p_k, & 0, & & \cdots, & & -q_{k-1} \\
0, & q_2, & q_4, & \cdots, & q_{2k-2}, & p_1, & p_3, & p_5, & \cdots, & p_{2k-3} \\
0, & q_1, & q_3, & \cdots, & q_{2k-3}, & 1, & p_2, & p_4, & \cdots, & p_{2k-4} \\
0, & 0, & q_2, & \cdots, & q_{2k-4}, & 0, & p_1, & p_3, & \cdots, & p_{2k-5} \\
 & \cdot & \cdot & \cdot & & & & \cdot\ \cdot\ \cdot & & \\
0, & & \cdots, & & q_k, & 0, & & \cdots, & & p_{k-1}
\end{vmatrix},$$

$$k = 2, 3, 4, \cdots, n, \quad p_k = q_k = 0 \quad \text{for} \quad k > n.$$

By Theorem 41.1, *the test-fraction for* $P(z)$ *exists if, and only if, these determinants* F_k *are different from zero* [**15**].

46. Polygonal Bounds for the Roots of a Polynomial. The considerations in § 26 furnish methods for determining bounds for the roots of a polynomial. We first write the test-fraction for $P(z)$ in the form

$$\frac{Q(z)}{P(z)} = \cfrac{c}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cdot \cdot - \cfrac{a_{n-1}^2}{b_n + z}}}. \tag{46.1}$$

This is a bounded J-fraction. Hence all the roots of $P(z)$ are contained in the convex set K_0 defined in § 26.

One may easily obtain a closed polygon, for instance a rectangle, containing K_0, and hence containing all the roots of $P(z)$. Let

$$\alpha_p(\theta) = \mathfrak{I}(a_p e^{i\theta}), \quad \beta_p(\theta) = \mathfrak{I}(b_p e^{i\theta}). \tag{46.2}$$

If $Y(\theta)$ is determined so that

$$\sum_{p=1}^{n} [\beta_p(\theta) + Y(\theta)]x_p^2 - 2\sum_{p=1}^{n-1} \alpha_p(\theta) x_p x_{p+1} \geq 0 \tag{46.3}$$

for all real values of the x_p, then K_0 is contained in the half-plane (cf. § 26)

$$x \sin \theta + y \cos \theta \leq Y(\theta), \quad (z = x + iy).$$

Hence, in particular, K_0 is contained in the rectangle,

$$\begin{aligned} y &\leq Y(0), & x &\leq Y\left(\frac{\pi}{2}\right), \\ y &\geq -Y(\pi), & x &\geq -Y\left(\frac{3\pi}{2}\right). \end{aligned} \tag{46.4}$$

Now, by Theorem 16.2, the number $Y(\theta)$ satisfies (46.3) for all real x_p if, and only if, numbers g_p between 0 and 1 can be found such that

$$\begin{gathered} \beta_p(\theta) + Y(\theta) \geq 0, \quad p = 1, 2, 3, \cdots, n, \\ \alpha_p^2(\theta) \leq [\beta_p(\theta) + Y(\theta)][\beta_{p+1}(\theta) + Y(\theta)](1 - g_{p-1})g_p, \\ p = 1, 2, 3, \cdots, n-1. \end{gathered} \tag{46.5}$$

If, for the sake of simplicity, we take $g_0 = 0$, $g_p = \frac{1}{2}$, $p = 1, 2, \cdots, n - 1$, we then have the following theorem.

THEOREM 46.1. *Let $P(z)$ be a polynomial of degree n having a test-fraction* (46.1). *Let $\alpha_p(\theta)$ and $\beta_p(\theta)$ be defined by* (46.2), *and let $Y(\theta)$ be any number such that*

$$\begin{aligned} &\beta_p(\theta) + Y(\theta) \geq 0, \quad p = 1, 2, 3, \cdots, n, \\ &2\alpha_1^2(\theta) \leq [\beta_1(\theta) + Y(\theta)][\beta_2(\theta) + Y(\theta)], \\ &4\alpha_p^2(\theta) \leq [\beta_p(\theta) + Y(\theta)][\beta_{p+1}(\theta) + Y(\theta)], \quad p = 2, 3, \cdots, n - 1. \end{aligned} \tag{46.6}$$

Then all the roots of $P(z)$ are contained in the rectangle (46.4).

Note. It is an easy matter to determine the least integral value of $Y(\theta)$ which satisfies (46.6).

To illustrate the application of Theorem 46.1 in a numerical example, let

$$\begin{aligned} P(z) &= (z + 1 + i)\left(z + \frac{1 + i\sqrt{7}}{2}\right)\left(z + \frac{1 - i\sqrt{7}}{2}\right) \\ &= z^3 + (2 + i)z^2 + (3 + i)z + (2i + 2). \end{aligned}$$

The test-fraction for $P(z)$ is given by (40.8). When this is written in the form (46.1), we find that

$$c = 2, \quad b_1 = 2 + \frac{i}{2}, \quad b_2 = -\frac{i}{6}, \quad b_3 = \frac{2i}{3},$$

$$a_1 = \frac{3i}{2}, \quad a_2 = \frac{2\sqrt{2}i}{3}.$$

The inequalities (46.6), to be satisfied for $\theta = 0, \pi/2, \pi, 3\pi/2$, are then

$$2 \sin \theta + \tfrac{1}{2} \cos \theta + Y(\theta) \geq 0,$$

$$-\tfrac{1}{6} \cos \theta + Y(\theta) \geq 0,$$

$$\tfrac{2}{3} \cos \theta + Y(\theta) \geq 0,$$

$$\tfrac{9}{2} \cos^2 \theta \leq (2 \sin \theta + \tfrac{1}{2} \cos \theta + Y(\theta))(-\tfrac{1}{6} \cos \theta + Y(\theta)),$$

$$\tfrac{32}{9} \cos^2 \theta \leq (-\tfrac{1}{6} \cos \theta + Y(\theta))(\tfrac{2}{3} \cos \theta + Y(\theta)).$$

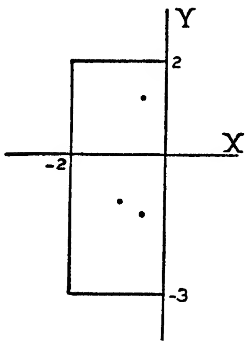

FIG. 7.

These are satisfied for

$$Y(0) = 2, \quad Y\left(\frac{\pi}{2}\right) = 0,$$

$$Y(\pi) = 3, \quad Y\left(\frac{3\pi}{2}\right) = 2.$$

Hence, the roots are contained in the rectangle shown in the accompanying figure.

47. Polynomials Whose Roots Are in a Given Half-Plane. We shall now solve the following problem.

To determine necessary and sufficient conditions, depending upon the coefficients δ_p, for a polynomial $P(z) = z^n + \delta_1 z^{n-1} + \cdots + \delta_n$ with complex coefficients, to have all its roots in a given open half-plane.

Let the given half-plane be defined by the inequality

$$\bar{a}z + a\bar{z} - 2p < 0,$$

where $|a| = 1$ and $p \geq 0$. Under the substitution $z = a(z' + p)$, this half-plane goes over into the half-plane $\Re(z') < 0$, and the polynomial $P(z)$ goes over into another polynomial of degree n. We may therefore assume without loss in generality that the given open half-plane is $\Re(z) < 0$.

We shall prove the following theorem.

THEOREM 47.1.[15] *Let $P(z) = z^n + \delta_1 z^{n-1} + \cdots + \delta_n$ be a polynomial of degree $n > 0$ with complex coefficients $\delta_k = p_k + iq_k$, $k = 1, 2, \cdots, n$. Let $Q(z) = p_1 z^{n-1} + iq_2 z^{n-2} + q_3 z^{n-3} + iq_4 z^{n-4} + \cdots$ be the alternant of $P(z)$. All the roots of $P(z)$ have negative real parts if, and only if, $P(z)$ has a test-fraction of the form*

$$\frac{Q(z)}{P(z)} = \cfrac{a_0}{z + a_0 + b_1 + \cfrac{a_1}{z + b_2 + \cfrac{a_2}{z + b_3 + \cdots + \cfrac{a_{n-1}}{z + b_n}}}}, \tag{47.1}$$

[15] This was proved for real polynomials by Wall [134] and extended to polynomials with complex coefficients by Frank [15].

in which $a_0, a_1, \cdots, a_{n-1}$ *are real and positive, and* $b_1, b_2, \cdots, b_n$ *are pure imaginary or zero* [**134, 15**].

Remark. If the test-fraction of $P(z)$ is written in the form (45.1), so that $a_0 = 1/c_1$, $a_p = 1/c_pc_{p+1}$, $p = 1, 2, 3, \cdots, n-1$, $b_p = k_p/c_p$, $p = 1, 2, 3, \cdots, n$, then the condition of the theorem is obviously equivalent to the condition that the c_p be real and positive and the k_p pure imaginary.

Proof of Theorem 47.1. We shall suppose first that $P(z)$ has a test-fraction with the stated properties, and prove that $P(z) \neq 0$ for $\Re(z) \geq 0$. The J-fraction (47.1) may be regarded as generated by the sequence of transformations

$$t = t_0(z; w) = \frac{a_0}{z + a_0 + b_1 + w}, \quad t = t_p(z; w) = \frac{a_p}{z + b_{p+1} + w},$$
$$p = 1, 2, \cdots, n-1.$$

The pth approximant is then equal to $t_0t_1 \cdots t_{p-1}(z; 0)$. If $\Re(z) \geq 0$, the transformation $t = t_0(z; w)$ takes the half-plane $\Re(w) \geq 0$ into the circular region

$$\left| t - \frac{a_0}{2(a_0 + x)} \right| \leq \frac{a_0}{2(a_0 + x)}, \quad \text{where} \quad x = \Re(z) \geq 0,$$

which is contained in the circular region

$$|t - \tfrac{1}{2}| \leq \tfrac{1}{2}. \tag{47.2}$$

The transformations $t = t_p(z; w)$ transform $\Re(w) \geq 0$ into all or a part of $\Re(t) \geq 0$. Consequently, $t = t_0t_1 \cdots t_{n-1}(z; w)$ transforms $\Re(w) \geq 0$ into a circular region inside (47.2), and, in particular, the nth approximant is in (47.2):

$$\left| \frac{Q(z)}{P(z)} - \frac{1}{2} \right| \leq \frac{1}{2} \quad \text{if} \quad \Re(z) \geq 0. \tag{47.3}$$

From the determinant formula (1.5) it follows that $Q(z)$ and $P(z)$ cannot vanish together. We therefore conclude that $P(z) \neq 0$ for $\Re(z) \geq 0$.

We now suppose, conversely, that $P(z)$ is a polynomial whose roots are all in the left half-plane, $\Re(z) < 0$. We denote by

$\bar{P}(z)$ the polynomial obtained from $P(z)$ by replacing each coefficient by its complex conjugate:

$$\bar{P}(z) = z^n + \bar{\delta}_1 z^{n-1} + \bar{\delta}_2 z^{n-2} + \cdots + \bar{\delta}_n.$$

The set of roots of $P(z)$ is evidently symmetrical to the set of roots of $\bar{P}(-z)$ with respect to the imaginary axis. If we regard the modulus of the polynomial as the product of the lengths of the vectors from z to its roots, we then conclude immediately that

$$|P(z)| > |\bar{P}(-z)| \quad \text{if} \quad \Re(z) > 0,$$

and

$$|P(z)| < |\bar{P}(-z)| \quad \text{if} \quad \Re(z) < 0.$$

Consequently, we have

$$|P(z) \pm \bar{P}(-z)| > 0 \quad \text{for} \quad \Re(z) \neq 0. \tag{47.4}$$

Now, the alternant of $P(z)$ is

$$Q(z) = \frac{P(z) + \bar{P}(-z)}{2} \quad \text{or} \quad Q(z) = \frac{P(z) - \bar{P}(-z)}{2},$$

according as the degree n of $P(z)$ is odd or even, respectively. *Therefore, by* (47.4), *the roots of* $Q(z)$ *are all on the axis of imaginaries.* Moreover, since, for $\Re(z) \geq 0$,

$$\left|\frac{\bar{P}(-z)}{P(z)}\right| = \left|\frac{\bar{P}(-z)}{P(z)} + (-1)^{n+1} + (-1)^n\right| = \left|2\frac{Q(z)}{P(z)} - 1\right| \leq 1,$$

it follows that (47.3) holds.

Since the roots of $P(z)$ are in the half-plane $\Re(z) < 0$, and those of $Q(z)$ are on the line $\Re(z) = 0$, it follows that $Q(z)/P(z)$ is irreducible.

By ordinary long division we now find that

$$\frac{Q(z)}{P(z)} = \cfrac{a_0}{z + a_0 + b_1 + \cfrac{C(z)}{Q(z)}} \tag{47.5}$$

where a_0 is the negative of the sum of the real parts of the roots of $P(z)$, and is therefore positive, b_1 is 0 or pure imaginary, and $C(z)/Q(z)$ is an irreducible rational fraction in which the denominator is of degree $n - 1$, and the degree of the numerator is less than $n - 1$. By (47.3) and (47.5) it follows that

$$\Re\left(\frac{C(z)}{Q(z)}\right) \geq -x, \quad \text{where} \quad x = \Re(z) > 0.$$

Consequently,

$$\Re\left(\frac{C(z)}{Q(z)}\right) \geq 0 \quad \text{for} \quad \Re(z) \geq 0. \tag{47.6}$$

In fact, if the left-hand member of (47.6) is equal to a number $-k$, $k > 0$, for some $z = z_0$ on the interior of the right half-plane, we can contradict the theorem that a nonconstant harmonic function cannot take on its minimum value in the interior of a region in which it is harmonic. It suffices to take for the region the portion of the right half-plane exterior to circles of radius $k/2$ with centers at the roots of $Q(z)$, and interior to a circle $|z| = r > |z_0|$ so large that $|C(z)/Q(z)| \leq k/2$ upon this circle.

From (47.6) we conclude that the roots of $Q(z)$ are simple, and that the residues of $C(z)/Q(z)$ are positive. Otherwise we can choose a path of z in the right half-plane approaching a pole of $C(z)/Q(z)$ in such a way that (47.6) is violated. We therefore have a partial fraction development of the form

$$\frac{C(z)}{Q(z)} = \sum_{p=1}^{n-1} \frac{L_p}{z + ix_p},$$

where the x_p are real and distinct, and the L_p are positive. On replacing z by $-iz$ and then multiplying by $-i$, we see that this becomes

$$\frac{-iC(-iz)}{Q(-iz)} = \sum_{p=1}^{n-1} \frac{L_p}{z - x_p}.$$

Therefore, by Theorem 43.1,

$$\frac{-iC(-iz)}{Q(-iz)} = \cfrac{a_1}{b_2 i + z - \cfrac{a_2}{b_3 i + z - \cdots - \cfrac{a_{n-1}}{b_n i + z}}},$$

where the a_p are positive, and the b_p are pure imaginary or zero. On replacing z by iz and dividing both members by $-i$, we then get

$$\frac{C(z)}{Q(z)} = \cfrac{a_1}{b_2 + z + \cfrac{a_2}{b_3 + z + \cdots + \cfrac{a_{n-1}}{b_n + z}}}.$$

On substituting this expression into (47.5) we obtain (47.1), and the proof of the theorem is complete.

48. Determination of the Number of Roots of $P(z)$ in Each of the Half-planes $\Re(z) \gtrless 0$. Let $P(z)$ be a polynomial with complex coefficients for which the test-fraction (45.1) exists. We have seen that all the roots of $P(z)$ are in the left half-plane $\Re(z) < 0$ if, and only if, the c_p are all positive in this test-fraction. On changing z to $-z$, we see that all the roots are in the right half-plane $\Re(z) > 0$ if, and only if, the c_p are all negative. We shall now prove the following theorem.

THEOREM 48.1.[16] *If $P(z)$ has a test-fraction* (45.1) *in which m of the c_p are positive and $n - m$ are negative, then m of the roots of $P(z)$ have negative real parts and $n - m$ have positive real parts* [**134, 15**].

Proof. As noted in § 45,

$$\frac{Q(z)}{P(z) - Q(z)} = \cfrac{1}{c_1 z + k_1 + \cfrac{1}{c_2 z + k_2 + \cdots + \cfrac{1}{c_n z + k_n}}}.$$

Therefore,

$$\frac{-iQ(-iz)}{P(-iz) - Q(-iz)} = \cfrac{1}{c_1 z + k_1 i - \cfrac{1}{c_2 z + k_2 i - \cdots - \cfrac{1}{c_n z + k_n i}}}.$$

Since the c_p and the $k_p i$ are real, this is a rational function of z with real coefficients. Consequently, there exists a constant

[16] For polynomials with real coefficients, the practical application of this theorem by means of the algorithm (45.2) is known as **Routh's method** [**81**].

$c \neq 0$ such that $u(z) = -icQ(-iz)$ and $v(z) = c[P(-iz) - Q(-iz)]$ are polynomials in z with real coefficients. Moreover,

$$cP(-iz) = u(z) + iv(z). \tag{48.1}$$

If we put $k_p i = d_p$, $p = 1, 2, 3, \cdots, n$, then

$$\frac{v(z)}{u(z)} = \cfrac{1}{c_1 z + d_1 - \cfrac{1}{c_2 z + d_2 - \cdot \cdot \cdot - \cfrac{1}{c_n z + d_n}}}. \tag{48.2}$$

Under the hypothesis that the test-fraction for $P(z)$ exists, the fraction $Q(z)/P(z)$ is irreducible. If $P(z)$ has a root of the form ri, where r is real, then

$$Q(ri) = \frac{P(ri) \pm \bar{P}(-ri)}{2} = \frac{P(ri) \pm \overline{P(+ri)}}{2} = 0,$$

which is contradictory. Hence, $P(z)$ has no root on the line $\Re(z) = 0$, and we may therefore write, for $\Re(z) = 0$,

$$P(z) = Ae^{i\pi\theta}, \quad A > 0.$$

If we regard $P(z)$ as the product of vectors from its roots to the point z, we then conclude immediately (cf. Fig. 8) that as z ranges along the axis of imaginaries from $+i \cdot \infty$ to $-i \cdot \infty$, then θ decreases by the integral amount

$$\Delta = N - P, \tag{48.3}$$

where P and N are the number of roots of $P(z)$ having positive and negative real parts, respectively. This same conclusion evidently results if we introduce the constant factor c, and consider $cP(z) = Ae^{i\pi\theta}$.

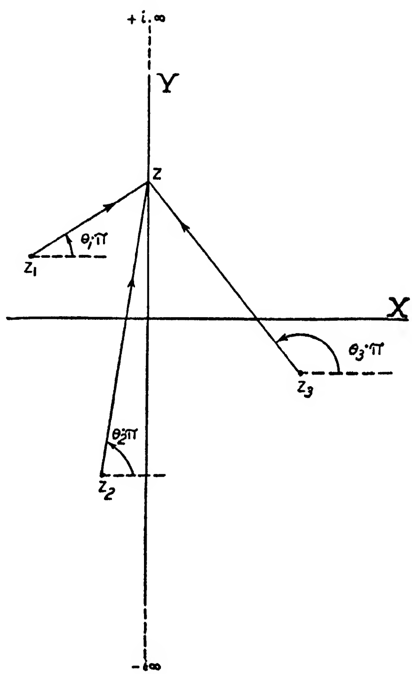

FIG. 8.

By (48.1) and the preceding argument, we then conclude that as z increases over the *real* axis from $-\infty$ to $+\infty$,

$$\theta = \frac{1}{\pi} \arc \tan \frac{v(z)}{u(z)}$$

decreases by the amount Δ.

Let $x_1, x_2, \cdots, x_k$ denote the distinct real zeros of $u(z)$. Let $s_p = +1$, 0, or -1, according as $v(z)/u(z)$ increases from $-\infty$ to $+\infty$, does not change sign, or decreases from $+\infty$ to $-\infty$, respectively, as z increases through the value x_p. We must then have

$$\Delta = \sum_{p=1}^{k} s_p.$$

To compute the number Δ, let polynomials $f_0 = 1, f_1 = c_n z + d_n, \cdots, f_n$, be defined by the recurrence formulas

$$f_{p+1} = (c_{n-p}z + d_{n-p})f_p - f_{p-1}, \quad p = 1, 2, 3, \cdots, n-1, \tag{48.4}$$

and define polynomials $F_0 = 0, F_1 = 1, \cdots, F_n$, by the formulas

$$F_{p+1} = (c_{n-p}z + d_{n-p})F_p - F_{p-1}, \quad p = 1, 2, 3, \cdots, n-1. \tag{48.5}$$

On multiplying (48.5) by f_p and (48.4) by $-F_p$ and then adding, we get $F_{p+1}f_p - F_pf_{p+1} = f_{p-1}F_p - f_pF_{p-1}$, from which it follows that

$$F_{p+1}f_p - F_pf_{p+1} \equiv 1, \quad p = 0, 1, \cdots, n-1. \tag{48.6}$$

Consider now the sequence

$$f_0, f_1, \cdots, f_n. \tag{48.7}$$

From (48.6) it follows that two successive members of this sequence cannot vanish for one and the same value of z. From (48.4) it follows that when f_p, $1 \leq p \leq n-1$, vanishes, then f_{p-1} and f_{p+1} have opposite signs. Hence, as z increases through a real root of f_p, $1 \leq p \leq n-1$, there can be no loss or gain in the number of variations in signs in the sequence (48.7). Therefore, as z increases through real values from $-\infty$ to $+\infty$, any change in the number of variations must be due to the vanishing of f_n. Moreover, there will be a loss or a gain in the number of variations according as the quotient f_{n-1}/f_n changes from negative to positive or from positive to negative, respectively, as z passes through a root of f_n. But, $f_{n-1}/f_n = v(z)/u(z)$, and consequently the number Δ is precisely the net loss in the number of variations in signs in the sequence (48.7) as z ranges from $-\infty$ to $+\infty$ through real values.

Now, when z is negative, the signs of the leading terms of the polynomials (48.7) are the same as the signs of

$$1, \quad -c_n, \quad +c_{n-1}c_n, \quad -c_{n-2}c_{n-1}c_n, \quad \cdots, \quad (-1)^n c_1 c_2 \cdots c_n, \tag{48.8}$$

while for positive z, the signs are those of

$$1, c_n, c_{n-1}c_n, \cdots, c_1 c_2 \cdots c_n. \tag{48.9}$$

If there are $n - m$ variations in signs in (48.9), then there are m variations in signs in (48.8). Therefore

$$\Delta = m - (n - m) = 2m - n.$$

By (48.3) and the relation $N + P = n$, it then follows that

$$N = m \quad \text{and} \quad P = n - m.$$

Since m is clearly equal to the number of positive terms in the sequence $c_1, c_2, \cdots, c_n$, Theorem 48.1 is established.

49. Computation of the Roots of Polynomials.[17] We shall now give a method, based upon Theorem 48.1, for computing the roots of a polynomial by successive approximations. Let $P(z) = z^n + (p_1 + iq_1)z^{n-1} + (p_2 + iq_2)z^{n-2} + \cdots + p_n + iq_n$ be the given polynomial, and put $P_h(z) = P(z + h)$. Let $Q_h(z)$ be the alternant of $P_h(z)$, and let

$$c_p(h) = \frac{a_{p-1,p-1}(h)}{a_{p,p}(h)}, \quad p = 1, 2, 3, \cdots, n,$$

(cf. (45.2)), be the coefficients of z in the test-fraction (45.1) for $P_h(z)$. By Theorem 48.1, if $k = k(h)$ of the coefficients $c_p(h)$ are positive for a given real value of h, then $P_h(z)$ has just $k(h)$

[17] The method of computation of the roots given here is related to the method of Hitchcock [38].

roots in the half-plane $\Re(z) < 0$, so that $P(z)$ has just $k(h)$ roots in the half-plane $\Re(z) < h$.

In general, the method for computing the roots of $P(z)$ consists in varying h in such a way that $a_{n,n}(h) \to 0$ and $k(h)$ changes by one unit. This means that the last remainder in the division process used in forming the test-fraction for $P_h(z)$, which is simply the Euclidean algorithm for the greatest common divisor of $P_h(z) - Q_h(z)$ and $Q_h(z)$, approaches zero. If $z_0(h)$ is the root of the *next* to the last remainder, $a_{n-1,n-1}(h)z + a_{n-1,n}(h)$, then $h + z_0(h)$ approaches a root of $P(z)$ as $a_{n,n}(h)$ approaches zero. If two or more roots of $P(z)$ have a common real part, the process must be suitably modified (cf. Example 2, following).

We shall now show how the computation can be so arranged that the roots of $P(z)$ can be effectively determined by this method. The polynomials $P_h(z)$ can be formed exactly as in **Horner's method.** The Euclidean algorithm can be reduced to the computation in the table (45.2).

Example 1. *To compute the roots of the polynomial*

$$P(z) = z^3 + (1 + 6i)z^2 - (13 - 5i)z - (7 + 10i).$$

We first compute the test-fraction for $P(z)$ by means of (45.2):

$$\begin{array}{llllll}
 & & a_{00} = 1, & a_{01} = 6i, & a_{02} = -13, & a_{03} = -10i, \\
-c_1 = -1, & -k_1 = -i & a_{11} = 1, & a_{12} = 5i, & a_{13} = -7 & \\
 & & b_{11} = i, & b_{12} = -6, & b_{13} = -10i, & \\
-c_2 = 1, & -k_2 = 2i & a_{22} = -1, & a_{23} = -3i, & & \\
 & & b_{22} = 2i, & b_{23} = -7, & & \\
-c_3 = -1, & -k_3 = -3i & a_{33} = -1, & & & \\
 & & b_{33} = -3i. & & &
\end{array}$$

Since $c_1 > 0$, $c_2 < 0$, $c_3 > 0$, there are two roots in $\Re(z) < 0$ and one in $\Re(z) > 0$ (Theorem 48.1).

By Theorem 46.1 we find that the roots of $P(z)$ are contained in the rectangle

$$y \leq -1, \quad x \leq 2, \quad y \geq -3, \quad x \geq -2, \quad (z = x + yi).$$

We now compute the polynomial $P_1(z) = P(z + 1)$, as in Horner's method.

1	$1 + 6i$	$-13 + 5i$	$-7 - 10i$	$h = 1,$
	1	$2 + 6i$	$-11 + 11i$	
1	$2 + 6i$	$-11 + 11i$	$-18 + i$	
	1	$3 + 6i$		
1	$3 + 6i$	$-8 + 17i$		
	1			
1	$4 + 6i$			

Hence, $P_1(z) = z^3 + (4 + 6i)z^2 + (-8 + 17i)z + (-18 + i)$.

With the aid of the calculating machine, we now form the table (45.2) for $P_1(z)$.

1	$6i$	-8	i
4	$17i$	-18	
$1.75i$	-3.5	i	
3.9375	$8.8750i$		
$7.98413i$	-18.00		
$-.003985$			

We have not recorded the values of the c_p, k_p or of b_{33}. The signs of c_1, c_2 and c_3 are $+$, $+$, and $-$, so that $P(z)$ has one root in the half-plane $\Re(z) > 1$. In the above table, $a_{33}(1) = -.003985$.

We next form

$$P_2(z) = z^3 + (7 + 6i)z^2 + (3 + 29i)z + (-21 + 24i),$$

and find that $c_1(2)$, $c_2(2)$ and $c_3(2)$ are all positive, so that all the roots of $P(z)$ are in the half-plane $\Re(z) < 2$. There is one root in the strip $1 < \Re(z) < 2$. We find that $a_{33}(2) = 8.98$. Since we had $a_{33}(1) = -.003985$, it would appear that this root has real part very nearly equal to 1. If we assume that $a_{33}(h)$ varies linearly with h, we find, by interpolation, that we should have $a_{33}(1.0004) = 0$. In the light of this information, we now form $P(z + 1.001) = z^3 + (4.003 + 6i)z^2 + (-7.991997 + 17.012i)z + (-18.007995999 + 1.017006i)$, and construct table (45.2) for this polynomial. We find that $a_{33}(1.001) = -.000076233$, and that $a_{22}(1.001)z + a_{23}(1.001) = 3.944596397z + 8.8904426667i$. On setting the latter equal to zero we find for the imaginary part of the root the approximate value $-2.254i$. We thus have as an approximate value of the root

$$1.001 - 2.254i.$$

Now,

$$P(d + 1.001 - 2.254i) = d^3 + (4.003 - .7621i)d^2 + (3.814455 - 1.033524i)d - (.000253547 + .000645698i).$$

If we neglect the terms in d^3 and d^2, and set the linear part equal to zero, we obtain the correction $d = .0000192 + .0001745i$. Then $d + 1.001 - 2.254i = 1.0010192 - 2.2538255i$ is the value of the root. This is actually correct to the number of places given, inasmuch as it was found by additional calculation that the root is $1.0010192258 - 2.2538255167i$, where the last digits 8 and 7 are in doubt. For the other two roots of $P(z)$ we find the values $-1.520324 - 1.39987916i$ and $-.480695 - 2.3462953i$, correct to the number of places given. As a check, we find that the sum and the product of these values of the roots are $-1 - 6i$ and $7 + 10i$, respectively, correct to six decimal places.

It should be emphasized that, except for the supplementary calculation of the table (45.2) at each step, the computation is exactly the same as in Horner's method.

Example 2. *To compute the roots of the polynomial* $P(z) = z^3 + 2z + 20$.

This polynomial has a pair of conjugate imaginary roots. Since the coefficient of z^2 is zero, the test-fraction does not exist. This is of little concern inasmuch as the test-fraction exists for $P_h(z) = P(z + h)$ when h is near the real parts of the roots. We find by Theorem 48.1, applied to $P_1(z)$ and $P_2(z)$, that the imaginary roots are in the strip $1 < \Re(z) < 2$. In the following table, the numbers ρ_p are the *next* to the last remainders obtained in applying the Euclidean algorithm to the polynomials $P_h(z) - Q_h(z)$ and $Q_h(z)$.

1	0	2	20
	1	1	3
1	1	3	23
	1	2	
1	2	5	
	1		
1	3		

$h = 1,$

$$\rho_1 = 5 - \tfrac{23}{3} = -2.7,$$

	1	4	9	$\frac{1}{h = 2}$	$\rho_2 = 14 - \frac{32}{6} = 8.7$
1	4	9	32		
	1	5			
1	5	14			
	1				
1	6				
	−.8	−4.16	−7.872	$\frac{-.8}{h = 1.2}$	$\rho_{1.2} = -.38$
1	5.2	9.84	24.128		
	−.8	−3.52			
1	4.4	6.32			
	−.8				
1	3.6				
	.1	.37	.669	$\frac{.1}{h = 1.3}$	$\rho_{1.3} = .71$
1	3.7	6.69	24.797		
	.1	.38			
1	3.8	7.07			
	.1				
1	3.9				
	−.07	−.2681	−.476133	$\frac{-.07}{h = 1.23}$	$\rho_{1.23} = -.041$
1	3.83	6.8019	24.320867		
	−.07	−.2632			
1	3.76	6.5387			
	−.07				
1	3.69				
	.01	.037	.065757	$\frac{.01}{h = 1.24}$	$\rho_{1.24} = .073$
1	3.7	6.5757	24.386624		
	.01	.0371			
1	3.71	6.6128			
	.01				
1	3.72				
	−.006	−.022284	−.039543096	$\frac{-.006}{h = 1.234}$	$\rho_{1.234} = -.0084$
1	3.714	6.590516	24.347080904		
	−.006	−.022248			
1	3.708	6.568268			
	−.006				
1	3.702				
	.001	.003703	.006571971	$\frac{.001}{h = 1.235}$	$\rho_{1.235} = .0024$
1	3.703	6.571971	24.353652875		
	.001	.003704			
1	3.704	6.575675			
	.001				
1	3.705				

$$3.705k^2 + 24.354 = 0, \quad k = \pm 2.564i$$

$$P(d + 1.235 + 2.564i) = d^3 + (3.705 + 7.692i)d^2 - (13.146613 - 18.99924i)d - (.003572805 - .004048556i).$$

The real part of the imaginary roots has the value $h = 1.235$, correct to three decimal places. The imaginary parts are $\pm 2.564i$, as indicated above. On equating to zero the linear part of $P(d + 1.235 + 2.564i)$ we obtain the correction $d = -.000237 + .000247i$. Hence, the imaginary roots are approximately equal to $1.234773 \pm 2.564247i$.

Since the sum of the roots is equal to zero, the real root must be -2.469546. One may readily verify that this is correct to six decimal places.

Exercise 10

10.1. A polynomial $P(z)$ of degree n has all its roots in the left half-plane, $\Re(z) < 0$, if, and only if, the determinants F_k, $k = 1, 2, 3, \cdots, n$, of § 45, are all positive [**15**].

10.2. A polynomial $P(z) = z^n + p_1 z^{n-1} + \cdots + p_n$, with *real* coefficients, has all its roots in the left half-plane, $\Re(z) < 0$, if, and only if, the determinants

$$H_k = \begin{vmatrix} p_1, & p_3, & p_5, & \cdots, & p_{2k-1} \\ 1, & p_2, & p_4, & \cdots, & p_{2k-2} \\ 0, & p_1, & p_3, & \cdots, & p_{2k-3} \\ 0, & 1, & p_2, & \cdots, & p_{2k-4} \\ & & \cdot \quad \cdot \quad \cdot & & \\ 0, & & \cdots, & & p_k \end{vmatrix}, \quad k = 1, 2, 3, \cdots, n,$$
$$(p_r = 0 \quad \text{if} \quad r > n)$$

are all positive [**39**].

10.3. Using the fact that the transformation

$$z = r \cdot \frac{1 + w}{1 - w}, \quad (r > 0),$$

maps the half-plane $\Re(w) < 0$ into the circular region $|z| < r$, derive an algorithm for determining whether or not a polynomial has all its roots in the circle $|z| < r$.

10.4. Show that if a is not on the real interval $-\infty < x \leq -\frac{1}{4}$, then one of the roots of the quadratic equation $az^2 + z - 1 = 0$ is within the circle $|z - 1| = 1$, and the other is outside this circle. The root within the circle is the value of the periodic continued fraction

$$\cfrac{1}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \ddots}}}$$

10.5. Theorem 48.1 cannot be applied to the polynomial $P(z) = z^4 - (\frac{9}{4})z^3 + z^2 - (\frac{9}{4})z + (\frac{5}{2})$ because in this case the test-fraction fails to exist. Show that the theorem can be applied, however, to the polynomial $(z + \frac{7}{4})P(z)$, and prove that $P(z)$ has two zeros in the right half-plane and two zeros in the left half-plane. Show that the same conclusion can be reached by applying the theorem to the polynomial $(\frac{2}{5})z^4P(1/z)$.

10.6. By a theorem of Hurwitz, the zeros of a polynomial are continuous functions of its coefficients. Apply this fact and Theorem 48.1 to the polynomial

$$P(t; z) = z^4 - (t + \tfrac{9}{4})z^3 + z^2 - (\tfrac{9}{4})z + (\tfrac{5}{2})$$

to determine the number of zeros of $P(0; z)$ in each of the half-planes $\Re(z) \gtrless 0$.

10.7. The test-fraction for $P(z) = z^5 - 3z^4 - 9z^3 - 27z^2 - 32z - 30$ is

$$\frac{Q(z)}{P(z)} = \cfrac{1}{(-\frac{1}{3})z + 1 + \cfrac{1}{(\frac{1}{6})z + \cfrac{1}{(\frac{9}{10})z + \cfrac{1}{(\frac{4}{3})z + \cfrac{1}{(\frac{1}{2})z}}}}}$$

Determine a polygon in which the roots of $P(z)$ lie.

10.8. Write the test-fraction for $P(z)$ in the form with all its partial denominators equal to unity. For $|z| \geq M$, let the partial numerators after the first have moduli not greater than $(1 - g_{p-1})g_p$, $p = 1, 2, 3, \cdots, n - 1$, respectively $(0 < g_k < 1)$. Then $P(z) \neq 0$ for $|z| \geq M$.

Chapter XI

J-FRACTION EXPANSIONS FOR POWER SERIES

In this chapter we consider the problem of expanding a power series $\Sigma(c_p/z^{p+1})$ into a J-fraction, with emphasis upon the computational aspects of the problem. We first show that the expansion exists if, and only if, a sequence of polynomials can be constructed which are orthogonal relative to the sequence of coefficients $c_0, c_1, c_2, \cdots$ of the power series. These polynomials are the denominators of the J-fraction. The algorithm for constructing the orthogonal polynomials gives immediately the J-fraction expansion for the power series. This idea goes back to Tschebycheff [**105**] (cf. also [**133**] and [**21**]).

In contrast with this step-by-step method for expanding a power series into a J-fraction, is a method due to Stieltjes [**93**] which will, in certain cases, give the complete expansion in one stroke. Stieltjes showed that the expansion can be written down immediately when one has at hand a suitable formal reduction of the quadratic form $\Sigma c_{p+q}x_px_q$ to a sum of squares. (Cf. § 53.)

In the concluding section of the chapter, we consider questions of convergence and of equality between the power series and its J-fraction expansion.

50. Polynomials Orthogonal Relative to a Sequence. Let c: $c_0, c_1, c_2, \cdots$ be any sequence of numbers. Relative to this sequence we define an operation called **formal integration,** which replaces u^p by c_p, $(p = 0, 1, 2, \cdots)$, in any polynomial in u upon which it operates:

$$\int(k_0 + k_1u + \cdots + k_nu^n)d\phi_c(u) = k_0c_0 + k_1c_1 + \cdots + k_nc_n.$$

The symbol $\phi_c(u)$ is used only to signify that u is the variable whose powers are to be replaced by the c_p. When the operation is performed upon the product of two or more polynomials, the polynomials must be multiplied together, and the product written as a polynomial, before the operation is performed.

A finite or infinite sequence of polynomials $B_p(u)$, $p = 0, 1, 2, \cdots$, is called **orthogonal relative to c** if

$$\int B_p(u)B_q(u)d\phi_c(u) = 0 \quad \text{for} \quad p \neq q.$$

We now consider the following problem.

To construct a set of $m + 1$ *polynomials* $B_p(u) = u^p + \delta_{p,1}u^{p-1} + \delta_{p,2}u^{p-2} + \cdots + \delta_{p,p}$, $p = 0, 1, 2, \cdots, m$, $m > 1$, *such that*

$$\int B_p(u)B_q(u)d\phi_c(u) \begin{cases} = 0 & \text{for} \quad p \neq q, \quad p \leq m, \quad q \leq m, \\ \neq 0 & \text{for} \quad p = q < m. \end{cases} \tag{50.1}$$

A necessary condition for the polynomials to exist is that the determinants

$$\Delta_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix} \neq 0, \quad p = 0, 1, 2, \cdots, m - 1. \tag{50.2}$$

In fact, since

$$0 \neq \int B_0^2(u)d\phi_c(u) = \int 1 \cdot d\phi_c(u) = c_0,$$

it is necessary that $\Delta_0 = c_0 \neq 0$. Now, if $q > 0$, $q < m$, then

$$\int B_0(u)B_q(u)d\phi_c(u) = \int B_q(u)d\phi_c(u) = 0,$$

$$\int B_1(u)B_q(u)d\phi_c(u) = \int uB_q(u)d\phi_c(u) = 0,$$

$$\cdot \quad \cdot \quad \cdot$$

$$\int B_{q-1}(u)B_q(u)d\phi_c(u) = \int u^{q-1}B_q(u)d\phi_c(u) = 0,$$

$$\int B_q^2(u)d\phi_c(u) = \int u^qB_q(u)d\phi_c(u) = h_q \neq 0.$$

From these equations we find immediately that

$$\Delta_q = h_q\Delta_{q-1}, \quad q = 0, 1, 2, \cdots, m - 1, \quad (\Delta_{-1} = 1),$$

and therefore (50.2) holds.

Suppose, conversely, that (50.2) holds. Then

$$\int B_0^2(u)d\phi_c(u) = \int 1 \cdot d\phi_c(u) = c_0 = \Delta_0 \neq 0.$$

We write $a_0 = c_0$. Let $B_1(u) = u + b_1$. Then

$$\int B_0(u)B_1(u)d\phi_c(u) = c_1 + c_0b_1 = 0$$

if, and only if,

$$\int B_0(u)d\phi_c(u) = a_0, \quad \int uB_0(u)d\phi_c(u) = -a_0b_1.$$

Since $a_0 \neq 0$, this determines b_1, and therefore B_1, uniquely.

Using induction, we now suppose that $B_0(u), B_1(u), \cdots, B_n(u)$, $n < m$, have been uniquely determined, such that (50.1) holds for $p \leq n$, $q \leq n$, and that constants a_p and b_p have been determined by the relations

$$\int u^pB_p(u)d\phi_c(u) = a_0a_1 \cdots a_p \neq 0,$$

$$\int u^{p+1}B_p(u)d\phi_c(u) = -a_0a_1 \cdots a_p(b_1 + b_2 + \cdots + b_{p+1}), \quad (50.3)$$
$$p = 0, 1, 2, \cdots, n - 1,$$

such that

$$B_p(u) = (b_p + u)B_{p-1}(u) - a_{p-1}B_{p-2}(u),$$
$$p = 1, 2, 3, \cdots, n, \quad (B_{-1}(u) = 0). \quad (50.4)$$

We shall prove that $B_{n+1}(u)$ is then uniquely determined.

It is easy to see that an arbitrary polynomial of degree $n + 1$ in which the coefficient of u^{n+1} is unity, can be expressed uniquely in the form

$$B_{n+1}(u) = (b_{n+1} + u)B_n(u) - a_nB_{n-1}(u) + k_0B_0(u) + k_1B_1(u) + \cdots + k_{n-2}B_{n-2}$$

where $b_{n+1}, a_n, k_0, k_1, \cdots, k_{n-2}$ are constants. The conditions

$$\int u^p B_{n+1}(u) d\phi_c(u) = 0, \quad p = 0, 1, 2, \cdots, n-2,$$

give in succession $k_0 a_0 = 0$, $k_1 a_0 a_1 = 0, \cdots, k_{n-2} a_0 a_1 \cdots a_{n-2} = 0$, so that $k_0 = k_1 = \cdots = k_{n-2} = 0$. From the conditions

$$\int u^{n-1} B_{n+1}(u) d\phi_c(u) = 0,$$

$$\int u^n B_{n+1}(u) d\phi_c(u) = 0,$$

we then obtain (50.3) for $p = n$. We then find immediately from the equations

$$\int u^p B_n(u) d\phi_c(u) = 0, \quad p = 0, 1, 2, \cdots, n-1,$$

$$\int u^n B_n(u) d\phi_c(u) = a_0 a_1 \cdots a_n,$$

that $\Delta_n = a_0 a_1 \cdots a_n \Delta_{n-1}$, and therefore $a_n \neq 0$. Accordingly, b_{n+1} is uniquely determined. Moreover,

$$\int B_p(u) B_q(u) d\phi_c(u) = 0 \quad \text{for} \quad p \neq q, \quad p \leq n+1, \quad q \leq n+1,$$

and, if $n + 1 < m$, then

$$\int B_{n+1}{}^2(u) d\phi_c(u) = \int u^{n+1} B_{n+1}(u) d\phi_c(u) \neq 0,$$

for otherwise we would have $\Delta_{n+1} = 0$.

We have completed the proof of the following theorem.

THEOREM 50.1. *Let $c = \{c_p\}$ be a sequence of constants such that for some $m > 1$, (50.2) holds. Then there exist uniquely determined polynomials $B_p(u) = u^p + \delta_1 u^{p-1} + \cdots + \delta_p$, $p = 0, 1, 2, \cdots, m$, such that (50.1) holds. These polynomials can be found recurrently by means of the formulas*

$$B_0(u) = 1, \quad B_1(u) = b_1 + u,$$
$$B_p(u) = (b_p + u) B_{p-1}(u) - a_{p-1} B_{p-2}(u), \quad p = 2, 3, 4, \cdots, m; \tag{50.5}$$

$$\int u^p B_p(u) d\phi_c(u) = a_0 a_1 \cdots a_p,$$
$$\int u^{p+1} B_p(u) d\phi_c(u) = -a_0 a_1 \cdots a_p (b_1 + b_2 + \cdots + b_{p+1}), \tag{50.6}$$
$$p = 0, 1, 2, \cdots, m-1.$$

Conversely, if (50.2) fails to hold, then such polynomials do not exist.

Remark. If the a_p and b_p are given, $a_p \neq 0$, then it is easy to see that the relations (50.5) and (50.6) serve to determine $c_0, c_1, \cdots, c_{2m-1}$ uniquely in terms of the a_p and b_p.

51. Algorithm for Expanding a Power Series into a J-fraction. By (50.5) it follows that the polynomials $B_p(z)$ are the denominators of the J-fraction

$$\cfrac{a_0}{b_1 + z - \cfrac{a_1}{b_2 + z - \cdot \, \cdot \, \cdot - \cfrac{a_{m-1}}{b_m + z}}}. \tag{51.1}$$

Let

$$P\left(\frac{1}{z}\right) = \sum_{p=0}^{\infty} \frac{c_p}{z^{p+1}} \tag{51.2}$$

be the formal power series whose coefficients are the given numbers c_p. The coefficients of $1/z, 1/z^2, \cdots, 1/z^p$ in the product $P(1/z)B_p(z)$ are

$$\int u^r B_p(u) d\phi_c(u) = 0, \quad r = 0, 1, 2, \cdots, p-1;$$

and the coefficients of $1/z^{p+1}$ and of $1/z^{p+2}$ in this product are

$$\int u^p B_p(u) d\phi_c(u) = a_0 a_1 \cdots a_p,$$

and

$$\int u^{p+1} B_p(u) d\phi_c(u) = -a_0 a_1 \cdots a_p(b_1 + b_2 + \cdots + b_{p+1}),$$

respectively. Consequently, we have

$$P\left(\frac{1}{z}\right) B_p(z) - A_p(z) = \frac{a_0 a_1 \cdots a_p}{z^{p+1}} - \frac{a_0 a_1 \cdots a_p(b_1 + b_2 + \cdots + b_{p+1})}{z^{p+2}} + \cdots,$$

$$p = 0, 1, 2, \cdots, m-1, \tag{51.3}$$

where $A_p(z)$ is a polynomial in z. It follows that (cf. (42.10))

$$\frac{A_p(z)}{B_p(z)} = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_{2p-1}}{z^{2p}} + \frac{c_{2p}{}^{(p)}}{z^{2p+1}} + \cdots. \quad (51.4)$$

By the argument used in proving (43.4) we see at once that $A_p(z)/B_p(z)$ is the pth approximant of (51.1), $(p \leq m)$.

If, conversely, (51.1) is any terminating J-fraction in which the partial numerators are different from zero, then we see immediately that its denominators $B_p(z)$ satisfy (50.1), the sequence c being $c_0, c_1, \cdots, c_{2m-1}$, determined by means of (51.4).

If (50.2) holds for all m, and only then, we can construct an *infinite* sequence of orthogonal polynomials. In this case we have, instead of (51.1), an infinite J-fraction; and the relation (51.4) holds for all values of p.

The power series $P(1/z)$ with which the power series expansion in descending powers of z of the pth approximant of the J-fraction agrees term by term for the first $2p$ terms ($p = 1, 2, 3, \cdots$), is called the **power series expansion** of the J-fraction; the coefficients c_p are the **moments** of the J-fraction. From the preceding discussion it follows that the following theorem is true.

THEOREM 51.1. *There is a one-to-one correspondence between infinite J-fractions and power series* $P(1/z)$ *for which the determinants* Δ_p *of* (50.2) *are different from zero for* $p = 0, 1, 2, 3, \cdots$. *This correspondence is completely characterized by the fact that* (51.3) *holds for* $p = 0, 1, 2, \cdots$. *The determinants* Δ_p *are related to the coefficients* a_p *in the J-fraction by the formulas*

$$\Delta_p = a_0 a_1 \cdots a_p \Delta_{p-1}, \quad p = 0, 1, 2, \cdots, \quad (\Delta_{-1} = 1). \quad (51.5)$$

If a power series $P(1/z)$ represents a rational function of z, then the determinants Δ_p are clearly zero from and after some value of p. Hence, we have

THEOREM 51.2. *The power series expansion of an infinite J-fraction cannot represent a rational function of* z.

The formulas (50.6) for constructing the orthogonal polynomials, and hence of the J-fraction, can be expressed in matrix notation in such a way as to facilitate the computation. We shall write down the formulas and explain them afterward.

$$B_p(z) = z^p + \delta_{p,1}z^{p-1} + \delta_{p,2}z^{p-2} + \cdots + \delta_{p,p},$$
$$A_p(z) = \sigma_{p,0}z^{p-1} + \sigma_{p,1}z^{p-2} + \cdots + \sigma_{p,p-1};$$

$$\begin{aligned}
\delta_{00} &= 1,\\
c_0\delta_{00} &= a_0,\\
c_1\delta_{00} &= -a_0b_1 = h_0;\\
b_1 &= -\frac{h_0}{a_0},\\
(\delta_{10}, \delta_{11}) &= (1, b_1),\\
(c_2, c_1)\begin{pmatrix}\delta_{10}\\ \delta_{11}\end{pmatrix} &= a_0a_1,\\
(c_3, c_2)\begin{pmatrix}\delta_{10}\\ \delta_{11}\end{pmatrix} &= -a_0a_1(b_1 + b_2) = h_1;\\
b_2 &= \frac{h_0}{a_0} - \frac{h_1}{a_0a_1},\\
(\delta_{20}, \delta_{21}, \delta_{22}) &= (\delta_{10}, \delta_{11})\begin{pmatrix}1, & b_2, & 0\\ 0, & 1, & b_2\end{pmatrix} - a_1(0, 0, \delta_{00}),\\
(c_4, c_3, c_2)\begin{pmatrix}\delta_{20}\\ \delta_{21}\\ \delta_{22}\end{pmatrix} &= a_0a_1a_2,\\
(c_5, c_4, c_3)\begin{pmatrix}\delta_{20}\\ \delta_{21}\\ \delta_{22}\end{pmatrix} &= -a_0a_1a_2(b_1 + b_2 + b_3) = h_2;\\
b_3 &= \frac{h_1}{a_0a_1} - \frac{h_2}{a_0a_1a_2},\\
(\delta_{30}, \delta_{31}, \delta_{32}, \delta_{33}) &= (\delta_{20}, \delta_{21}, \delta_{22})\begin{pmatrix}1, & b_3, & 0, & 0\\ 0, & 1, & b_3, & 0\\ 0, & 0, & 1, & b_3\end{pmatrix} - a_2(0, 0, \delta_{10}, \delta_{11}),\\
&\cdots
\end{aligned} \qquad (51.6)$$

$$(\sigma_{p,0}, \sigma_{p,1}, \sigma_{p,2}, \cdots) = (\delta_{p,0}, \delta_{p,1}, \delta_{p,2}, \cdots). \qquad (51.7)$$
$$\begin{pmatrix}c_0, & c_1, & \cdots, & c_{p-1}\\ 0, & c_0, & \cdots, & c_{p-2}\\ & \cdot & \cdot & \cdot\\ 0, & 0, & \cdots, & 0, & c_0\end{pmatrix}.$$

The second relation in each group, after the first, is (50.5) for the appropriate value of p. The last two relations in each group are the relations (50.6). We understand the usual *row times column* matrix multiplication. The equation (51.7) can be obtained by equating the coefficients of $z^0, z^1, z^2, \cdots, z^{p-1}$ on either side of (51.3).

Example. By way of illustration, we shall carry out several steps of the calculation for the case where

$$c_p = \frac{1}{p+1}, \quad p = 0, 1, 2, \cdots.$$

Here, $P(1/z) = \log [z/(z-1)]$. Hence, we shall have obtained the first few partial quotients in the J-fraction expansion for this function.

Following (51.6), we get:

$$\delta_{00} = 1,$$

$$c_0\delta_{00} = a_0 = 1,$$

$$c_1\delta_{00} = -a_0b_1 = \tfrac{1}{2};$$

$$b_1 = -\tfrac{1}{2},$$

$$(\delta_{10}, \delta_{11}) = (1, -\tfrac{1}{2}),$$

$$(c_2, c_1)\begin{pmatrix}\delta_{10}\\ \delta_{11}\end{pmatrix} = (\tfrac{1}{3}, \tfrac{1}{2})\begin{pmatrix}1\\ -\tfrac{1}{2}\end{pmatrix} = a_0a_1 = \tfrac{1}{12}, \quad a_1 = \tfrac{1}{12},$$

$$(\tfrac{1}{4}, \tfrac{1}{3})\begin{pmatrix}1\\ -\tfrac{1}{2}\end{pmatrix} = -a_0a_1(b_1 + b_2) = \tfrac{1}{12} = h_1;$$

$$b_2 = -\tfrac{1}{2},$$

$$(\delta_{20}, \delta_{21}, \delta_{22}) = (1, -\tfrac{1}{2})\begin{pmatrix}1, & -\tfrac{1}{2}, & 0\\ 0, & 1, & -\tfrac{1}{2}\end{pmatrix} - (\tfrac{1}{12})(0, 0, 1)$$
$$= (1, -1, \tfrac{1}{6}),$$

$$(\tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3})\begin{pmatrix}1\\ -1\\ \tfrac{1}{6}\end{pmatrix} = a_0a_1a_2 = \tfrac{1}{180}, \quad a_2 = \tfrac{1}{15},$$

$$(\tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4})\begin{pmatrix}1\\ -1\\ \tfrac{1}{6}\end{pmatrix} = -a_0a_1a_2(b_1 + b_2 + b_3) = \tfrac{1}{120} = h_2;$$

$$b_3 = -\tfrac{1}{2},$$

$$(\delta_{30}, \delta_{31}, \delta_{32}, \delta_{33}) = (1, -\tfrac{3}{2}, \tfrac{3}{5}, -\tfrac{1}{20}),$$

$$(\sigma_{30}, \sigma_{31}, \sigma_{32}) = (1, -1, \tfrac{11}{60}).$$

The first three partial quotients of the J-fraction for $\log [z/(z-1)]$ are then found to be

$$\cfrac{1}{-\frac{1}{2} + z - \cfrac{(\frac{1}{12})}{-\frac{1}{2} + z - \cfrac{(\frac{1}{15})}{-\frac{1}{2} + z}}} = \frac{z^2 - z + (\frac{11}{60})}{z^3 - (\frac{3}{2})z^2 + (\frac{3}{5})z - (\frac{1}{20})}.$$

We note that for $z = -1$, this gives $\log 2 = .69312 \cdots$, which is correct to *four* decimal places.

52. Stieltjes Type Continued Fraction Expansions for Power Series. In Chapter VI we considered continued fractions of the form (28.1), which we called *Stieltjes type* continued fractions. These continued fractions have the property that their even parts are J-fractions. For the purpose of abbreviation, we shall call any continued fraction of the form (28.1) in which the k_p are different from zero, or any continued fraction which can be obtained from this by an equivalence transformation or simple change of variable, an **S-fraction.** According to this definition, a J-fraction in which $b_p = 0$, $p = 1, 2, 3, \cdots$, is an S-fraction:

$$\cfrac{a_0}{z - \cfrac{a_1}{z - \cfrac{a_2}{z - \cdots}}} \tag{52.1}$$

By means of an equivalence transformation, this can be thrown into the form

$$\cfrac{a_0 z}{z^2 - \cfrac{a_1}{1 - \cfrac{a_2}{z^2 - \cfrac{a_3}{1 - \cdots}}}} \tag{52.2}$$

Hence, it follows that if $P(1/z)$ is the power series expansion of (52.1), then

$$\frac{1}{z}P\left(\frac{1}{z}\right) = \frac{c_0}{z^2} + \frac{c_1}{z^4} + \frac{c_2}{z^6} + \cdots \tag{52.3}$$

contains only even powers of $1/z$. It is therefore convenient to remove a factor z from (52.2) and then replace z^2 by z. This gives the S-fraction

$$\cfrac{a_0}{z - \cfrac{a_1}{1 - \cfrac{a_2}{z - \cfrac{a_3}{1 - \cdots}}}}, \tag{52.4}$$

and its power series expansion is

$$\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots. \tag{52.5}$$

Inasmuch as $c_0, 0, c_1, 0, c_2, 0, \cdots$ are the moments of a J-fraction, the determinants

$$c_0, \quad \begin{vmatrix} c_0, & 0 \\ 0, & c_1 \end{vmatrix}, \quad \begin{vmatrix} c_0, & 0, & c_1 \\ 0, & c_1, & 0 \\ c_1, & 0, & c_2 \end{vmatrix}, \quad \cdots \tag{52.6}$$

must be different from zero. By making suitable interchanges among rows and corresponding columns, the determinants (52.6) may be written as

$$\Delta_0, \quad \Delta_0\Omega_0, \quad \Delta_1\Omega_0, \quad \Delta_1\Omega_1, \cdots$$

where the Δ_p are the determinants introduced before, and

$$\Omega_p = \begin{vmatrix} c_1, & c_2, & \cdots, & c_{p+1} \\ c_2, & c_3, & \cdots, & c_{p+2} \\ & \cdot & \cdot & \cdot \\ c_{p+1}, & & \cdots, & c_{2p+1} \end{vmatrix}, \quad p = 0, 1, 2, \cdots. \tag{52.7}$$

Hence, a necessary condition for (52.5) to have an S-fraction expansion (52.4) is that

$$\Delta_p \neq 0, \quad \Omega_p \neq 0, \quad p = 0, 1, 2, \cdots. \tag{52.8}$$

Conversely, if this condition is satisfied, then the determinants (52.6) are all different from zero, so that $c_0, 0, c_1, 0, c_2, 0, \cdots$ are the moments of a J-fraction. To construct this J-fraction, it is only necessary to replace c_{2p} by c_p and c_{2p+1} by 0, $p = 0, 1, 2, \cdots$, in the formulas (51.6). If we do this, we find that

$$\begin{array}{llll} b_1 = 0, & \delta_{11} = 0, & & \\ b_2 = 0, & \delta_{21} = 0, & & \\ b_3 = 0, & \delta_{31} = 0, & \delta_{33} = 0, & \\ b_4 = 0, & \delta_{41} = 0, & \delta_{43} = 0, & \\ b_5 = 0, & \delta_{51} = 0, & \delta_{53} = 0, & \delta_{55} = 0, \\ & & \cdots & \end{array}$$

Thus the J-fraction has the form (52.1), so that (52.5) has the expansion (52.4).

We shall state this result as

THEOREM 52.1. *There is a one-to-one correspondence between power series* (52.5) *for which* (52.8) *holds, and S-fractions* (52.4) *in which* $a_p \neq 0$, $p = 0, 1, 2, \cdots$. *The correspondence is completely characterized by the fact that the expansion in descending powers of z of the* pth *approximant of the S-fraction agrees term by term with the power series for the first* p *terms* ($p = 1, 2, 3, \cdots$).

Remark. If we remember that the determinants (52.6) are products $\Delta_0, \Delta_0\Omega_0, \Delta_1\Omega_1, \Delta_2\Omega_1, \Delta_2\Omega_2, \cdots$ of determinants (52.8), we see by (51.5) that the coefficients a_p in (52.4) are positive if, and only if,

$$\Delta_p > 0, \quad \Omega_p > 0, \quad p = 0, 1, 2, \cdots. \tag{52.9}$$

Just as in the case of J-fractions, we have

THEOREM 52.2. *An infinite S-fraction cannot have a power series expansion which represents a rational function of z.*

53. Stieltjes' Expansion Theorem. We have given a step-by-step method for expanding a power series into a J-fraction. We shall now describe a method due to Stieltjes [93] which will, in certain cases, give the complete expansion in one stroke. Stieltjes showed that the problem of expanding a power series into a J-fraction is equivalent to the problem of obtaining a decomposition of a certain quadratic form into a sum of squares. The theorem may be stated as follows.

THEOREM 53.1. *The coefficients in the J-fraction*

$$\cfrac{1}{b_1 + z - \cfrac{a_1}{b_2 + z - \cfrac{a_2}{b_3 + z - \cdot_{\cdot_{\cdot}}}}} \tag{53.1}$$

and its power series expansion

$$P\left(\frac{1}{z}\right) = \sum_{p=0}^{\infty} \frac{(-1)^p c_p}{z^{p+1}} \tag{53.2}$$

are connected by the relations

$$c_{p+q} = k_{0,p}k_{0,q} + a_1 k_{1,p}k_{1,q} + a_1 a_2 k_{2,p}k_{2,q} + \cdots, \tag{53.3}$$

where

$$k_{0,0} = 1, \quad k_{r,s} = 0 \quad \text{if} \quad r > s,$$

and where the $k_{r,s}$ *for* $s \geq r$ *are given recurrently by the matrix equation*

$$\begin{pmatrix} k_{00}, & 0, & 0, & 0, & \cdots \\ k_{01}, & k_{11}, & 0, & 0, & \cdots \\ k_{02}, & k_{12}, & k_{22}, & 0, & \cdots \\ & \cdot & \cdot & \cdot & \end{pmatrix} \cdot \begin{pmatrix} b_1, & 1, & 0, & 0, & 0, & \cdots \\ a_1, & b_2, & 1, & 0, & 0, & \cdots \\ 0, & a_2, & b_3, & 1, & 0, & \cdots \\ & & \cdot & \cdot & \cdot & \end{pmatrix}$$

$$= \begin{pmatrix} k_{01}, & k_{11}, & 0, & 0, & 0, & \cdots \\ k_{02}, & k_{12}, & k_{22}, & 0, & 0, & \cdots \\ k_{03}, & k_{13}, & k_{23}, & k_{33}, & 0, & \cdots \\ & & \cdot & \cdot & \cdot & \end{pmatrix}. \tag{53.4}$$

Moreover, there is the formal decomposition into a sum of squares:

$$\begin{aligned} \sum_{p,\,q=0}^{\infty} c_{p+q}x_p x_q = {} & (x_0 + k_{01}x_1 + k_{02}x_2 + \cdots)^2 \\ & + a_1(x_1 + k_{12}x_2 + k_{13}x_3 + \cdots)^2 \\ & + a_1 a_2(x_2 + k_{23}x_3 + k_{24}x_4 + \cdots)^2 + \cdots. \end{aligned} \tag{53.5}$$

Conversely, if we have a decomposition (53.5), *where the* a_p *are not zero, then* $P(1/z)$ *is the power series expansion of the J-fraction* (53.1), *where*

$$b_1 = k_{01}, \quad b_{p+1} = k_{p,p+1} - k_{p-1,p}, \quad p = 1, 2, 3, \cdots. \tag{53.6}$$

Finally, the problem of expanding the power series (53.2) *into the J-fraction* (53.1) *is equivalent to the problem of obtaining a power series identity of the form* [18]

$$Q(x + y) = Q(x)Q(y) + a_1Q_1(x)Q_1(y) + a_1a_2Q_2(x)Q_2(y) + \cdots, \tag{53.7}$$

where the a_p *are constants different from zero, and*

$$\begin{aligned} Q(z) &= \sum_{p=0}^{\infty} c_p \frac{z^p}{p!}, \quad (c_0 = 1), \\ Q_r(z) &= \sum_{p=r}^{\infty} k_{r,p} \frac{z^p}{p!}. \end{aligned} \tag{53.8}$$

The coefficients a_p *in the J-fraction are the* a_p *of* (53.7), *and the* b_p *of the J-fraction are given by* (53.6) *in terms of the* $k_{r,s}$ *of* (53.8). **[93.]**

Proof. We remark, in the first place, that the identity (53.7) can be written as

$$\sum_{p,q=0}^{\infty} c_{p+q} \frac{x^p}{p!}\frac{y^q}{q!} = Q(x)Q(y) + a_1Q_1(x)Q_1(y) + a_1a_2Q_2(x)Q_2(y) + \cdots,$$

so that the identity is equivalent to a decomposition (53.5).

Denote the right-hand member of (53.3) by $C_{p,q}$. Then from (53.4) we obtain the relations

$$\begin{aligned} c_{p,q+1} = {} & k_{0,p}(k_{0,q} + a_1k_{1,q}) + a_1k_{1,p}(k_{0,q} + b_2k_{1,q} + a_2k_{2,q}) \\ & + a_1a_2k_{2,p}(k_{1,q} + b_3k_{2,q} + a_3k_{3,q}) + \cdots, \end{aligned}$$

and

$$\begin{aligned} C_{p+1,q} = {} & k_{0,q}(k_{0,p} + a_1k_{1,p}) + a_1k_{1,q}(k_{0,p} + b_2k_{1,p} + a_2k_{2,p}) \\ & + a_1a_2k_{2,q}(k_{1,p} + b_3k_{2,p} + a_3k_{3,p}) + \cdots. \end{aligned}$$

Consequently,

$$\begin{aligned} C_{p,q+1} &= C_{p+1,q} = C_{p+2,q-1} = \cdots = C_{p+q+1,0} \\ &= C_{p-1,q+2} = C_{p-2,q+3} = \cdots = C_{0,p+q+1}. \end{aligned}$$

[18] The formulation by means of this identity was given by Rogers [80].

Thus,

$$C_{p,q} = C_{r,s} \quad \text{if} \quad p + q = r + s,$$

and we are therefore justified in writing $C_{p,q} = C_{p+q}$.

Considering now the bilinear form

$$C = \sum_{p,\,q=0}^{\infty} C_{p+q} x_p y_q,$$

we readily verify that

$$\begin{aligned} C = {} & (k_{0,0}x_0 + k_{0,1}x_1 + k_{0,2}x_2 + \cdots) \\ & \qquad (k_{0,0}y_0 + k_{0,1}y_1 + k_{0,2}y_2 + \cdots) \\ & + a_1(k_{1,1}x_1 + k_{1,2}x_2 + k_{1,3}x_3 + \cdots) \\ & \qquad (k_{1,1}y_1 + k_{1,2}y_2 + k_{1,3}y_3 + \cdots) \\ & + a_1a_2(k_{2,2}x_2 + k_{2,3}x_3 + k_{2,4}x_4 + \cdots) \\ & \qquad (k_{2,2}y_2 + k_{2,3}y_3 + k_{2,4}y_4 + \cdots) \\ & + \cdots. \end{aligned}$$

If we put $x_p = 0$, $y_q = 0$ for $p > n$, $q > n$, we then conclude that the linear transformation

$$\begin{aligned} U_0 &= k_{0,0}x_0 + k_{0,1}x_1 + \cdots + k_{0,n}x_n, \\ U_1 &= \qquad\quad\; k_{1,1}x_1 + \cdots + k_{1,n}x_n, \\ & \cdots \\ U_n &= \qquad\qquad\qquad\qquad\quad k_{n,n}x_n, \\ V_0 &= k_{0,0}y_0 + k_{0,1}y_1 + \cdots + k_{0,n}y_n, \\ V_1 &= \qquad\quad\; k_{1,1}y_1 + \cdots + k_{1,n}y_n, \\ & \cdots \\ V_n &= \qquad\qquad\qquad\qquad\quad k_{n,n}y_n, \end{aligned}$$

carries the bilinear form

$$\sum_{p,\,q=0}^{n} C_{p+q} x_p y_q$$

into the bilinear form

$$\sum_{p=0}^{n} a_0 a_1 \cdots a_p U_p V_p, \quad (a_0 = 1).$$

Since the determinant of the linear transformation is equal to unity, it follows that

$$\Delta_p = \begin{vmatrix} C_0, & C_1, & \cdots, & C_p \\ C_1, & C_2, & \cdots, & C_{p+1} \\ & & \cdot \quad \cdot \quad \cdot & \\ C_p, & C_{p+1}, & \cdots, & C_{2p} \end{vmatrix} = a_0^{p+1}a_1^{p}a_2^{p-1} \cdots a_{p-1}^{2}a_p,$$

and therefore, since the a_p are not zero,

$$\Delta_p = a_0a_1 \cdots a_p\Delta_{p-1}, \quad p = 0, 1, 2, \cdots, (\Delta_{-1} = 1). \quad (53.9)$$

Likewise, if $x_n = y_n = 0$, $x_p = 0$, $y_q = 0$ for $p > n + 1$, $q > n + 1$, we find that

$$\begin{aligned} \Delta_p' &= k_{p,p+1}a_0^{p+1}a_1^{p}a_2^{p-1} \cdots a_{p-1}^{2}a_p \\ &= (a_0a_1 \cdots a_p)(b_1 + b_2 + \cdots + b_{p+1})\Delta_{p-1}, \end{aligned} \quad (53.10)$$

where Δ_p' is the determinant obtained from Δ_p by advancing the subscripts of all the elements of the last row by unity.

Now (53.9) and (53.10) are precisely the relations one obtains by equating the coefficients of $1/z, 1/z^2, \cdots, 1/z^{p+2}$ on either side of (51.3), but with C_p replaced by c_p, $p = 0, 1, 2, \cdots$. Since those relations determine the c_p uniquely in terms of the a_p and b_p (cf. the remark following Theorem 50.1), it follows that $C_p = c_p$, $p = 0, 1, 2, \cdots$. This establishes the relation (53.3).

Let us suppose, finally, that we have a decomposition (53.5) of the quadratic form whose coefficients c_{p+q} are the coefficients in a given power series; and that the a_p are not zero. We can then conclude that (53.9) and (53.10) hold with $C_p = c_p$; and hence that the power series has the J-fraction expansion (53.1) in which the b_p are defined by (53.6). This completes the proof of Theorem 53.1.

Example. We shall use this theorem to obtain the expansion [**93, 80**]

$$\int_0^\infty \operatorname{sech}^k u \, e^{-zu} du = \cfrac{1}{z + \cfrac{1 \cdot k}{z + \cfrac{2(k+1)}{z + \cfrac{3(k+2)}{z + \cdots}}}} \quad (53.11)$$

Consider the identity

$$\begin{aligned}
\operatorname{sech}^k (x + y) &= (\cosh x \cosh y + \sinh x \sinh y)^{-k} \\
&= \operatorname{sech}^k x \operatorname{sech}^k y - k \operatorname{sech}^{k+1} x \sinh x \operatorname{sech}^{k+1} y \sinh y \\
&\quad + \frac{k(k+1)}{1 \cdot 2} \operatorname{sech}^{k+2} x \sinh^2 x \operatorname{sech}^{k+2} y \sinh^2 y - \cdots \\
&= Q(x)Q(y) + a_1 Q_1(x) Q_1(y) + a_1 a_2 Q_2(x) Q_2(y) + \cdots,
\end{aligned}$$

where

$$Q_p(z) = \frac{\operatorname{sech}^{k+p} z \sinh^p z}{p!} = \frac{z^p}{p!} + k_{p,p+1} \frac{z^{p+1}}{(p+1)!} + \cdots,$$

$$a_p = -p(k + p - 1), \quad k_{p,p+1} = 0.$$

The expansion (53.11) now results immediately by application of the last part of Theorem 53.1.

The formula (53.4) furnishes what is perhaps the most convenient means for expanding a J-fraction into a power series. By way of illustration, let us find the first few coefficients in the power series for

$$\cfrac{1}{z + 1 - \cfrac{1^2 t}{z + t + 2 - \cfrac{2^2 t}{z + 2t + 3 - \cfrac{3^2 t}{z + 3t + 4 - \ddots}}}}$$

We find for (53.4), (to three rows),

$$\begin{pmatrix} 1, & 0, & 0, & \cdots \\ 1, & 1, & 0, & \cdots \\ t+1, & t+3, & 1, & \cdots \\ & \cdot\ \cdot\ \cdot & & \end{pmatrix} \begin{pmatrix} 1, & 1, & 0, & 0, & \cdots \\ t, & t+2, & 1, & 0, & \cdots \\ 0, & 4t, & 2t+3, & 1, & \cdots \\ & & \cdot\ \cdot\ \cdot & & \end{pmatrix}$$

$$= \begin{pmatrix} 1, & 1, & 0, & 0, & \cdots \\ t+1, & t+3, & 1, & 0, & \cdots \\ t^2 + 4t + 1, & t^2 + 10t + 7, & 3t + 6, & 1, & \cdots \\ & \cdot\ \cdot\ \cdot & & & \end{pmatrix}.$$

Then, by (53.3) we get immediately,

$$c_0 = 1, \qquad c_3 = t^2 + 4t + 1,$$

$$c_1 = 1, \qquad c_4 = t^3 + 11t^2 + 11t + 1,$$

$$c_2 = t + 1, \quad c_5 = t^4 + 26t^3 + 66t^2 + 26t + 1,$$

$$c_6 = t^5 + 57t^4 + 302t^3 + 302t^2 + 57t + 1.$$

54. Convergence Questions. There is one important case where we can assert that the power series $P(1/z)$ is equal to the J-fraction or S-fraction of which it is the expansion. It will be convenient to replace z by $1/z$ and then divide by z, in the power series and the J-fraction or S-fraction. The J-fraction takes the form

$$\cfrac{a_0}{b_1 z + 1 - \cfrac{a_1 z^2}{b_2 z + 1 - \cfrac{a_2 z^2}{b_3 z + 1 - \cdots}}}; \tag{54.1}$$

the S-fraction becomes

$$\cfrac{a_0}{1 - \cfrac{a_1 z}{1 - \cfrac{a_2 z}{1 - \cdots}}}; \tag{54.2}$$

while the power series becomes

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots. \tag{54.3}$$

THEOREM 54.1. *If the J-fraction* (54.1) *converges uniformly for* $|z| \leq M$, *then its power series expansion* (54.3) *has radius of convergence at least equal to* M, *and the sum of the series is the value of the J-fraction. The corresponding statement holds for S-fractions* [**111**].

Proof. The approximants of the J-fraction or S-fraction form a sequence of rational functions $f_p(z)$, converging uniformly for $|z| \leq M$. There must exist an index k such that for $p \geq k$, the power series expansion of $f_p(z)$ in ascending powers of z converges for $|z| \leq M$. Put

$$u_1(z) = f_k(z), \quad u_p(z) = f_{k+p-1}(z) - f_{k+p-2}(z), \quad p = 2, 3, 4, \cdots.$$

Then

$$\sum_{p=1}^{\infty} u_p(z) = \lim_{p=\infty} f_p(z) = u(z),$$

uniformly for $|z| \leq M$, where $u(z)$ is an analytic function for $|z| < M$. By a theorem of Weierstrass, the series of nth derivatives

$$\sum_{p=1}^{\infty} u_p^{(n)}(z)$$

converges for $|z| < M$ to the sum $u^{(n)}(z)$. Now, since the expansion in powers of z of $f_p(z)$ agrees term by term with the series $P(z)$ for a number of consecutive terms beginning with the first, which increases to ∞ with p, it follows that

$$\sum_{p=1}^{\infty} u_p^{(n)}(0) = n!c_n, \quad n = 0, 1, 2, \cdots,$$

and therefore, if $|z| < M$,

$$u(z) = \sum_{p=0}^{\infty} \frac{u^{(p)}(0)}{p!} z^p = \sum_{p=0}^{\infty} c_p z^p = P(z),$$

as was to be proved.

THEOREM 54.2. *If*

$$\lim_{p=\infty} a_p = 0, \quad (a_p \neq 0), \tag{54.4}$$

then the S-fraction (54.2) *converges to a meromorphic function of* z. *The convergence is uniform over every closed bounded region containing none of the poles of this function. If*

$$\lim_{p=\infty} a_p = a \neq 0, \tag{54.5}$$

then the S-fraction (54.2) *converges in the domain exterior to the rectilinear cut running from* $1/4a$ *to* ∞ *in the direction of the vector from* 0 *to* $1/4a$, *to a function having at most polar singularities in this domain. The con-*

vergence is uniform over every closed bounded region exterior to the cut which contains no poles of the function[19] **[111, 110]**.

Remark. This theorem is due to E. B. Van Vleck. Stieltjes **[95]** proved that if the a_p are real and positive, then (54.2) converges to a meromorphic function *if, and only if*, (54.4) holds. One can find examples to show that this conclusion does not hold if the a_p are not restricted to be real and positive **[121, 122]**.

Proof of Theorem 54.2. Let G be a bounded closed region containing the origin on the interior, which is otherwise arbitrary in case (54.4) holds, and which is exterior to the specified cut in case (54.5) holds. By Theorem 33.1, there exists an index N, depending only upon G, such that if $n > N$, then the continued fraction

$$\cfrac{a_n z}{1 - \cfrac{a_{n+1} z}{1 - \cfrac{a_{n+2} z}{1 - \cdots}}} \tag{54.6}$$

converges uniformly over G to an analytic function $F_n(z)$. Since G contains the origin on the interior, the power series expansion of (54.6) converges in the neighborhood of the origin to $F_n(z)$, by Theorem 54.1. Hence, by Theorem 52.2, $F_n(z)$ is not a rational function of z. We then conclude that (54.2) converges over G to the function

$$\frac{A_n(z) - F_n(z)A_{n-1}(z)}{B_n(z) - F_n(z)B_{n-1}(z)},$$

which is analytic over G except possibly for poles, inasmuch as $B_n(z) - F_n(z)B_{n-1}(z)$ cannot be identically zero. The convergence is clearly uniform over the region obtained from G by removing the interiors of small circles with centers at the poles of the function. This completes the proof of the theorem.

Exercise 11

11.1. Let $a_{3p} = c_p \neq 0$, $\lim c_p = 0$, $a_{3p+1} = -a_{2p-1} = a \neq 0$. Then the continued fraction (54.2) converges to a meromorphic function of z **[121]**.

[19] The continued fraction converges uniformly in the neighborhood of the origin (cf. end of § 10), and is equal to its power series expansion by Theorem 54.1.

11.2. Let $x_1, x_2, \cdots, x_m$ be real numbers such that $x_1 < x_2 < \cdots < x_m$, and suppose that $M_p > 0$, $p = 1, 2, 3, \cdots, m$. Put

$$\sum_{p=1}^{m} \frac{M_p}{z - x_p} = \cfrac{a_0}{z + b_1 - \cfrac{a_1}{z + b_2 - \cdots - \cfrac{a_{m-1}}{z + b_m}}}.$$

The polynomial $P_n(z)$ which renders the sum $\sum_{p=1}^{m} [f(x_p) - P_n(x_p)]^2 M_p$ a minimum is given by

$$P_n(z) = \frac{1}{a_0 a_1 \cdots a_n} \sum_{p=1}^{m} \frac{B_{n+1}(z)B_n(x_p) - B_{n+1}(x_p)B_n(z)}{z - x_p} f(x_p). \quad \text{[105, 106.]}$$

11.3. Let $x_p = x_1 + (p - 1)d$, $M_p = 1$, $p = 1, 2, 3, \cdots, m$, $d \neq 0$. Then the constants a_p and b_p are given by

$$a_0 = md, \quad a_p = \frac{p^2 d^2 (m^2 - p^2)}{4(4p^2 - 1)}, \quad b_p = -(x_1 + \tfrac{1}{2}(m - 1)d). \quad \text{[94.]}$$

11.4. If $x_1 = -k$, $x_2 = -k + 2$, $x_3 = -k + 4$, $\cdots$, $x_{k+1} = +k$, k a positive integer, and $M_p = \binom{k}{p-1} = k!/(p - 1)!(k - p + 1)!$, then

$$a_0 = 2, \quad a_p = p(k - p + 1), \quad b_p = 0, \quad p = 1, 2, 3, \cdots, k + 1. \quad \text{[93.]}$$

11.5. Obtain the formal expansion

$$\int_0^\infty cnue^{-zu}du = \cfrac{1}{z + \cfrac{1^2}{z + \cfrac{2^2k^2}{z + \cfrac{3^2}{z + \cfrac{4^2k^2}{z + \cfrac{5^2}{z + \cdots}}}}}}. \quad \text{[93.]}$$

Chapter XII

MATRIX THEORY OF CONTINUED FRACTIONS

Up to this point we have regarded the J-fraction as being generated by a sequence of linear fractional transformations in *one* variable. We may also regard the J-fraction as arising from a single linear transformation in *infinitely many* variables. In fact, let us consider the system of linear equations

$$\begin{aligned}(b_1 + z)\xi_1 - a_1\xi_2 \qquad\qquad\qquad\qquad &= \eta_1,\\ -a_1\xi_1 + (b_2 + z)\xi_2 - a_2\xi_3 \qquad\qquad &= \eta_2,\\ -a_2\xi_2 + (b_3 + z)\xi_3 - a_3\xi_4 &= \eta_3,\\ \cdots\cdot \qquad\qquad\qquad\qquad&\end{aligned} \tag{a}$$

This may be looked upon as a transformation of the point $\xi = (\xi_1, \xi_2, \xi_3, \cdots)$ in the space of infinitely many variables into the point $\eta = (\eta_1, \eta_2, \eta_3, \cdots)$ of the space. If $\eta_1 = 1$, $\eta_p = 0$ for $p > 1$, we may write the equations in the form

$$\xi_1 = \cfrac{1}{b_1 + z - \cfrac{a_1\xi_2}{\xi_1}}, \quad \frac{a_1\xi_2}{\xi_1} = \cfrac{a_1^2}{b_2 + z - \cfrac{a_2\xi_3}{\xi_2}},$$

$$\frac{a_2\xi_3}{\xi_2} = \cfrac{a_2^2}{b_3 + z - \cfrac{a_3\xi_4}{\xi_3}} \qquad \cdots.$$

If we then substitute in succession from each into the preceding, we obtain the formal expansion of ξ_1 into a J-fraction

$$\xi_1 = \cfrac{1}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cfrac{a_2^2}{b_3 + z - \cdot_{\cdot_{\cdot}}}}}$$

Suppose now that we have an "inverse" of the transformation (a),

$$\begin{aligned} \rho_{11}\eta_1 + \rho_{12}\eta_2 + \rho_{13}\eta_3 + \cdots &= \xi_1, \\ \rho_{21}\eta_1 + \rho_{22}\eta_2 + \rho_{23}\eta_3 + \cdots &= \xi_2, \\ \rho_{31}\eta_1 + \rho_{32}\eta_2 + \rho_{33}\eta_3 + \cdots &= \xi_3, \\ \cdot \quad \cdot \quad \cdot_{\cdot} \end{aligned} \tag{b}$$

On setting $\eta_1 = 1$, $\eta_p = 0$ for $p = 2, 3, 4, \cdots$, we then see that $\xi_1 = \rho_{11}$. We shall call the matrix (ρ_{pq}) a **right reciprocal** of the matrix J of the transformation (a). *From the preceding we now see that the leading coefficient, ρ_{11}, in a right reciprocal of the matrix J is formally equal to the J-fraction.*

To make this a little more precise, we may understand that (b) is an inverse of (a) if, when the values of the ξ_p from (b) are substituted in (a), the latter is formally satisfied. On making this substitution, we find that the ρ_{pq} can be any numbers such that

$$-a_{p-1}\rho_{p-1,q} + (b_p + z)\rho_{p,q} - a_p\rho_{p+1,q} = \delta_{p,q},$$

where $\delta_{p,q}$ is the **Kronecker delta,** and is equal to zero or unity according as $p \neq q$ or $p = q$, respectively. Here we agree to take $a_0 = 0$. Since $a_1, a_2, a_3, \cdots$ are not zero, it follows that, for a fixed q, if $\rho_{1,q}$ is chosen *arbitrarily*, then $\rho_{p,q}$, $p = 2, 3, 4, \cdots$, are uniquely determined. *Therefore, there are infinitely many different right reciprocals.* This makes it obvious that one must restrict in some way the class of right reciprocals to be considered in order that the statement "ρ_{11} is formally equal to the J-fraction" shall have any significance.

The principal goal of the present chapter is to investigate the right reciprocals of the J-matrix, and to determine those which have an essential relationship to the J-fraction. To that end, we first develop some of the fundamental ideas of the **matrix calculus.**

The introduction to the matrix calculus given here is based upon notes of a seminar directed by Ernst Hellinger. The principal references are Hellinger [**30, 31**], Hellinger and Toeplitz [**32, 33, 34**], Hilbert [**37**], and Stone [**97**].

55. Linear Forms. The expression $\sum_{p=1}^{\infty} a_p x_p$, in which the a_p are given constants and the x_p are independent variables, is called a **linear form.** The linear form is called **bounded** if there exists a constant N such that for every positive integer n and every system of values x_p,

$$\left| \sum_{p=1}^{n} a_p x_p \right| \leq N \cdot \sqrt{\sum_{p=1}^{n} | x_p |^2}, \tag{55.1}$$

where N is independent of n and of the variables x_p. The smallest value of N for which (55.1) holds is called the **norm** of the linear form.

THEOREM 55.1. *The linear form* $\sum_{p=1}^{\infty} a_p x_p$ *is bounded if, and only if, the infinite series* $\sum_{p=1}^{\infty} | a_p |^2$ *is convergent. If the linear form is bounded, then its norm is given by*

$$N = \sqrt{\sum_{p=1}^{\infty} | a_p |^2}. \tag{55.2}$$

The infinite series

$$\sum_{p=1}^{\infty} a_p x_p$$

converges absolutely and uniformly in any domain of the variables such that

$$\sum_{p=1}^{\infty} | x_p |^2 \leq r^2,$$

where r^2 *is a positive constant.*

Proof. If (55.1) holds, then on substituting $x_p = \bar{a}_p$ in that inequality we obtain

$$\sum_{p=1}^{n} |a_p|^2 \leq N \cdot \sqrt{\sum_{p=1}^{n} |a_p|^2},$$

so that

$$\sum_{p=1}^{n} |a_p|^2 \leq N^2,$$

and consequently the series

$$\sum_{p=1}^{\infty} |a_p|^2$$

converges, and its sum does not exceed N^2. Conversely, if this series converges, then, by Schwarz's inequality,

$$\left| \sum_{p=1}^{n} a_p x_p \right| \leq \sqrt{\sum_{p=1}^{n} |a_p|^2} \sqrt{\sum_{p=1}^{n} |x_p|^2},$$

so that (55.1) holds when N has the value (55.2), and it cannot hold if N has a smaller value. Again, by Schwarz's inequality,

$$\left| \sum_{p=m}^{m+n} a_p x_p \right| \leq r \sqrt{\sum_{p=m}^{m+n} |a_p|^2},$$

if $\sum_{p=1}^{\infty} |x_p|^2 \leq r^2$. Hence, when the linear form is bounded, the series $\sum_{p=1}^{\infty} a_p x_p$ converges absolutely and uniformly in the prescribed domain of the variables.

A sequence $x_1, x_2, x_3, \cdots$ such that $\sum_{p=1}^{\infty} |x_p|^2$ converges, is called a **point of Hilbert space,** and may be conveniently designated by the single letter x. The Hilbert space is denoted by $\mathfrak{H}$; the set of points x such that $\sum_{p=1}^{\infty} |x_p|^2 \leq r^2$ is the **Hilbert sphere of radius** r, and is denoted by $\mathfrak{H}_r$. The *value* of the linear form $\sum_{p=1}^{\infty} a_p x_p$ is the sum of this infinite series in case it converges. A bounded linear form has a value for every point of $\mathfrak{H}$; its norm is the least upper bound of its absolute value for all x in $\mathfrak{H}_1$, and is actually attained by the form for at least one point of $\mathfrak{H}_1$.

56. Bilinear Forms. The expression

$$A(x, y) = \sum_{p,q=1}^{\infty} a_{pq} x_p y_q,$$

in which the a_{pq} are given constants forming the infinite matrix $A = (a_{pq})$, is called a **bilinear form** in the infinitely many variables x_p, y_q. It is called **bounded** if there exists a number N such that for every positive integer n and every system of values x_p and y_q,

$$\left| \sum_{p,q=1}^{n} a_{pq} x_p y_q \right| \leq N \cdot \sqrt{\sum_{p=1}^{n} | x_p |^2 \sum_{q=1}^{n} | y_q |^2}, \qquad (56.1)$$

where N is independent of n and of the values x_p, y_q.

It is easily seen by application of Schwarz's inequality that the convergence of the double series $\Sigma | a_{pq} |^2$ is sufficient for the bilinear form to be bounded. That this is not a necessary condition is seen, for instance, from the fact that the **unit form**

$$I(x, y) = \sum_{p,q=1}^{\infty} \delta_{pq} x_p y_q = \sum_{p=1}^{\infty} x_p y_p \qquad (56.2)$$

is bounded, although the double series $\Sigma \delta_{pq}{}^2$ is divergent.

THEOREM 56.1. *A necessary condition for $A(x, y)$ to be bounded is the convergence of the series*

$$\sum_{p=1}^{\infty} | a_{pq} |^2, \quad q = 1, 2, 3, \cdots, \quad \textit{and} \quad \sum_{q=1}^{\infty} | a_{pq} |^2,$$
$$p = 1, 2, 3, \cdots. \qquad (56.3)$$

Proof. For a fixed q, let $y_q = 1$ and let $y_p = 0$ for $p \neq q$. Then

$$A(x, y) = \sum_{p=1}^{\infty} a_{pq} x_p$$

is a linear form which, by Theorem 55.1, is not bounded unless the series $\sum_{p=1}^{\infty} | a_{pq} |^2$ is convergent. It follows, a fortiori, that the convergence of this series is necessary for the bilinear form to be bounded. Similarly, the second series in (56.3) must be convergent for each fixed p. That the condition is not sufficient may be seen from the example $a_{pq} = p \cdot \delta_{pq}$.

THEOREM 56.2. *A bounded bilinear form converges for each x and y in* $\mathfrak{H}$, *by rows, that is, for each p, the series*

$$\sum_{q=1}^{\infty} a_{pq} y_q \tag{56.4}$$

converges, and the series

$$\sum_{p=1}^{\infty} \left(\sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \tag{56.5}$$

converges. The sum of the latter series, which will be denoted by $A(x, y)$, *and called the value of the bilinear form, satisfies the inequality*

$$| A(x, y) | \leq N \sqrt{\sum_{p=1}^{\infty} | x_p |^2 \sum_{q=1}^{\infty} | y_q |^2}, \tag{56.6}$$

where N is the constant appearing in (56.1).

Proof. The convergence of the series (56.4) follows at once by Schwarz's inequality and Theorem 56.1. Let m and n be two positive integers such that $n > m$. Then by (56.1) we have:

$$\left| \sum_{p=m}^{n} \left(\sum_{q=1}^{k} a_{pq} y_q \right) x_p \right| \leq N \sqrt{\sum_{p=m}^{n} | x_p |^2 \sum_{q=1}^{k} | y_q |^2},$$

$$k = 1, 2, 3, \cdots,$$

and therefore, by the convergence of (56.4),

$$\left| \sum_{p=m}^{n} \left(\sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right| \leq N \sqrt{\sum_{p=m}^{\infty} | x_p |^2 \sum_{q=1}^{\infty} | y_q |^2}. \tag{56.7}$$

Since x is in $\mathfrak{H}$, the right-hand member of this inequality approaches zero as m tends to ∞, so that (56.5) converges. On putting m equal to unity, and then letting n tend to ∞ in (56.7), we then obtain (56.6).

THEOREM 56.3. *A bounded bilinear form converges by segments for each x and y in $\mathfrak{H}$, to the value $A(x, y)$ defined in Theorem 56.2, that is:*

$$\lim_{n=\infty} \sum_{p,q=1}^{n} a_{pq}x_p y_q = A(x, y) = \sum_{p=1}^{\infty}\left(\sum_{q=1}^{\infty} a_{pq} y_q\right) x_p. \tag{56.8}$$

Proof. Put

$$A_n(x, y) = \sum_{p,q=1}^{n} a_{pq}x_p y_q, \quad S_n(x, y) = \sum_{p=1}^{n}\left(\sum_{q=1}^{\infty} a_{pq} y_q\right) x_p.$$

Then,

$$S_n(x, y) - A_n(x, y) = \sum_{p=1}^{n}\left(\sum_{q=n+1}^{\infty} a_{pq} y_q\right) x_p,$$

which is the bilinear form $A(x, y)$ evaluated for $x_{n+1} = x_{n+2} = \cdots = 0, y_1 = y_2 = \cdots = y_n = 0$, and therefore

$$| S_n(x, y) - A_n(x, y) | \leq N \sqrt{\sum_{p=1}^{n} | x_p |^2 \sum_{q=n+1}^{\infty} | y_q |^2}.$$

Since the right-hand member of this inequality approaches zero as n tends to ∞, and since, by Theorem 56.2,

$$\lim_{n=\infty} S_n(x, y) = A(x, y),$$

it follows that

$$\lim_{n=\infty} A_n(x, y) = A(x, y).$$

As an immediate corollary we have

THEOREM 56.4. *The summation of a bounded bilinear form by rows, by columns, or by segments, give one and the same value $A(x, y)$ when x and y are points of $\mathfrak{H}$.*

57. Bounded Matrices. A matrix

$$\mathbf{A} = (a_{pq}) = \begin{pmatrix} a_{11}, & a_{12}, & a_{13}, & \cdots \\ a_{21}, & a_{22}, & a_{23}, & \cdots \\ a_{31}, & a_{32}, & a_{33}, & \cdots \\ & \cdot & \cdot & \cdot \end{pmatrix}$$

is called **bounded** if the bilinear form $A(x, y) = \Sigma a_{pq}x_p y_q$ is bounded in the sense of (56.1). By the **norm** of the matrix we shall understand the least number N which can be used in (56.1).

If c is any number, then $c\mathbf{A}$ is the matrix (ca_{pq}); and if $\mathbf{B} = (b_{pq})$, then the *sum* $\mathbf{A} + \mathbf{B}$ is the matrix $(a_{pq} + b_{pq})$. One sees immediately that if $\mathbf{A}$ is bounded, then $c\mathbf{A}$ is bounded; and that if $\mathbf{A}$ and $\mathbf{B}$ are bounded, then $\mathbf{A} + \mathbf{B}$ is bounded. We have:

$$N_{c\mathbf{A}} = |c| \cdot N_{\mathbf{A}} \quad N_{\mathbf{A}+\mathbf{B}} \leq N_{\mathbf{A}} + N_{\mathbf{B}},$$

where $N_{\mathbf{X}}$ denotes the norm of the matrix $\mathbf{X}$.

The *product* $\mathbf{AB}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is the matrix $\mathbf{C} = (c_{pq})$, where

$$c_{pq} = \sum_{r=1}^{\infty} a_{pr}b_{rq}, \quad p, q = 1, 2, 3, \cdots,$$

provided all these series are convergent. Otherwise, the product is undefined.

THEOREM 57.1. *The product of two bounded matrices exists and is bounded. Moreover, the norm of the product is not greater than the product of the norms of the factors.*

Proof. It is required to show that the bilinear form

$$\sum_{p,q=1}^{\infty} c_{pq}x_p y_q, \quad \text{where} \quad c_{pq} = \sum_{r=1}^{\infty} a_{pr}b_{rq},$$

is bounded when (a_{pq}) and (b_{pq}) are bounded matrices. By Theorem 56.1 and Schwarz's inequality, the series defining c_{pq} is *absolutely convergent*. We may therefore write

$$\sum_{p,q=1}^{n} c_{pq}x_p y_q = \sum_{p,q=1}^{n} \left(\sum_{r=1}^{\infty} a_{pr}b_{rq} \right) x_p y_q$$
$$= \sum_{r=1}^{\infty} \left(\sum_{p=1}^{n} a_{pr}x_p \right) \left(\sum_{q=1}^{n} b_{rq}y_q \right),$$

so that, by Schwarz's inequality,

$$\left| \sum_{p,q=1}^{n} c_{pq}x_p y_q \right| \leq \sqrt{\sum_{r=1}^{\infty} \left| \sum_{p=1}^{n} a_{pr}x_p \right|^2 \sum_{r=1}^{\infty} \left| \sum_{q=1}^{n} b_{rq}y_q \right|^2}. \tag{57.1}$$

Put $e_r = \sum_{p=1}^{n} a_{pr}x_p$. Since $\mathbf{A}$ is bounded we then have:

$$\left| \sum_{r=1}^{m} e_r w_r \right| = \left| \sum_{r=1}^{m} \left(\sum_{p=1}^{n} a_{pr}x_p \right) w_r \right| \leq N_{\mathbf{A}} \sqrt{\sum_{p=1}^{n} |x_p|^2 \sum_{r=1}^{m} |w_r|^2},$$

where $N_{\mathbf{A}}$ is the norm of **A**. On putting $w_r = \bar{e}_r$ in this inequality, we get

$$\sqrt{\sum_{r=1}^{m}\left|\sum_{p=1}^{n} a_{pr}x_p\right|^2} \leq N_{\mathbf{A}}\sqrt{\sum_{p=1}^{n}|x_p|^2}.$$

For the same reasons,

$$\sqrt{\sum_{r=1}^{m}\left|\sum_{q=1}^{n} b_{rq}y_q\right|^2} \leq N_{\mathbf{B}}\sqrt{\sum_{q=1}^{n}|y_q|^2}.$$

On letting m tend to ∞ in these inequalities, and then substituting from the resulting inequalities into (57.1) we then have:

$$\left|\sum_{p,\,q=1}^{n} c_{pq}x_p y_q\right| \leq N_{\mathbf{A}}N_{\mathbf{B}}\sqrt{\sum_{p=1}^{n}|x_p|^2\sum_{q=1}^{n}|y_q|^2},$$

which establishes the theorem.

If (a_{pq}) is a bounded matrix, the transformation

$$y_p = \sum_{q=1}^{\infty} a_{pq}x_q, \quad p = 1, 2, 3, \cdots, \tag{57.2}$$

has the property that if x is in $\mathfrak{H}$ then y is in $\mathfrak{H}$. In fact, as in the preceding proof,

$$\left|\sum_{p=1}^{n}\left(\sum_{q=1}^{m} a_{pq}x_q\right)w_p\right| \leq N_{\mathbf{A}}\sqrt{\sum_{p=1}^{n}|w_p|^2\sum_{q=1}^{m}|x_q|^2},$$

so that, on letting m increase to ∞, we have:

$$\left|\sum_{p=1}^{n} y_p w_p\right| \leq N_{\mathbf{A}}\sqrt{\sum_{q=1}^{\infty}|x_q|^2\sum_{p=1}^{n}|w_p|^2}.$$

Thus, $\sum_{p=1}^{\infty} y_p w_p$ is a bounded linear form, and therefore $\sum_{p=1}^{\infty}|y_p|^2$ converges. Moreover,

$$\sum_{p=1}^{\infty}|y_p|^2 \leq N_{\mathbf{A}}^2\sum_{q=1}^{\infty}|x_q|^2. \tag{57.3}$$

The transformation (57.2) which takes points of $\mathfrak{H}$ into points of $\mathfrak{H}$ is called a **bounded transformation.**

THEOREM 57.2. *The bilinear form whose matrix is the product of two bounded matrices* (a_{pq}) *and* (b_{pq}) *in this order has the value*

$$\sum_{r=1}^{\infty}\left(\sum_{p=1}^{\infty}a_{pr}x_p\right)\left(\sum_{q=1}^{\infty}b_{rq}y_q\right). \tag{57.4}$$

Proof. Let $C(x, y)$ be the bilinear form whose matrix is $(a_{pq})(b_{pq}) = (c_{pq})$. Then, by the definition of a bilinear form,

$$C(x, y) = \sum_{p,\,q=1}^{\infty}c_{pq}x_py_q = \sum_{p=1}^{\infty}x_p\left(\sum_{q=1}^{\infty}y_qc_{pq}\right)$$
$$= \sum_{p=1}^{\infty}x_p\sum_{q=1}^{\infty}y_q\left(\sum_{r=1}^{\infty}a_{pr}b_{rq}\right).$$

The coefficient of x_p in this last expression is the value of the bilinear form $B(x, y)$ with matrix (b_{pq}), where $x_r = a_{pr}$, the summation being by columns. On summing the same by rows, we get

$$C(x, y) = \sum_{p=1}^{\infty}x_p\sum_{r=1}^{\infty}a_{pr}\left(\sum_{q=1}^{\infty}b_{rq}y_q\right).$$

But this is the value of the bilinear form $A(x, y)$ with matrix (a_{pq}) and with $y_r = \sum_{q=1}^{\infty}b_{rq}y_q$, the same being summed by rows. On summing by columns we then have:

$$C(x, y) = \sum_{r=1}^{\infty}\left(\sum_{q=1}^{\infty}b_{rq}y_q\right)\left(\sum_{p=1}^{\infty}a_{pr}x_p\right),$$

as was to be proved.

We shall now prove that the bilinear form with matrix **A**(**BC**) is equal to the bilinear form with matrix (**AB**)**C**, when **A**, **B**, and **C** are bounded matrices. *That is to say, multiplication of bounded matrices satisfies the associative law.* It is required to show that

$$\sum_{r=1}^{\infty}a_{pr}\left(\sum_{s=1}^{\infty}b_{rs}c_{sq}\right) = \sum_{s=1}^{\infty}\left(\sum_{r=1}^{\infty}a_{pr}b_{rs}\right)c_{sq},$$
$$p, q = 1, 2, 3, \cdots. \tag{57.5}$$

Now, the left-hand member of this proposed equality is the value of the bilinear form $B(x, y)$ with $x_r = a_{pr}, y_s = c_{sq}$, the summation being by rows, while the right-hand member is the same but with the summation being by columns. It follows that (57.5) is a true equality.

A matrix (a_{pq}) is called **symmetric** if $a_{pq} = a_{qp}$, for $p, q = 1, 2, 3, \cdots$.

THEOREM 57.3. *A real symmetric matrix (a_{pq}) is bounded if for each real point x in $\mathfrak{H}$:*

$$\left| \sum_{p,q=1}^{n} a_{pq} x_p x_q \right| \leq N \sum_{p=1}^{n} x_p^2, \quad n = 1, 2, 3, \cdots, \tag{57.6}$$

where N is a constant independent of n and of the x_p; and the norm relative to real $\mathfrak{H}$ is equal to the least value of N which can be used in (57.6).

Proof. If x and y are real points of $\mathfrak{H}$, then for the nth segment $A_n(x, y)$ of $A(x, y)$, we have:

$$A_n(x, y) = \tfrac{1}{4}[A_n(x + y, x + y) - A_n(x - y, x - y)],$$

so that, if x and y are in $\mathfrak{H}_1$,

$$| A_n(x, y) | \leq \frac{N}{4}\left(\sum_{p=1}^{n}(x_p + y_p)^2 + \sum_{p=1}^{n}(x_p - y_p)^2\right)$$
$$= \frac{N}{4} \cdot 2\left(\sum_{p=1}^{n} x_p^2 + \sum_{p=1}^{n} y_p^2\right) \leq N.$$

Therefore, $A(x, y)$ is bounded relative to **real** $\mathfrak{H}$, and its norm relative to real $\mathfrak{H}$ is the smallest number N that can be used in (57.6). It is also bounded relative to complex $\mathfrak{H}$. For, if $x_p = x_p' + i x_p''$, $y_q = y_q' + i y_q''$, then

$$A_n(x, y) = A_n(x', y') - A_n(x'', y'') + iA_n(x', y'') + iA_n(x'', y'),$$

so that, by the above inequality,

$$| A_n(x, y) | \leq \frac{N}{2} \sum_{p=1}^{n}(x_p'^2 + y_p'^2 + x_p''^2 + y_p''^2) \cdot 2 \leq 2N,$$

if x and y are in $\mathfrak{H}_1$.

If $A(x, y)$ is a bilinear form whose matrix is symmetric, then $A(x, x)$ is called a **quadratic form**; it is called **bounded** if its matrix is bounded. The value of a bounded **real** quadratic form is the sum of the double series

$$A(x, x) = \sum_{p, q=1}^{\infty} a_{pq}x_p x_q,$$

x being a **real** point of H.

58. Bounded Reciprocals of Bounded Matrices. If $\mathbf{AB} = \mathbf{I}$, where $\mathbf{I} = (\delta_{pq})$ is the unit matrix, then $\mathbf{B}$ is called a **right reciprocal** of $\mathbf{A}$; and if $\mathbf{CA} = \mathbf{I}$, then $\mathbf{C}$ is called a **left reciprocal** of $\mathbf{A}$. If $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{B}$ is called a **reciprocal** of $\mathbf{A}$.

THEOREM 58.1. *If* $\mathbf{A}$ *is a bounded matrix having a bounded right reciprocal* $\mathbf{B}$, *then the system of equations*

$$\sum_{q=1}^{\infty} a_{pq}x_q = y_p, \quad p = 1, 2, 3, \cdots, \tag{58.1}$$

has at least one solution x in $\mathfrak{H}$ for each y in $\mathfrak{H}$; and if there is a bounded left reciprocal $\mathbf{C}$, *then the system of equations* (58.1) *has at most one solution x in $\mathfrak{H}$ for each y in $\mathfrak{H}$.*

Proof. It will be convenient here and in subsequent developments to write systems of linear equations such as (58.1) as a single matrix equation. We represent a point x as a one-column matrix:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

The system of equations (58.1) is then equivalent to the single matrix equation $\mathbf{Ax} = \mathbf{y}$. Suppose that $\mathbf{A}$ has a bounded right reciprocal $\mathbf{B}$; we see by direct substitution that $\mathbf{x} = \mathbf{By}$ constitutes a bounded transformation of y into x, so that (58.1) has at least one solution x for each y in $\mathfrak{H}$. Suppose now that (58.1) has a solution x in $\mathfrak{H}$ when y is a given point of $\mathfrak{H}$, and that $\mathbf{A}$ has a bounded left reciprocal $\mathbf{C}$. We then have:

$$\mathbf{Ax} = \mathbf{y}, \quad \mathbf{C}(\mathbf{Ax}) = \mathbf{Cy}, \quad (\mathbf{CA})\mathbf{x} = \mathbf{Cy}, \quad \mathbf{x} = \mathbf{Cy}.$$

Thus, if there is one solution, there is *only* one solution.

Theorem 58.2. *If a bounded matrix* **A** *has a bounded left reciprocal* **C**, *and a bounded right reciprocal* **B**, *then* **B** = **C**, *and the equation* **Ax** = **y** *has one and only one solution x in* $\mathfrak{H}$ *for each y in* $\mathfrak{H}$.

Proof. We have: **AB** = **I** and **CA** = **I**. On multiplying the first of these equations on the left by **C** and the second on the right by **B**, we then have: **C**(**AB**) = **C**, (**CA**)**B** = **C**(**AB**) = **B**, so that **B** = **C**. By Theorem 58.1, the equation **Ax** = **y** has at least one solution x in $\mathfrak{H}$ for each y in $\mathfrak{H}$, and at the same time it has at most one. Therefore, it has *just* one solution x in $\mathfrak{H}$ for each y in $\mathfrak{H}$.

Theorem 58.3. *For a bounded symmetric matrix* **A**, *there are two and only two possibilities, namely:*

(a) **A** *has neither a right nor a left bounded reciprocal;*

(b) **A** *has a unique bounded reciprocal* $\mathbf{A}^{-1}$ *such that*

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

Proof. If **A** has a bounded right reciprocal **B**, then **AB** = **I**. If we denote by **X**′ the **transpose** of a matrix **X**, i.e., the matrix obtained from **X** by interchanging corresponding rows and col umns, then (**AB**)′ = **B**′**A**′ = **I**′ = **I**, or **B**′**A** = **I**, since **A** is symmetric. Hence, by Theorem 58.2, **B** = **B**′. Similarly, if **A** has a bounded left reciprocal, then it has a unique bounded reciprocal. There is but one other possibility, namely, (a).

Theorem 58.4. *If a bounded matrix* **A** *has a unique right (left) bounded reciprocal, then this is the reciprocal.*

Proof. Let **B** be the unique right reciprocal of **A**. Then, **AB** = **I**. Consider **BA** − **I**. Since **A**(**BA** − **I**) = (**AB**)**A** − **AI** = **O**, we conclude that **A**(**B** + **BA** − **I**) = **I**. Since **A** has, by hypothesis, a unique bounded right reciprocal **B**, and since **B** + **BA** − **I** is bounded and is a right reciprocal of **A**, we have: **B** + **BA** − **I** = **B**, or **BA** = **I**. Hence **B** is *the* bounded reciprocal of **A**.

Theorem 58.5. *If* **A** = **I** − **D**, *where* **D** *is a matrix with norm less than unity, then* **A** *has a bounded reciprocal.*

Proof. Since the norm of **D** is a number r less than 1, then the norm of $\mathbf{D}^k$ is less than or equal to r^k, $k = 1, 2, 3, \cdots$. Hence, if x and y are points of $\mathfrak{H}_1$, the series

$$\sum_{p=1}^{\infty} D^{p-1}(x, y) = F(x, y), \quad (D^0 = I),$$

converges absolutely and defines a function $F(x, y)$ such that

$$| F(x, y) | \leq \frac{1}{1 - r}.$$

Let $x_p = y_p = 0$ for $p > n$. Then, $F(x, y) = B_n(x, y)$ is the nth segment of a bounded bilinear form $B(x, y)$ whose matrix $\mathbf{B} = (b_{pq})$ is given by

$$b_{pq} = \delta_{pq} + \sum_{k=1}^{\infty} d_{pq}{}^{(k)}, \quad p, q = 1, 2, 3, \cdots,$$

where we have put $\mathbf{D}^k = (d_{pq}{}^{(k)})$. We shall prove that $B(x, y) = F(x, y)$. Determine m so that, for all x and y in $\mathfrak{H}_1$,

$$\left| \sum_{k=m}^{\infty} D^k(x, y) \right| \leq \frac{r^m}{1 - r} < \epsilon,$$

where $\epsilon > 0$ is a preassigned number. Let $D^k(x, y)_n$ denote the value of $D^k(x, y)$ when $x_p = y_p = 0$ for $p > n$. Then,

$$\begin{aligned} F(x, y) - B_n(x, y) &= \sum_{k=1}^{\infty} D^{k-1}(x, y) - \sum_{k=1}^{\infty} D^{k-1}(x, y)_n \\ &= \sum_{k=1}^{m} [D^{k-1}(x, y) - D^{k-1}(x, y)_n] \\ &\quad + \sum_{k=m+1}^{\infty} D^{k-1}(x, y) \\ &\quad - \sum_{k=m+1}^{\infty} D^{k-1}(x, y)_n. \end{aligned}$$

The modulus of each of the last two sums does not exceed ϵ for $n = 1, 2, 3, \cdots$; and, m being fixed, we may choose n_0 so large that for $n \geq n_0$, the modulus of the first sum does not exceed ϵ. Here, n_0 depends in general upon x, y and ϵ. We then have:

$$| F(x, y) - B_n(x, y) | \leq 3\epsilon \quad \text{for} \quad n \geq n_0,$$

i.e., $F(x, y) = B(x, y)$.

Put

$$B(x, y) = \sum_{k=1}^{m} D^{k-1}(x, y) + R_m(x, y).$$

Then,

$$(I - D)(B - R_m)(x, y) = (I - D)\sum_{k=1}^{m} D^{k-1}(x, y)$$

$$= I(x, y) - D^m(x, y),$$

$$(I - D)B(x, y) - I(x, y) = (I - D)R_m(x, y) - D^m(x, y).$$

For any values of x and y in $\mathfrak{H}_1$, the modulus of the right-hand member does not exceed

$$\frac{(1 + r)r^m}{1 - r} + r^m,$$

which approaches zero as $m \to \infty$. Since the left-hand member is independent of m, we conclude that it must be identically zero. Therefore, $(I - D)B(x, y) = I(x, y)$, or $(\mathbf{I} - \mathbf{D})\mathbf{B} = \mathbf{I}$. Thus $\mathbf{B}$ is a bounded right reciprocal of $\mathbf{A} = \mathbf{I} - \mathbf{D}$. In like manner, we find that $\mathbf{BA} = \mathbf{I}$. Hence, $\mathbf{A}$ has a unique bounded reciprocal.

59. The Bounded Reciprocal of a Bounded J-matrix. The matrix of a J-form $\Sigma(b_p + z)x_p^2 - 2\Sigma a_p x_p x_{p+1}$ is called a **J-matrix.** We shall denote this matrix by

$$\mathbf{J} = \mathbf{J}_0 + z\mathbf{I} = \begin{bmatrix} b_1 + z, & -a_1, & 0, & 0, & 0, & \cdots \\ -a_1, & b_2 + z, & -a_2, & 0, & 0, & \cdots \\ 0, & -a_2, & b_3 + z, & -a_3, & 0, & \cdots \\ 0, & 0, & -a_3, & b_4 + z, & -a_4, & \cdots \\ & & \cdot & \cdot & \cdot & \end{bmatrix}.$$

If $\mathbf{J}_0$ is bounded and has norm N, and if $|z| > N$, then the norm of $\mathbf{J}_0/z$ is a number r less than unity. By Theorem 58.5 it then follows that $\mathbf{I} + (\mathbf{J}_0/z)$ has a unique bounded reciprocal for $|z| > N$, which is given by

$$\left(\mathbf{I} + \frac{\mathbf{J}_0}{z}\right)^{-1} = \mathbf{I} - \frac{\mathbf{J}_0}{z} + \frac{\mathbf{J}_0^2}{z^2} - \frac{\mathbf{J}_0^3}{z^3} + \cdots.$$

Therefore, $\mathbf{J} = \mathbf{J}_0 + z\mathbf{I}$ has a unique bounded reciprocal for $|z| > N$, which is given by

$$\mathbf{J}^{-1} = (\mathbf{J}_0 + z\mathbf{I})^{-1} = \sum_{p=0}^{\infty} \frac{(-1)^p \mathbf{J}_0{}^p}{z^{p+1}}, \quad (\mathbf{J}_0{}^0 = \mathbf{I}). \quad (59.1)$$

This is a matrix whose elements are power series in $1/z$, convergent for $|z| > N$. We remark that even if $\mathbf{J}_0$ is unbounded, this matrix of power series exists formally, inasmuch as all powers of $\mathbf{J}_0$ exist.

Let us put $\mathbf{J}^{-1} = (P_{pq}(1/z)) = (P_{pq})$. Inasmuch as $\mathbf{J}\mathbf{J}^{-1} = \mathbf{I}$, we have the power series identities

$$\begin{aligned} P_{11}(b_1 + z) - P_{12}a_1 &= 1, \\ -P_{11}a_1 + P_{12}(b_2 + z) - P_{13}a_2 &= 0, \\ -P_{12}a_2 + P_{13}(b_3 + z) - P_{14}a_3 &= 0, \\ &\cdot \quad \cdot \quad \cdot, \end{aligned}$$

and therefore

$$P_{11} = P_{11}\left(\frac{1}{z}\right) \quad (59.2)$$

$$= \cfrac{a_1{}^{-1}}{a_1{}^{-1}(b_1+z) - \cfrac{a_1a_2{}^{-1}}{a_2{}^{-1}(b_2+z) - \cdots - \cfrac{a_na_{n+1}{}^{-1}}{a_{n+1}{}^{-1}(b_{n+1}+z) - \cfrac{P_{1,n+2}}{P_{1,n+1}}}}}.$$

We may write this in the form (cf. (17.13)),

$$P_{11}\left(\frac{1}{z}\right) = \frac{X_{n+2}(z) - \dfrac{P_{1,n+2}}{P_{1,n+1}} X_{n+1}(z)}{Y_{n+2}(z) - \dfrac{P_{1,n+2}}{P_{1,n+1}} Y_{n+1}(z)},$$

and therefore

$$P_{11}\left(\frac{1}{z}\right) - \frac{X_{n+1}(z)}{Y_{n+1}(z)} = \frac{a_{n+1}{}^{-1}}{Y_{n+1}(z)Y_{n+2}(z) - \dfrac{P_{1,n+2}}{P_{1,n+1}} Y_{n+1}(z)}. \quad (59.3)$$

One may verify that $P_{1,n+2}/P_{1,n+1} = (a_{n+1}/z) + (1/z^2)$, where $(1/z^2)$ denotes a power series in descending powers of z beginning with the term in $1/z^2$. Hence we conclude that the power series expansion in descending powers of z of the difference in the left-hand member of (59.3) begins with the term in $1/z^{2n+1}$. *Therefore,* $P_{11}(1/z) = P(1/z)$ *is the power series expansion of the J-fraction* (cf. § 51),

$$\cfrac{1}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cfrac{a_2^2}{b_3 + z - \cdots}}}$$

In § 26, we showed that a bounded J-fraction of norm N converges uniformly for $|z| \geq N + c$ for every positive constant c. *By Theorem* 54.1 *and the preceding conclusion, it then follows that the leading coefficient in the matrix* (59.1) *is equal to the value of the J-fraction for* $|z| > N$.

60. Reciprocals of an Arbitrary J-matrix. As pointed out at the beginning of this chapter, any matrix (ρ_{pq}) whose coefficients satisfy the equations

$$-a_{p-1}\rho_{p-1,q} + (b_p + z)\rho_{p,q} - a_p\rho_{p+1,q} = \delta_{p,q}, \quad p, q = 1, 2, 3, \cdots, \tag{60.1}$$

$(a_0 = 0,\ a_p \neq 0,\ p = 1, 2, 3, \cdots)$, is a right reciprocal of the matrix $\mathbf{J}$. These equations are equivalent to the matrix equation $\mathbf{J}(\rho_{pq}) = \mathbf{I}$. Since the $a_p \neq 0$ for $p \geq 1$, it follows that, for a fixed q, $\rho_{p,q}$, $p = 2, 3, 4, \cdots$, are determined uniquely in terms of an arbitrarily chosen initial value $\rho_{1,q}$. We shall now obtain explicit formulas for the $\rho_{p,q}$, $p = 2, 3, 4, \cdots$, in terms of $\rho_{1,q}$.

For a fixed q, the equations (60.1), for $p = 1, 2, 3, \cdots, q$, are the same as the recurrence formulas in (17.10) for the polynomials $Y_p(z)$, which determine the $Y_p(z)$ uniquely up to a factor independent of p. Consequently,

$$\rho_{p,q} = \rho_{1,q}Y_p(z), \quad p = 1, 2, 3, \cdots, q.$$

If $p > q$, it is clear that $\rho_{p,q}$ must have the form

$$\rho_{p,q} = \rho_{1,q}Y_p(z) + C_{p,q}(z),$$

where $C_{p,q}(z)$ is a polynomial in z; and that the $C_{p,q}(z)$ are uniquely determined by means of the system of equations

$$-a_qC_{q+1,q} = 1,$$
$$(b_{q+1} + z)C_{q+1,q} - a_{q+1}C_{q+2,q} = 0,$$
$$-a_{q+r}C_{q+r,q} + (b_{q+r+1} + z)C_{q+r+1,q} - a_{q+r+1}C_{q+r+2,q} = 0,$$
$$r = 1, 2, 3, \cdots.$$

We observe that, by the determinant formula,

$$C_{q+1,q} = -a_q^{-1} = X_qY_{q+1} - X_{q+1}Y_q,$$

and also,

$$C_{q+2,q} = a_{q+1}^{-1}(b_{q+1} + z) = X_qY_{q+2} - X_{q+2}Y_q.$$

One may now readily show by mathematical induction that

$$C_{q+r,q} = X_qY_{q+r} - X_{q+r}Y_q, \quad r = 1, 2, 3, \cdots.$$

Hence, we have proved

THEOREM 60.1. *The general right reciprocal of the J-matrix is* (ρ_{pq}), *where* $\rho_{1,q}$, $q = 1, 2, 3, \cdots$, *are arbitrary functions of* z, *and*

$$\rho_{pq}(z) = \begin{cases} \rho_{1,q}(z)Y_p(z), & p = 1, 2, 3, \cdots, q; \\ \rho_{1,q}(z)Y_p(z) + X_q(z)Y_p(z) - X_p(z)Y_q(z), \end{cases} \tag{60.2}$$
$$p = q + 1, q + 2, q + 3, \cdots.$$

If we introduce a new function $w_q = w_q(z)$, by means of the equation

$$\rho_{1,q} = Y_qw_q - X_q, \quad q = 1, 2, 3, \cdots,$$

then the formulas (60.2) take the symmetrical form [3]

$$\rho_{pq}(z) = \begin{cases} Y_p(z)Y_q(z)\left(w_q(z) - \dfrac{X_q(z)}{Y_q(z)}\right), & p = 1, 2, 3, \cdots, q; \\ Y_p(z)Y_q(z)\left(w_q(z) - \dfrac{X_p(z)}{Y_p(z)}\right), & p = q + 1, q + 2, \cdots. \end{cases} \tag{60.3}$$

From (60.3) we see immediately that the following theorem is true.

THEOREM 60.2. *The matrix* (ρ_{pq}) *is symmetric if, and only if, the functions* $w_q(z)$ *are all equal to one another; or if, and only if,*

$$\frac{\rho_{n+1,q}}{\rho_{n,q}} = v_n(z), \quad \text{for} \quad n \geq q, \tag{60.4}$$

where $v_n(z)$ *is independent of* q *for* $n \geq q$, $(n, q = 1, 2, 3, \cdots)$.

If the J-fraction is bounded, and we put $w_q(z) = P(1/z)$, the power series expansion of the J-fraction, then (60.3) gives the unique bounded reciprocal, $(P_{pq}(1/z))$, of the J-matrix. In the general case, it follows from Theorem 56.1 that a necessary condition for the matrix (ρ_{pq}) to be bounded is that the series $\sum_{q=1}^{\infty} |\rho_{1,q}(z)|^2$ converge. In the indeterminate case (Definition 22.1) it follows from Theorem 22.1, (60.2), and Schwarz's inequality, that this condition is also sufficient. For then the double series $\Sigma|\rho_{pq}(z)|^2$ is convergent. Since the $\rho_{1,q}(z)$ are arbitrary functions, we then see that in the indeterminate case the J-matrix has infinitely many bounded right reciprocals. In the determinate case, we are unable to say, in general, whether or not a bounded right reciprocal exists. In the next section we shall find that if the J-fraction is positive definite, then there is a unique bounded reciprocal in the determinate case, which has an essential relationship to the J-fraction. Also, in the indeterminate case, there are among the infinitely many bounded reciprocals certain ones which have an essential relationship to the J-fraction.

61. Reciprocals of the J-matrix Associated with a Positive Definite J-fraction. We now consider a positive definite J-fraction

$$\cfrac{1}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cfrac{a_2^2}{b_3 + z - \cdots}}}, \qquad (a_p \neq 0), \tag{61.1}$$

and the symmetrical reciprocals of its associated J-matrix. We shall employ the ideas and notation of Chapter IV.

The symmetrical reciprocals are given by (60.3) with $w_q(z) = f(z)$, $q = 1, 2, 3, \cdots$, an arbitrary function of z. We shall impose certain restrictions upon the function $f(z)$. We recall that for $\mathfrak{J}(z) \geq \delta > 0$, the approximants of the J-fraction are *uniformly bounded* (cf. (17.7)). Hence, every infinite sequence of its approximants contains an infinite subsequence which is uniformly convergent over every bounded closed region lying entirely within the upper half-plane, $\mathfrak{J}(z) > 0$, to an analytic limit-function $f(z)$. The values of this function must lie in *all* the circles $K_p(z)$ (cf. (17.5)). Any function which for $\mathfrak{J}(z) > 0$ is analytic and has its values in all the circles $K_p(z)$ will be called an **equivalent function** of the J-fraction. In particular, every function which is the limit of a convergent sequence of approximants of the J-fraction is an equivalent function of the J-fraction. *We shall require that* $w_q(z) = f(z)$, $q = 1, 2, 3, \cdots$, *where* $f(z)$ *is an equivalent function of the J-fraction.*

Let $\xi_1, \xi_2, \xi_3, \cdots, \xi_n$ be arbitrary real numbers. On multiplying the equation (60.1) by ξ_q and summing over q from 1 to n, we obtain

$$-a_{p-1}\eta_{p-1} + (b_p + z)\eta_p - a_p\eta_{p+1} = \xi_p, \tag{61.2}$$

where

$$\eta_p = \sum_{q=1}^{n} \rho_{pq}\xi_q. \tag{61.3}$$

By Theorem 60.2, the quotient $\rho_{n+1,q}/\rho_{n,q}$ is independent of q for $q = 1, 2, 3, \cdots, n$. Moreover, if we put

$$w = \frac{a_n\rho_{n,q}}{\rho_{n+1,q}},$$

then

$$w = a_n\frac{Y_n(z)f(z) - X_n(z)}{Y_{n+1}(z)f(z) - X_{n+1}(z)},$$

so that, by (17.13),

$$f(z) = t_0t_1 \cdots t_n(w) = \frac{X_{n+1}(z)w - a_nX_n(z)}{Y_{n+1}(z)w - a_nY_n(z)}.$$

Consequently, the function $f(z)$ has its values for $\mathfrak{J}(z) > 0$ in the circle $K_n(z)$ if, and only if, $\mathfrak{J}(w) \geq \beta_{n+1}g_n$ (cf. (17.12)). By our assumption that $f(z)$ is an equivalent function of the J-fraction, we therefore conclude from Lemma 17.1 that

$$\beta_n + y - \Im\left(\frac{a_n^2}{w}\right) \geq \beta_n g_{n-1} + y, \quad \text{where} \quad y = \Im(z) > 0,$$

or

$$\Im\left(\frac{a_n^2}{w}\right) \leq \beta_n(1 - g_{n-1}). \tag{61.4}$$

Now

$$a_n \rho_{n+1,q} = \frac{a_n^2}{w} \rho_{n,q}.$$

On multiplying this by ξ_q and summing over q from 1 to n, we have

$$a_n \eta_{n+1} = \frac{a_n^2}{w} \eta_n. \tag{61.5}$$

We now multiply (61.2) by $\bar{\eta}_p$, sum over p from 1 to n, and then eliminate the quantity $a_n \eta_{n+1} \bar{\eta}_n$ by means of (61.5). This gives immediately the relation

$$\sum_{p=1}^{n} (b_p + z) | \eta_p |^2 - \sum_{p=1}^{n-1} a_p (\eta_p \bar{\eta}_{p+1} + \bar{\eta}_p \eta_{p+1}) = \frac{a_n^2}{w} | \eta_n |^2 + \sum_{p=1}^{n} \xi_p \bar{\eta}_p.$$

If we consider only the imaginary part and make use of the inequality (61.4), we then have the relation (cf. (16.10))

$$y \sum_{p=1}^{n} | \eta_p |^2 + \sum_{p=1}^{n-1} | \sqrt{\beta_p(1 - g_{p-1})} \eta_p - \sqrt{\beta_{p+1} g_p} \eta_{p+1} |^2 + \sum_{p=1}^{n} \xi_p \Im(\eta_p) \leq 0. \tag{61.6}$$

Hence, in particular,

$$y \sum_{p=1}^{n} | \eta_p |^2 + \sum_{p=1}^{n} \xi_p \Im(\eta_p) \leq 0. \tag{61.7}$$

Turning now to the quadratic form

$$R_n(\xi, \xi) = \sum_{p,q=1}^{n} \rho_{pq}(z)\xi_p\xi_q = \sum_{p=1}^{n} \xi_p\eta_p,$$

we have, by Schwarz's inequality and (61.7),

$$\begin{aligned} | R_n(\xi, \xi) |^2 = \left| \sum_{p=1}^{n} \xi_p\eta_p \right|^2 &\leq \frac{1}{y} \sum_{p=1}^{n} \xi_p^2 \cdot y \sum_{p=1}^{n} | \eta_p |^2 \\ &\leq \frac{1}{y} \sum_{p=1}^{n} \xi_p^2 \cdot \left(- \sum_{p=1}^{n} \xi_p \mathfrak{J}(\eta_p) \right) \\ &= \frac{1}{y} \sum_{p=1}^{n} \xi_p^2 \cdot (-\mathfrak{J}[R_n(\xi, \xi)]). \end{aligned}$$

Therefore,

$$| R_n(\xi, \xi) |^2 \leq \frac{1}{y} \sum_{p=1}^{n} \xi_p^2 \cdot | R_n(\xi, \xi) |,$$

or

$$| R_n(\xi, \xi) | \leq \frac{1}{y} \sum_{p=1}^{n} \xi_p^2. \tag{61.8}$$

For the related bilinear form we then have, by the identity used in the proof of Theorem 57.3,

$$| R_n(u, v) | \leq \frac{1}{y}, \quad \text{if} \quad \sum_{p=1}^{n} u_p^2 \leq 1, \quad \sum_{p=1}^{n} v_p^2 \leq 1, \tag{61.9}$$

for all *real* u and v.

We have completed the proof of the following theorem.[20]

THEOREM 61.1. *Any reciprocal of the J-matrix associated with a positive definite J-fraction, which is given by* (60.3) *with* $w_q(z) = f(z)$, $q = 1, 2, 3, \cdots$, *where* $f(z)$ *is an equivalent function of the J-fraction, is bounded for* $\mathfrak{J}(z) > 0$. *The norm of this matrix, relative to real Hilbert space, is not greater than* $1/\mathfrak{J}(z)$. *That is,*

$$\left| \sum_{p,q=1}^{n} \rho_{pq} u_p v_q \right| \leq \frac{1}{y} \sqrt{\sum_{p=1}^{n} u_p^2 \cdot \sum_{q=1}^{n} v_q^2}, \quad y = \mathfrak{J}(z) > 0, \tag{61.10}$$

for all real u_p *and* v_q. **[136.]**

[20] This theorem and the next were proved for real J-fractions by Hellinger [**31**]. For related theorems, where a modification of the boundedness condition was used, cf. [**35**] and [**138**].

In the determinate case, the circles $K_p(z)$ have one and only one point in common for $\mathfrak{J}(z) > 0$, namely, the value of the J-fraction. We shall prove

THEOREM 61.2. *In the determinate case for a positive definite J-fraction, the associated J-matrix has one and only one bounded reciprocal for* $\mathfrak{J}(z) > 0$. *This is given by* (60.3) *with* $w_q(z) = f(z)$, $q = 1, 2, 3, \cdots$, *where* $f(z)$ *is the value of the J-fraction* [136].

Proof. That the said reciprocal is bounded for $\mathfrak{J}(z) > 0$ has been proved. It remains to be shown that any other reciprocal is unbounded for at least one z in $\mathfrak{J}(z) > 0$. For any other (right) reciprocal there must be at least one q and at least one z in $\mathfrak{J}(z) > 0$ such that

$$\left| w_q(z) - \frac{X_p(z)}{Y_p(z)} \right| \geq k > 0$$

for all sufficiently large values of p, k being a positive constant.

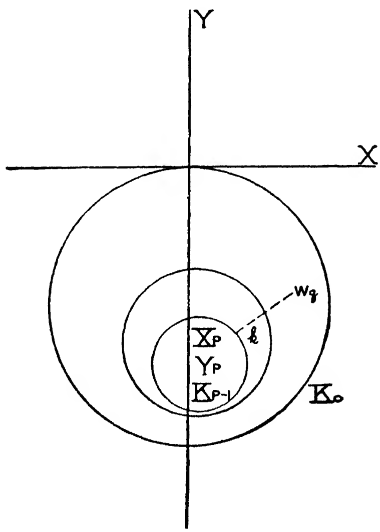

FIG. 9.

Hence, it follows by (60.3) that, for these values of q and z,

$$| \rho_{pq}(z) |^2 \geq | Y_q(z) |^2 k^2 \cdot | Y_p(z) |^2,$$

for all sufficiently large values of p. Since $| Y_q(z) | > 0$, and since the series $\sum_{p=1}^{\infty} | Y_p(z) |^2$ diverges in the determinate case, it then follows that the series $\sum_{p=1}^{n} | \rho_{pq}(z) |^2$ diverges. Therefore, by Theorem 56.1 the matrix $(\rho_{pq}(z))$ is unbounded.

62. Estimates for the Equivalent Functions. If in (61.10) we let $u_p = v_q = 1, u_r = 0$ for $r \neq p$, $v_r = 0$ for $r \neq q$, that inequality becomes

$$| \rho_{pq}(z) | \leq \frac{1}{y}, \quad y = \mathfrak{J}(z) > 0.$$

By (60.1),

$$\rho_{pq}(z) = \frac{a_{p-1}\rho_{p-1,q} - b_p\rho_{p,q} + a_p\rho_{p+1,q}}{z} + \frac{\delta_{p,q}}{z}. \quad (62.1)$$

Consequently,

$$\rho_{pq}(z) = \frac{\delta_{pq}}{z} + \frac{0(1)}{yz}, \quad (62.2)$$

where we use $0(1)$ to represent a function of z which is numerically less than a constant independent of z for all z with $\mathfrak{J}(z) > 0$. In particular, since $\rho_{11}(z) = f(z)$, we have, for any equivalent function of the J-fraction,[21]

$$f(z) = \frac{1}{z} + \frac{0(1)}{yz}. \quad (62.3)$$

If we substitute the values of $\rho_{p-1,q}$, $\rho_{p,q}$, $\rho_{p+1,q}$ from (62.2) into the right-hand member of (62.1), we obtain

$$\rho_{pq}(z) = \frac{\delta_{pq}}{z} + \frac{\delta_{pq}^{(1)}}{z^2} + \frac{0(1)}{yz^2}, \quad p \geq 1,$$

where $\delta_{pq}^{(1)} = a_{p-1}\delta_{p-1,q} - b_p\delta_{p,q} + a_p\delta_{p+1,q}$, $(a_0 = 0)$. On substituting from this into (62.1), and continuing the same process, we then find that

$$\rho_{pq}(z) = \frac{\delta_{pq}}{z} + \frac{\delta_{pq}^{(1)}}{z^2} + \cdots + \frac{\delta_{pq}^{(k)}}{z^{k+1}} + \frac{0(1)}{yz^{k+1}}, \quad p > k - 1, \quad (62.4)$$

where

$$\delta_{pq}^{(k)} = a_{p-1}\delta_{p-1,q}^{(k-1)} - b_p\delta_{p,q}^{(k-1)} + a_p\delta_{p+1,q}^{(k-1)}.$$

[21] Cf. Beth [3] for the case of real J-fractions, and Hellinger and Wall [35] for the case a_p real and $\mathfrak{J}(b_p) \geq 0$.

Since $\delta_{pq}^{(k)} = 0$ if $p > q + k$, we then have

$$\rho_{q+k,q} = \frac{0(1)}{z^{k+1}} + \frac{0(1)}{yz^{k+1}}, \quad k = 0, 1, 2, \cdots. \quad [138.] \quad (62.5)$$

EXERCISE 12

12.1. A bilinear form $A(x, y)$ is called *completely continuous* if for every $\epsilon > 0$, there exists an N, such that

$$| A_n(x, y) - A_m(x, y) | < \epsilon \quad \text{if} \quad m, n \geq N,$$

for all points x and y in $\mathfrak{H}_1$, the number N being independent of x and y. A matrix is called completely continuous if it is the matrix of a completely continuous bilinear form. A completely continuous bilinear form is necessarily bounded, but not all bounded forms are completely continuous. For example, the unit form is not completely continuous. Show that if the double series $\Sigma| a_{pq} |^2$ is convergent, then $A(x, y)$ is completely continuous. This is not a necessary condition. A necessary but not sufficient condition is that $\lim a_{pq} = 0$ as p and q tend to ∞.

12.2. If we are given two sequences of points $x^{(n)}$ and $y^{(n)}$ in the Hilbert sphere $\mathfrak{H}_1$, such that for each p:

$$\lim_{n=\infty} x_p^{(n)} = x_p, \quad \lim_{n=\infty} y_p^{(n)} = y_p,$$

and if $A(x, y)$ is completely continuous, then

$$\lim_{n=\infty} A(x^{(n)}, y^{(n)}) = A(x, y).$$

12.3. If $A(x, y)$ is a bounded form such that $A(x, y)$ approaches zero whenever all coordinates x_p and y_p of x and y, respectively, approach zero, then $A(x, y)$ is completely continuous.

Hint. Show first that if M is the least upper bound of $| A(x, y) |$ for x and y in $\mathfrak{H}_1$, then there exist points z and w in $\mathfrak{H}_1$ such that $| A(z, w) | = M$. Apply this to the form

$$A^{(n)}(x, y) = \sum_{p,\, q=n+1}^{\infty} a_{pq}x_p y_q,$$

which may be regarded as $A(x, y)$ evaluated for $x_p = y_p = 0$ for $p \leq n$.

12.4. The product of two bounded matrices is completely continuous if at least one of the factors is completely continuous.

12.5. If $\mathbf{A}$ is a bounded real matrix, and if $\mathbf{AA}'$ is completely continuous, then $\mathbf{A}$ is completely continuous.

12.6. If $A(x, y)$ is symmetric and positive definite, and if Σa_{pp} converges, then the double series Σa_{pq}^2 converges and $A(x, y)$ is completely continuous.

12.7. If the matrix $\mathbf{D}$ of Theorem 58.5 is completely continuous, and if we put $(\mathbf{I} - \mathbf{D})^{-1} = \mathbf{I} - \mathbf{E}$, then $\mathbf{E}$ is completely continuous.

12.8. If a bounded real matrix **A** has a bounded right reciprocal, then there exists a positive constant m such that

$$AA'(x, x) \geq m\sum_{p=1}^{\infty} x_p^2$$

for all real x_p.

12.9. If **S** is a real bounded symmetric matrix, such that

$$S(x, x) \geq m\sum_{p=1}^{\infty} x_p^2$$

for all real x_p, where m is a positive constant, then **S** has a reciprocal, and its norm does not exceed $1/m$.

12.10. The condition in 12.8 is sufficient as well as necessary for **A** to have a bounded right reciprocal.

12.11. A system of linear equations

$$x_p - \sum_{q=1}^{\infty} d_{pq}x_q = y_p, \quad p = 1, 2, 3, \cdots,$$

which may be written as the single matrix equation

$$(\mathbf{I} - \mathbf{D})\mathbf{x} = \mathbf{x} - \mathbf{D}\mathbf{x} = \mathbf{y},$$

is called completely continuous if the matrix $\mathbf{D} = (d_{pq})$ is a completely continuous matrix. The solution of the completely continuous system may be made to depend entirely upon the solution of an algebraic system of linear equations in a finite number of unknowns.

12.12. The completely continuous system $(\mathbf{I} - \mathbf{D})\mathbf{x} = \mathbf{y}$ has a unique solution x in $\mathfrak{H}$ for each y in $\mathfrak{H}$, or else the homogeneous system $(\mathbf{I} - \mathbf{D})\mathbf{x} = 0$ has at least one solution $x \neq 0$ in $\mathfrak{H}$.

12.13. A bounded J-matrix has a unique bounded reciprocal for any z not in the convex set K_0 introduced in § 26.

Chapter XIII

CONTINUED FRACTIONS AND DEFINITE INTEGRALS

In the earlier investigations of J-fractions, beginning with the classical work of Stieltjes [95], the coefficients a_p and b_p were supposed real. For this case, and with some additional restrictions, Stieltjes was able to connect the J-fraction with one or more integrals of the form

$$\int_0^\infty \frac{d\phi(u)}{z-u},$$

where $\phi(u)$ is a bounded nondecreasing (not necessarily continuous) function of u. For particular cases where these restrictions are relaxed, Van Vleck [109] obtained again a connection with an analogous integral, the range of which he had to extend over the whole real axis. Hilbert's [37] famous theory of bounded quadratic forms, in which the ideas of Stieltjes are in the background, allows immediate application to real J-fractions and their connection with integrals of the above form, but with a finite range of integration [32]. Grommer [23] showed that the process of Hilbert can be applied to more general cases where the integral extends from $-\infty$ to $+\infty$. A general theory of real J-fractions was first developed by Hamburger [26] following the pattern laid down by Stieltjes. At about the same time, the general case was treated by several other mathematicians. Hellinger [31] employed Hilbert's theory of infinite linear systems, R. Nevanlinna [62] used methods of function theory and asymptotic series, Carleman [6] used his theory of integral equations, and M. Riesz [79] used methods of successive approximation.

In this chapter, we first develop some of the properties of Stieltjes integrals. Then, using the asymptotic expression (62.3), found in the preceding chapter, we arrive at the fundamental theorem that any equivalent function of a positive definite J-fraction is equal to a Stieltjes integral of the form

$$\int_{-\infty}^{+\infty} \frac{d\phi(u)}{z-u},$$

in which $\phi(u)$ is a bounded nondecreasing function of u.

63. The Stieltjes Integral.[22] In his investigations of the continued fraction (28.1), in which the k_p are real and positive, Stieltjes found that in some cases the value of the continued fraction has the form

$$\int_{0}^{\infty} \frac{f(u)du}{z+u},$$

where $f(u)$ is a positive function of u, while in some cases the value of the continued fraction has the form

$$\sum_{p=1}^{\infty} \frac{L_p}{z+x_p},$$

where the L_p are positive and $0 \leq x_1 < x_2 < x_3 < \cdots$. Moreover, in other cases, the value of the continued fraction may be a sum of an integral and an infinite series of the above forms. This situation led Stieltjes to define an integral of the form

$$\int_{0}^{\infty} \frac{d\phi(u)}{z+u},$$

embracing all the functions of the three diverse types.

Let $f(u)$ and $\phi(u)$ be two real or complex valued functions of the real variable u, defined on a finite interval $a \leq u \leq b$. We subdivide this interval into $n+1$ subintervals by interpolating n points $u_1, u_2, \cdots, u_n$ between a and b, so that if $u_0 = a$, $u_{n+1} = b$, then

$$u_0 < u_1 < u_2 < \cdots < u_{n+1}.$$

[22] References: Stieltjes [95, Chap. VI], Bray [5], F. Riesz [78], Evans [13], Widder [142], Shohat and Tamarkin [90].

For $k = 1, 2, 3, \cdots, n+1$, let v_k be chosen arbitrarily in the interval $u_{k-1} \leq u \leq u_k$, i.e., $u_{k-1} \leq v_k \leq u_k$, $k = 1, 2, 3, \cdots, n+1$, and form the sum

$$S(u, v) = \sum_{k=1}^{n+1} f(v_k)[\phi(u_k) - \phi(u_{k-1})].$$

If this sum tends to a finite limit L as n tends to ∞ in such a way that the maximum of the differences $u_k - u_{k-1}$ tends to zero, the limit being independent of the manner in which the successive subdivisions are made and the interpolated points v_k chosen, then L is called the **Stieltjes integral** of $f(u)$ with respect to $\phi(u)$, and is denoted by the symbol

$$\int_a^b f(u)d\phi(u).$$

We shall now develop some of the properties of this integral.

There is a duality between the functions $f(u)$ and $\phi(u)$ appearing in this definition:

THEOREM 63.1. *If the Stieltjes integral of $f(u)$ with respect to $\phi(u)$ exists, then the Stieltjes integral of $\phi(u)$ with respect to $f(u)$ exists, and there is the formula of "integration by parts,"*

$$\int_a^b f(u)d\phi(u) = f(b)\phi(b) - f(a)\phi(a) - \int_a^b \phi(u)df(u). \tag{63.1}$$

Proof. Inasmuch as, by hypothesis, the Stieltjes integral of $f(u)$ with respect to $\phi(u)$ exists, we may regard it as the limit as $\max(u_k - u_{k-1}) \to 0$ of the sum $S(u, v)$, wherein we agree to take $v_1 = a$, $v_{n+1} = b$. We then have:

$$S(u, v) = f(b)\phi(b) - f(a)\phi(a) - \sum_{k=1}^{n} \phi(u_k)[f(v_{k+1}) - f(v_k)].$$

In passing to the limit, we may choose $u_1, u_2, \cdots, u_n$ at pleasure in the intervals $a \leq u \leq v_2$, $v_2 \leq u \leq v_3$, $\cdots$, $v_n \leq u \leq b$, respectively, and the condition $\max(u_k - u_{k-1}) \to 0$ is equivalent to the condition $\max(v_k - v_{k-1}) \to 0$. It is therefore evident that the Stieltjes integral of $\phi(u)$ with respect to $f(u)$ exists, and that (63.1) holds.

One may easily verify that the Stieltjes integral obeys the following rules of operation:

$$\int_a^b [f_1(u) + f_2(u)]d\phi(u) = \int_a^b f_1(u)d\phi(u) + \int_a^b f_2(u)d\phi(u);$$

$$\int_a^b f(u)d[\phi_1(u) + \phi_2(u)] = \int_a^b f(u)d\phi_1(u) + \int_a^b f(u)d\phi_2(u);$$

$$\int_a^b k_1 f(u)d[k_2\phi(u)] = k_1 k_2 \int_a^b f(u)d\phi(u),$$

k_1, k_2 being constants.

Under suitable conditions, the Stieltjes integral reduces to a Riemann integral.

THEOREM 63.2. *If $\phi(u)$ is continuous for $a \leq u \leq b$, and possesses a derivative $\phi'(u)$ such that $f(u)\phi'(u)$ is Riemann integrable for $a \leq u \leq b$, then*

$$\int_a^b f(u)d\phi(u) = \int_a^b f(u)\phi'(u)du.$$

This may be easily established with the aid of the mean value theorem.

If there exists a constant L such that for all subdivisions of the interval $a \leq u \leq b$, the sum

$$\sum_{p=1}^{n} | \phi(u_k) - \phi(u_{k-1}) | \leq L, \tag{63.2}$$

then $\phi(u)$ is said to be of **bounded variation** on the interval $a \leq u \leq b$. In this case, if $|f(u)| \leq M$ for $a \leq u \leq b$, then

$$| S(u, v) | \leq ML. \tag{63.3}$$

THEOREM 63.3. *If $f(u)$ is continuous and $\phi(u)$ is of bounded variation, on the interval $a \leq u \leq b$, then the Stieltjes integral of $f(u)$ with respect to $\phi(u)$ exists.*

Proof. Since $f(u)$ is continuous on the closed interval $a \leq u \leq b$, it is uniformly continuous, so that if $\epsilon > 0$, there exists a number $\delta > 0$, such that if $a < u_1 < u_2 < \cdots < u_n < b$ is any subdivision of the interval into subintervals of maximum length δ, and u', u'' are arbitrary points in an arbitrary one of the subintervals, then $|f(u') - f(u'')| < \epsilon$. For brevity, such a subdivision of the interval $a \leq u \leq b$ will be called an **ϵ-division.**

In any given ϵ-division, let new division points be interpolated, forming a new subdivision, which is obviously an ϵ-division. If $S(u, v)$ and $S(u', v')$ are sums formed for the given subdivision and the new one, then one may easily verify that

$$| S(u, v) - S(u', v') | \leq \epsilon L.$$

Suppose now that $S(u, v)$ and $S(u', v')$ are sums corresponding to arbitrary ϵ-divisions, and let $S(u'', v'')$ be an arbitrary sum corresponding to the ϵ-division obtained by superimposing the u and u' division points. Then, by the preceding, we have:

$$| S(u, v) - S(u'', v'') | \leq \epsilon L, \quad | S(u', v') - S(u'', v'') | \leq \epsilon L,$$

and therefore,

$$| S(u, v) - S(u', v') | \leq 2\epsilon L. \tag{63.4}$$

Let $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ be a sequence of positive numbers with the limit 0, and, for each p, let $S(u^{(p)}, v^{(p)})$ be a sum corresponding to an ϵ_n-division. Then if $\eta > 0$, it follows from (63.4) that there exists an index N, such that

$$| S(u^{(n+p)}, v^{(n+p)}) - S(u^{(n)}, v^{(n)}) | < \eta, \quad \text{if} \quad n > N,$$
$$p = 1, 2, 3, \cdots.$$

Therefore

$$\lim_{n=\infty} S(u^{(n)}, v^{(n)}) = I$$

exists and is finite. If $S(u, v)$ is an arbitrary sum corresponding to an ϵ-division, then

$$| S(u, v) - I | \leq | S(u, v) - S(u^{(n)}, v^{(n)}) | + | S(u^{(n)}, v^{(n)}) - I |$$
$$\leq 2\epsilon L + \eta,$$

for all sufficiently large values of n. Consequently, the Stieltjes integral of $f(u)$ with respect to $\phi(u)$ exists and is equal to I.

We shall suppose in what follows that $f(u)$ is continuous and $\phi(u)$ is of bounded variation on the intervals considered.

The symbol

$$\int_a^b | d\phi(u) |$$

is commonly used to denote the least upper bound of the sums in (63.2), taken with respect to all finite subdivisions of the interval. By (63.3), we than have the inequality

$$\left|\int_a^b f(u)d\phi(u)\right| \leq M\int_a^b |d\phi(u)|, \tag{63.5}$$

where M is the least upper bound of $|f(u)|$ on $a \leq u \leq b$.

Since the properties assumed for $f(u)$ and $\phi(u)$ in $a \leq u \leq b$ pertain as well to any subinterval of $a \leq u \leq b$, we have:

$$\int_a^b f(u)d\phi(u) = \int_a^c f(u)d\phi(u) + \int_c^b f(u)d\phi(u), \quad a < c < b.$$

Integrals over an infinite range are defined as in the case of Riemann integration:

$$\int_{-\infty}^{+\infty} f(u)d\phi(u) = \lim_{\substack{b=+\infty \\ a=-\infty}} \int_a^b f(u)d\phi(u),$$

provided the limit exists. Of course, but one of the limits of integration may be infinite.

If a constant c is added to $\phi(u)$, the value of the integral of $f(u)$ with respect to $\phi(u)$ is obviously unchanged. More generally, we have this theorem:

THEOREM 63.4. *In order that the integral*

$$\int_a^b f(u)d\phi(u) \tag{63.6}$$

vanish for every continuous function $f(u)$, it is necessary and sufficient that $\phi(u) = \phi(a)$ for $u = b$, and for all other values of u with the exception of at most a countable set included in the set of discontinuities of $\phi(u)$. [78.]

Proof. Suppose first that (63.6) vanishes for every continuous $f(u)$. Putting in succession $f(u) = 1$; and $f(u) = u$ for $a \leq u \leq v$, $f(u) = v$ for $u > v$, we get

$$\phi(b) = \phi(a), \quad 0 = \int_a^v ud\phi(u) + v\int_v^b d\phi(u)$$

$$= (v - a)\phi(a) - \int_a^v \phi(u)au.$$

On differentiating the last equality with respect to v, we then find that $\phi(v) = \phi(a)$ at all points v where $\phi(u)$ is continuous.

Suppose, conversely, that $\phi(a) = \phi(b) = \phi(u)$, except possibly for a countable set of values of u. Then it follows at once by the definition of a Stieltjes integral that (63.6) is equal to 0 for every continuous $f(u)$. In fact, the sums used in defining the integral will all be equal to 0 if the division points are chosen among the points for which $\phi(u) = \phi(a)$.

THEOREM 63.5. *Let $f(u)$ be continuous and nonnegative for $a \leq u \leq b$, and in this interval let $\phi(u)$ be bounded and nondecreasing. Suppose that there is at least one point u_0, $a < u_0 < b$, such that $f(u_0) > 0$ and $\phi(u_0 + t) > \phi(u_0 - t)$ for all sufficiently small values of $t > 0$. Then the integral $\int_a^b f(u)d\phi(u)$ is positive.*

We omit the proof.

THEOREM 63.6. *Let $P(u)$ be any real polynomial of degree $r - 1$ which is not identically equal to zero, and let $\phi(u)$ be a bounded nondecreasing function such that for at least r points u_0, $a < u_0 < b$, we have $\phi(u_0 + t) > \phi(u_0 - t)$ for all sufficiently small values of $t > 0$. Suppose further that*

$$\int_a^b u^k P(u)d\phi(u) = 0, \quad \textit{for} \quad k = 0, 1, 2, \cdots, n - 1.$$

Then $P(u)$ changes sign at least n times in the interval $a < u < b$.

Proof. If $P(u)$ does not change sign in the interval, then the relation $\int_a^b P(u)d\phi(u) = 0$ shows, by Theorem 63.5, that $P(u) \equiv 0$, contrary to hypothesis. Suppose that $P(u)$ changes sign just m times, where $n > m$, at the points $u_1, u_2, \cdots, u_m$, where $a < u_1 < u_2 < \cdots < u_m < b$. Then the polynomial

$$(u - u_1)(u - u_2) \cdots (u - u_m)P(u)$$

does not change sign in the interval. Therefore, inasmuch as

$$\int_a^b (u - u_1)(u - u_2) \cdots (u - u_m)P(u)d\phi(u) = 0,$$

we conclude, as before, that $P(u) \equiv 0$, contrary to hypothesis.

64. Sequences of Stieltjes Integrals.[23] If $f_p(u), p = 1, 2, 3, \cdots$, is a sequence of continuous functions, uniformly convergent for $a \leq u \leq b$, and if $\phi(u)$ is of bounded variation in this interval, then

$$\lim_{p=\infty} \int_a^b f_p(u) d\phi(u) = \int_a^b \lim_{p=\infty} f_p(u) d\phi(u).$$

The proof is analogous to the proof of the corresponding theorem for Riemann integrals. There is another type of double-limit theorem for Stieltjes integrals, namely:

THEOREM 64.1. *Let $f(u)$ be continuous for $a \leq u \leq b$. Let $\phi_p(u)$, $p = 1, 2, 3, \cdots$, be a sequence of functions of bounded variation on this interval, such that*

$$\int_a^b | d\phi_p(u) | < M, \quad p = 1, 2, 3, \cdots,$$

where M is a constant independent of p. Let

$$\lim_{p=\infty} \phi_p(u) = \phi(u),$$

over a set K of points everywhere dense in $a \leq u \leq b$, including $u = a$ and $u = b$, where $\phi(u)$ is of bounded variation. Then

$$\lim_{p=\infty} \int_a^b f(u) d\phi_p(u) = \int_a^b f(u) d\phi(u).$$

Proof. From the inequality (63.4) it follows that by choosing from K a suitable set of division points $u_0 = a < u_1 < u_2 < \cdots < u_n < u_{n+1} = b$, we can make

$$\left| \int_a^b f(u) d\phi_m(u) - \sum_{p=0}^{n+1} f(v_p) [\phi_m(u_{p+1}) - \phi_m(u_p)] \right| < \epsilon,$$

$$\left| \int_a^b f(u) d\phi(u) - \sum_{p=0}^{n+1} f(v_p) [\phi(u_{p+1}) - \phi(u_p)] \right| < \epsilon,$$

where $\epsilon > 0$ is a preassigned number. Since the u_p are fixed and

[23] The theorems of this section have been used in many investigations where Stieltjes integrals are involved. The ideas go back to Stieltjes, and were developed and extended by Hilbert as important tools in his theory of infinite quadratic forms. Cf. Hilbert [37 (book)], p. 113 and p. 116.

finite in number, we may choose m_0 sufficiently large in order that

$$\left| \sum_{p=0}^{n+1} f(v_p)[\phi_m(u_{p+1}) - \phi_m(u_p)] - \sum_{p=0}^{n+1} f(v_p)[\phi(u_{p+1}) - \phi(u_p)] \right| < \epsilon,$$

for $m > m_0$. We therefore conclude that

$$\left| \int_a^b f(u) d\phi_m(u) - \int_a^b f(u) d\phi(u) \right| < 3\epsilon \quad \text{if} \quad m > m_0,$$

which is what was to be proved.

As a companion to the preceding theorem we have

THEOREM 64.2. *Let $\phi_p(u)$, $p = 1, 2, 3, \cdots$, be a sequence of nondecreasing functions on the interval $-\infty < u < +\infty$, such that for all these values of u*

$$c \leq \phi_p(u) \leq C, \quad p = 1, 2, 3, \cdots,$$

where c and C are finite constants. Then there exists a nondecreasing function $\phi(u)$ such that $c \leq \phi(u) \leq C$, and a sequence of indices $p_1 < p_2 < p_3 < \cdots$, such that for all values of u,

$$\lim_{k=\infty} \phi_{p_k}(u) = \phi(u).$$

Proof. Let $v_1, v_2, v_3, \cdots$ be a countable set of points, everywhere dense along the real axis, which set will be denoted by V. Since the functions $\phi_p(u)$ are uniformly bounded, we can select a subsequence S_1, convergent at $u = v_1$; from S_1 we may select a subsequence S_2, convergent at $u = v_2$; from S_2 we may select a subsequence S_3, convergent at $u = v_3$; $\cdots$. Using the well-known diagonal process, we may then select a sequence S consisting of one function from each of the sets S_p, which converges at every point of V. We define $\phi(u)$ over V as the limit of the sequence S. If v' and v'' are points of V such that $v'' > v'$, then $\phi(v') \leq \phi(v'')$, so that we may complete the definition of $\phi(u)$ by writing

$$\phi(u) = \lim_{v=u} \phi(v),$$

where v approaches u over points of V which are less than u. The function $\phi(u)$ is now defined for all values of u; it is clearly nondecreasing; and $c \leq \phi(u) \leq C$.

Let u be any point where $\phi(u)$ is continuous. Then if $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\phi(u) - \epsilon \leq \phi(u') \leq \phi(u) \leq \phi(u'') \leq \phi(u) + \epsilon,$$

if

$$u - \delta \leq u' \leq u \leq u'' \leq u + \delta.$$

Since V is everywhere dense, we may take u' and u'' to be points of V. Then if K is sufficiently large:

$$\left.\begin{array}{l}\phi(u') - \epsilon \leq \phi_k(u') \leq \phi(u') + \epsilon, \\ \phi(u'') - \epsilon \leq \phi_k(u'') \leq \phi(u'') + \epsilon,\end{array}\right. \quad \text{if} \quad k \geq K, \quad \phi_k(u) \quad \text{in} \quad S.$$

Therefore, if $k \geq K$, $\phi_k(u)$ in S,

$$\phi(u) - 2\epsilon \leq \phi(u') - \epsilon \leq \phi_k(u') \leq \phi_k(u) \leq \phi_k(u'') \leq \phi(u'') + \epsilon \leq \phi(u) + 2\epsilon,$$

so that

$$| \phi_k(u) - \phi(u) | \leq 4\epsilon \quad \text{if} \quad k \geq K, \quad \phi_k(u) \quad \text{in} \quad S,$$

i.e., the sequence S converges to $\phi(u)$ at all points u where $\phi(u)$ is continuous. Now, we may use the diagonal process to select from S a subsequence $f_{p_k}(u)$, $k = 1, 2, 3, \cdots$, convergent at all the points of discontinuity of $\phi(u)$. If $\phi(u)$ is then suitably defined at these points, our sequence will converge to $\phi(u)$ everywhere.

65. The Stieltjes Inversion Formula. The Stieltjes integral

$$\int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u} \tag{65.1}$$

is called a **Stieltjes transform.**

THEOREM 65.1. *If $\phi(u)$ is bounded and nondecreasing on the interval $-\infty < u < +\infty$, then the integral* (65.1) *converges absolutely and uniformly over every bounded closed region whose distance from the real axis is positive, and represents a function $F(z)$ which is analytic for all nonreal values of z.*

Proof. If s and t are finite numbers and $s < t$, it follows from Theorem 63.3 that the integral

$$\int_{s}^{t} \frac{d\phi(u)}{z - u} \tag{65.2}$$

exists for all nonreal z. Let G be a simply connected closed region whose distance from the real axis is $\delta > 0$. Then if z is in G we have:

$$\left| \int_s^t \frac{d\phi(u)}{z-u} \right| \leq \int_s^t \frac{d\phi(u)}{|z-u|} \leq \frac{1}{\delta}[\phi(t) - \phi(s)].$$

Since $\phi(t) - \phi(s) \to 0$ as s and t both approach $+\infty$ or both approach $-\infty$, it follows that (65.2) converges for $t = +\infty$, $s = -\infty$, absolutely and uniformly over G.

To prove that (65.1) is an analytic function of z over G, it suffices, by a theorem of Weierstrass, to show that (65.2) is an analytic function of z over G for all finite values of s and t. To prove this, it is only necessary to note that the sums $S(u, v)$ which were used in defining a Stieltjes integral, when formed for (65.2), are analytic functions of z which are uniformly bounded over G, and then to apply Theorem 24.2.

Note. The integral in Theorem 65.1 does not, in general, define a single analytic function, but two analytic functions: one in the upper half-plane, $\mathfrak{J}(z) > 0$, and another in the lower half-plane $\mathfrak{J}(z) < 0$. The real axis may be a natural boundary separating the two functions. The two functions are one and the same analytic function in case there is an interval of constancy of $\phi(u)$. If $\phi(u)$ has discontinuities everywhere dense along the real axis, then the real axis is a natural boundary, and one function cannot be continued analytically into the other.

We now consider the problem of expressing the monotone function $\phi(u)$ in the integral (65.1) in terms of the analytic function $F(z)$ defined for $\mathfrak{J}(z) > 0$ by the integral. We form the integral of $F(z)$ along the rectilinear path from $z = s + iy$ to $z = t + iy$,

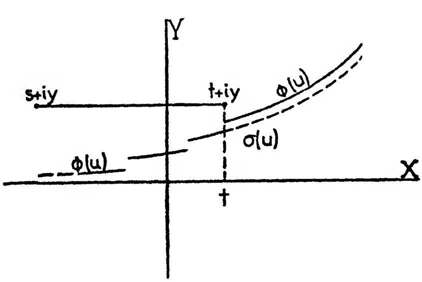

FIG. 10

$s < t, y > 0$, and obtain

$$\int_{s+iy}^{t+iy} F(z)dz = \int_{s+iy}^{t+iy} dz \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z-u} = \int_{-\infty}^{+\infty} d\phi(u) \int_{s+iy}^{t+iy} \frac{dz}{z-u},$$

the interchange of the order of integration being allowable by virtue of the uniform convergence of the integral (65.1) along the path of integration with respect to z. On equating imaginary parts, and performing one of the integrations, we then have, if $z = x + iy$,

$$\int_s^t \mathfrak{I}[F(z)]dx = \int_{-\infty}^{+\infty} \left(\arctan \frac{u-t}{y} - \arctan \frac{u-s}{y} \right) d\phi(u).$$

We now observe that the integral

$$\int_{-\infty}^{+\infty} \arctan \frac{u-t}{y}\, d\phi(u) \tag{65.3}$$

is unchanged if $\phi(u)$ is replaced by the function

$$\sigma(u) = \begin{cases} \phi(u) & \text{for} \quad u < t, \\ \phi(u) - \phi(t+0) + \phi(t-0) & \text{for} \quad u > t, \\ \phi(t-0) & \text{for} \quad u = t, \end{cases}$$

which is continuous at $u = t$. We may then write (65.3) as

$$\int_{-U}^{t-\delta} \arctan \frac{u-t}{y}\, d\sigma(u) + \int_{t+\delta}^{+U} \arctan \frac{u-t}{y}\, d\sigma(u) + \theta(U, \delta, y),$$

where, for any $\epsilon > 0$,

$$|\,\theta(U, \delta, y)\,| < \epsilon, \quad \text{if} \quad U > N, 0 < \delta \leq \delta_0,$$

N and δ_0 being numbers depending upon ϵ but not upon $y > 0$. On letting y approach 0 through positive values, the first of the above integrals has the limit

$$-\frac{\pi}{2}[\sigma(t-\delta) - \sigma(-U)],$$

and the second has the limit

$$+\frac{\pi}{2}[\sigma(+U) - \sigma(t+\delta)].$$

Therefore, we conclude immediately that as y approaches zero through positive values, the integral (65.3) has the limit

$$\frac{\pi}{2}[\sigma(+\infty)+\sigma(-\infty)-\sigma(t-0)-\sigma(t+0)]$$
$$=\pi\left(\frac{\phi(+\infty)+\phi(-\infty)}{2}-\frac{\phi(t-0)+\phi(t+0)}{2}\right).$$

On subtracting from this expression the same expression with t replaced by s, we then have the following formula:

$$\frac{1}{\pi}\lim_{y=+0}\int_s^t \Im[F(x+iy)]dx$$
$$=\frac{\phi(s-0)+\phi(s+0)}{2}-\frac{\phi(t-0)+\phi(t+0)}{2}. \qquad (65.4)$$

This is **Stieltjes' inversion formula** [95, No. 39].

66. Representation of an Equivalent Function of a Positive Definite J-fraction as a Stieltjes Transform. We now consider the question of obtaining an integral expression

$$\int_{-\infty}^{+\infty}\frac{d\phi(u)}{z-u} \qquad (66.1)$$

for an arbitrary equivalent function of a positive definite J-fraction. We recall that an equivalent function is analytic for $\Im(z) > 0$, and has a negative imaginary part in this domain. These are properties of the Stieltjes transform (66.1) if $\phi(u)$ is bounded and nondecreasing. In (62.3) we have an asymptotic expression for an equivalent function, namely,

$$f(z)=\frac{1}{z}+\frac{0(1)}{yz}, \quad y=\Im(z)>0. \qquad (66.2)$$

We shall prove that these properties are sufficient to guarantee that an equivalent function of a positive definite J-fraction is equal to a Stieltjes transform (66.1), *in which* $\phi(u)$ *is bounded, nondecreasing, and* $\phi(+\infty)-\phi(-\infty)=1$. [35.]

Suppose that $0 < y < c$, and consider the contour Γ in the

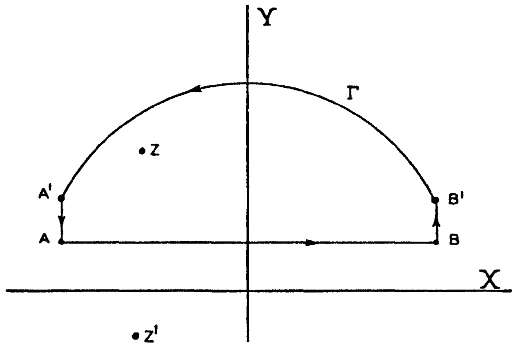

FIG. 11.

upper half of the z-plane, consisting of: the straight line segment from $A = -c^2 + iy$ to $B = c^2 + iy$, the straight line segment from B to $B' = c^2 + ic$, the arc of the circle with center at the origin through B' and $A' = -c^2 + ic$, and, finally, the straight line segment from A' to A. Inasmuch as $f(z)$ is analytic in the domain interior to Γ, we have:

$$\int_A^B f(z)dz = -\int_B^{B'} f(z)dz - \int_{B'}^{A'} f(z)dz - \int_{A'}^{A} f(z)dz. \qquad (66.3)$$

Now, by (66.2),

$$\int_B^{B'} f(z)dz = i\int_y^c \left(\frac{1}{c^2 + it} + \frac{0(1)}{t(c^2 + it)}\right) dt.$$

The modulus of this integral is therefore less than

$$\frac{1}{c} + \frac{K}{c^2}\log\frac{y}{c}, \quad K \text{a constant.} \qquad (66.4)$$

In like manner, we find that the modulus of the last integral in the right-hand member of (66.3) is less than the quantity (66.4). If we put $z = re^{i\theta}$, $r = c\sqrt{1 + c^2}$, in the second integral in the right-hand member of (66.3), and use (66.2), that integral becomes

$$\int_{B'}^{A'} f(z)dz = i\int_{\text{arc tan }(1/c)}^{\pi - \text{arc tan }(1/c)} d\theta + i\int_{\text{arc tan }(1/c)}^{\pi - \text{arc tan }(1/c)} \frac{0(1)}{r \sin\theta} d\theta.$$

Since $r \sin \theta \geq c$ along the path of integration, the modulus of the last integral does not exceed $\pi K/c$. Hence, we have

$$\int_{B'}^{A'} f(z)dz = \pi i - 2i \arctan\left(\frac{1}{c}\right) + \frac{K'}{c}, \quad |K'| \leq K.$$

From these estimates, we now conclude immediately that

$$\lim_{c=\infty} \int_{-c^2}^{+c^2} f(x + iy)dx = -\pi i, \quad \text{if} \quad y > 0. \tag{66.5}$$

Let $\mathfrak{I}[f(x + iy)] = v(x, y)$. By hypothesis, $v(x, y) < 0$ for $y > 0$. From (66.5) we now have

$$\lim_{c=\infty} \int_{-c^2}^{+c^2} -v(x, y)dx = \pi, \quad \text{if} \quad y > 0. \tag{66.6}$$

Let

$$\phi(y, u) = \frac{1}{\pi} \int_0^u -v(x, y)dx.$$

Since the integrand is positive, this is a nondecreasing function of u; and by (66.6) it follows that $\phi(y, u)$ is bounded, and

$$\phi(y, +\infty) - \phi(y, -\infty) = 1. \tag{66.7}$$

We now apply Theorem 64.2. There exists a sequence of positive values of y: $y_1, y_2, y_3, \cdots$, approaching 0, and a bounded nondecreasing function $\phi(u)$, such that $\lim_{n=\infty} \phi(y_n, u) = \phi(u)$, for all values of u. By (66.7) we must have, $\phi(+\infty) - \phi(-\infty) = 1$.

If z is any point within Γ, Cauchy's integral formula now gives:

$$f(z) = \frac{1}{2\pi i} \int \frac{f(s)ds}{s - z}.$$

With the aid of (66.2) one may readily verify that for $c = \infty$, this goes over into

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(u + iy)du}{u + iy - z}, \tag{66.8}$$

where the integral is to be regarded in the sense of Cauchy's principal value. Let z' be the point outside Γ which is symmetrical to the point z with respect to the line AB. Then we have

$$0 = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(u+iy)du}{u+iy-z'}, \quad \text{or} \quad 0 = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\overline{f(u+iy)}du}{u-iy-\bar{z}'}.$$

Inasmuch as $u + iy - z = u - iy - \bar{z}'$, we then have, on subtracting the last equation from the equation (66.8), and then introducing the function $\phi(y, u)$:

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{v(u, y)du}{u+iy-z} = \int_{-\infty}^{+\infty}\frac{d\phi(y, u)}{z-u-iy}.$$

On letting y approach 0 over the sequence $y_1, y_2, y_3, \cdots$, we then find, with the aid of Theorem 64.1 and an easy argument, that

$$f(z) = \int_{-\infty}^{+\infty}\frac{d\phi(u)}{z-u}.$$

Since $\phi(u)$ is bounded and nondecreasing, the integral converges absolutely (not merely as Cauchy's principal value) and uniformly for z in any region whose distance from the real axis is positive.

Since $f(z)$ is now expressed as a Stieltjes transform, it follows from Stieltjes' inversion formula, (65.4), that $\phi(u)$ is determined, except for an additive constant, at all its points of continuity, by the formula

$$\phi(u) = \lim_{y=+0}\int_0^u \mathfrak{J}[f(x+iy)]dx,$$

where y approaches 0 in any manner through positive values, and not merely over the sequence y_p. Thus, $\phi(u)$ is determined uniquely by $f(z)$ to an additive constant, at all points where $\phi(u)$ is continuous.

We shall state this result as

THEOREM 66.1. *If $f(z)$ is an equivalent function of a positive definite J-fraction, then $f(z)$ can be expressed as a Stieltjes transform,*

$$f(z) = \int_{-\infty}^{+\infty}\frac{d\phi(u)}{z-u},$$

where $\phi(u)$ is a bounded nondecreasing function of u such that $\phi(+\infty) - \phi(-\infty) = +1$. The function $\phi(u)$ is determined uniquely up to an additive constant at all its points of continuity by the function $f(z)$. **[138.]**

We have actually proved the following more general theorem.

THEOREM 66.2. *Any function $f(z)$ which is analytic and has a negative imaginary part in the domain $\mathfrak{I}(z) > 0$, and which there has the asymptotic form*

$$f(z) = \frac{1}{z} + \frac{0(1)}{zI(z)}, \tag{66.9}$$

can be represented as a Stieltjes transform as in Theorem 66.1. [**35.**]

Note. It should be remarked that the condition (66.9) is not necessary in order that $f(z)$ be expressible as a Stieltjes transform of the type under consideration. This may be seen from the example

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u} = \int_{1}^{+\infty} \frac{d(1 - u^{-1/2})}{z - u}. \tag{66.10}$$

Here

$$\phi(u) = \begin{cases} 0 & \text{for} \quad u \leq 1, \\ 1 - u^{-1/2} & \text{for} \quad u > 1. \end{cases}$$

The function (66.10) does not satisfy the condition (66.9). [**35.**]

67. Proper Equivalent Functions. We shall say that a function $f(z)$ is a **proper equivalent function** of a positive definite J-fraction if it is the limit of a subsequence of the approximants of the J-fraction. We may establish for these functions the integral representation of Theorem 66.1, *independently of the matrix theory of Chapter XII.* In fact, if we use (17.7) and consider the partial fraction development of the pth approximant of the J-fraction (cf. [**138**], we find that

$$\frac{A_p(z)}{B_p(z)} = \int_{-\infty}^{+\infty} \frac{d\phi_p(u)}{z - u}, \tag{67.1}$$

where $\phi_p(u)$ is a bounded nondecreasing function of u such that $\phi_p(+\infty) - \phi_p(-\infty) = 1$. If we apply Theorems 64.1 and 64.2 to (67.1), we are led at once to the desired integral representation for a proper equivalent function.

We note that in the case of a *real* J-fraction it follows from (27.3) that the function $\phi_p(u)$ in (67.1) is a simple step-function. If the poles of the approximants are confined to some part of the real axis, then it is clear that the integral need be extended only over that part. For example, a proper equivalent function of the S-fraction (28.1), in which the k_p are real and positive, has the form

$$\int_{-\infty}^{0} \frac{d\phi(u)}{z - u}. \tag{67.2}$$

If we replace u by $-u$ and put $\theta(u) = -\phi(-u)$, this becomes

$$\int_{0}^{+\infty} \frac{d\theta(u)}{z + u}. \tag{67.3}$$

The value of the continued fraction of Theorem 27.4 has the integral representation

$$\int_{-1}^{+1} \frac{d\phi(u)}{\zeta - u}, \tag{67.4}$$

and the value of (27.8) is of the form

$$\int_{0}^{1} \frac{d\theta(u)}{1 + zu}. \tag{67.5}$$

Exercise 13

13.1. Show by means of the Stieltjes inversion formula that

$$\frac{1}{1 - u + (1 + u)\sqrt{1 + z}} = \int_0^1 \frac{d\phi(t)}{1 + zt}, \quad (0 \leq u \leq 1),$$

where

$$\phi(t) = \frac{1}{\pi} \int_0^t \frac{1 + u}{(1 + u)^2 - 4us} \sqrt{\frac{1}{s} - 1}\,ds, \quad 0 \leq t \leq 1. \quad [88.]$$

13.2. Show that

$$\frac{u + 1}{u + (n + 1)^2} = \int_0^1 t^n d\phi(t), \quad n = 0, 1, 2, \cdots,$$

where

$$\phi(t) = t\left[\cos(\sqrt{u}\log t) - \frac{\sin(\sqrt{u}\log t)}{\sqrt{u}}\right], \quad 0 \leq u \leq 1, \quad 0 \leq t \leq 1. \quad [88.]$$

13.3. Let

$$f(z) = \int_0^1 \frac{d\phi(u)}{1 + zu},$$

where $\phi(u)$ is of bounded variation on the interval $0 \leq u \leq 1$. Show that $\phi(u)$ is continuous at $u = 1$ if, and only if, $\lim_{z=-1} (1 + z)f(z) = 0$, where z approaches -1 through values interior to or upon the circle $|z| = 1$; and that

$\phi(u)$ is continuous at $u = 0$ if, and only if, $\lim_{z=\infty} f(z) = 0$, where z approaches ∞ along any ray through the origin with the exception of the negative half of the real axis. Show that $\phi(u)$ is continuous at $u = r$, $0 < r \leq 1$, if, and only if, $\lim_{z=-1/r} (1 + zr)f(z) = 0$, where z approaches $-1/r$ along any ray with the exception of the portion of the real axis to the left of $-1/r$. [83.]

13.4. Show that if $n = 0, 1, 2, \cdots$, then

$$\lim_{n=\infty} \int_0^1 u^n d\phi(u) = \phi(1) - \phi(1 - 0).$$

13.5. Show that

$$\int_0^1 u^z d\phi(u)$$

is an analytic function of z for $\Re(z) > 0$, and is a continuous function of z for $\Re(z) \geq 0$.

13.6. Show that it is impossible to find a function $\phi(u)$ of bounded variation such that

$$\frac{1}{n!} = \int_0^1 u^n d\phi(u), \quad n = 0, 1, 2, \cdots.$$

13.7. Let

$$c_p = \int_0^1 u^p d\phi(u), \quad p = 0, 1, 2, \cdots,$$

where $\phi(u)$ is of bounded variation on the interval $0 \leq u \leq 1$. Let $c(z)$ be analytic and have modulus less than M for $\Re(z) > 0$. Let $c(p) = c_p$, $p = 0, 1, 2, \cdots$. Then

$$c(z) = \int_0^1 u^z d\phi(u).$$

13.8. Let

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u}, \quad \phi(+\infty) - \phi(-\infty) = 1,$$

where $\phi(u)$ is a bounded nondecreasing function of u. Suppose that the integral

$$\int_{-\infty}^{+\infty} u d\phi(u)$$

converges. Then, if $\Im(z) > 0$,

$$f(z) = \frac{1}{z} + \frac{g(z)}{z\Im(z)}, \quad |g(z)| < M,$$

where M is a constant independent of z.

13.9. Express the function $1/(1 + z^k)$, $0 < k \leq 1$, as a Stieltjes transform.

13.10. Let $\phi(u)$ be bounded and nondecreasing, and let $f(u)$ be bounded and Riemann integrable for $a \leq u \leq b$. Let

$$\theta(u) = \int_a^u f(t)dt.$$

The Stieltjes integral

$$\int_a^b \phi(u)d\theta(u)$$

exists and is equal to the Riemann integral

$$\int_a^b \phi(u)f(u)du.$$

13.11. Let $\phi(u)$ and $f(u)$ have the properties required in 13.10. Then there exists a number X in the interval $a \leq x \leq b$ such that

$$\int_a^b \phi(u)f(u)du = \phi(a)\int_a^X f(u)du + \phi(b)\int_X^b f(u)du.$$

13.12. Let $f(u)$ and $\phi(u)$ be functions of bounded variation on the interval $a \leq u \leq b$, which have no common discontinuities in the interval. Then the Stieltjes integral $\int_a^b f(u)d\phi(u)$ exists. The restriction on the discontinuities is necessary, for the integral does not exist if $f(u)$ and $\phi(u)$ have a point of discontinuity in common.

Chapter XIV

THE MOMENT PROBLEM FOR A FINITE INTERVAL

In § 50, we introduced an operation called *formal* integration,

$$c_p = \int u^p d\phi_c(u), \quad p = 0, 1, 2, \cdots,$$

which replaces u^p by a given constant c_p. By the **moment problem,** we shall understand the problem of expressing this *formal* operation by means of an *actual* integral operation. It is clear that the possibility of solving the moment problem and the character of the solution when it exists must depend upon the nature of the given sequence $\{c_p\}$. In the present chapter we shall determine those sequences $\{\mu_p\}$ which admit of a solution of the form

$$\mu_p = \int_0^1 u^p d\mu(u), \quad p = 0, 1, 2, \cdots,$$

where $\mu(u)$ is a bounded nondecreasing function, and the integral is to be understood in the sense of Stieltjes. That is, we replace here the formal integral by a Stieltjes integral extended over a finite interval. For the sake of simplicity, the interval is taken from 0 to 1.

68. Formulation of the Problem. We begin by proving the following uniqueness theorem.

THEOREM 68.1. *Let $\{c_p\}$ be a given sequence of complex constants, and suppose that there exist two functions $\phi_1(u)$ and $\phi_2(u)$ of bounded variation on a finite interval $a \leq u \leq b$, such that*

$$c_p = \int_a^b u^p d\phi_1(u) = \int_a^b u^p d\phi_2(u), \quad p = 0, 1, 2, \cdots,$$

Put $\phi(u) = \phi_1(u) - \phi_2(u)$. Then $\phi(u) = \phi(a)$ for $u = b$ and for all other values of u in the interval (a, b) with the exception of at most a countable set included in the set of discontinuities of $\phi(u)$.

Proof. Since $\int_a^b u^p d\phi(u) = 0$, $p = 0, 1, 2, \cdots$, it follows that $\int_a^b G(u) d\phi(u) = 0$ for every polynomial $G(u)$, and, therefore, by a theorem of Weierstrass, for every continuous function $G(u)$. The theorem now follows from Theorem 63.4.

DEFINITION 68.1. *By the moment problem for the interval* (0, 1), *we shall understand the problem of determining a bounded nondecreasing function $\mu(u)$ such that*

$$\mu(u) = \begin{cases} 0 \quad \text{for} \quad u \leq 0, \\ \dfrac{\mu(u-0) + \mu(u+0)}{2} \quad \text{for} \quad 0 < u < 1, \\ \mu(1) \quad \text{for} \quad u > 1, \end{cases} \tag{68.1}$$

which satisfies the system of equations

$$\mu_p = \int_0^1 u^p d\mu(u), \quad p = 0, 1, 2, \cdots, \tag{68.2}$$

where $\{\mu_p\}$ is a given sequence of constants. For the sake of brevity, we shall refer to this as "the moment problem (68.2)." *A function $\mu(u)$ satisfying* (68.1) *will be said to be* **normalized,** *and if it satisfies* (68.2) *it will be called a* **solution** *of the moment problem.*

From Theorem 68.1 we have

THEOREM 68.2. *If the moment problem for the interval* (0, 1) *has a solution, then the solution is unique.*

DEFINITION 68.2. *By the* ***symmetrical moment problem for the interval*** $(-1, +1)$, *we shall understand the problem of determining a bounded nondecreasing function $\theta(u)$ such that*

$$\theta(u) = \begin{cases} \theta(-1) \quad \text{for} \quad u < -1, \\ \dfrac{\theta(u-0) + \theta(u+0)}{2} \quad \text{for} \quad -1 < u < +1, \\ \theta(+1) \quad \text{for} \quad u > 1, \end{cases} \tag{68.3}$$

and such that

$$\theta(u) = -\theta(-u) \quad \text{for} \quad -\infty < u < +\infty, \tag{68.4}$$

which satisfies the system of equations

$$d_p = \int_{-1}^{+1} u^p d\theta(u), \quad p = 0, 1, 2, \cdots, \tag{68.5}$$

where $\{d_p\}$ *is a given sequence of constants. For the sake of brevity, we shall refer to this as "the moment problem* (68.5)."

One may readily verify that the following theorem is true.

THEOREM 68.3. *If the moment problem* (68.5) *has the solution* $\theta(u)$, *and if we put* $\mu_p = d_{2p}/2$, $p = 0, 1, 2, \cdots$, *then the moment problem* (68.2) *has the solution* $\mu(u) = \theta(\sqrt{u})$; *and if the moment problem* (68.2) *has the solution* $\mu(u)$, *and if we put* $d_{2p+1} = 0$, $d_{2p} = 2\mu_p$, $p = 0, 1, 2, \cdots$, *then the moment problem* (68.5) *has the solution* $\theta(u) = \mu(u^2)$, $u \geq 0$, $\theta(u) = -\mu(u^2)$, $u < 0$.

In view of this theorem, the solution of either of these moment problems can be reduced to that of the other.

69. Solution of the Moment Problem by Means of S-fractions. Excepting in the trivial case $d_p = 0$, $p = 0, 1, 2, \cdots$, we may evidently assume that $d_0 > 0$ in (68.5). Moreover, if $d_0 > 0$, there is no loss in generality in assuming $d_0 = 1$. Otherwise we could consider instead the sequence d_p/d_0, $p = 0, 1, 2, \cdots$, and the solution $\theta(u)/d_0$.

We shall now prove the following theorem.

THEOREM 69.1. *The symmetrical moment problem* (68.5), *in which* $\{d_p\}$ *is a given sequence of constants and* $d_0 = 1$, *has a solution* (*cf.* *Definition* 68.2) *if, and only if,* $d_0, d_1, d_2, \cdots$ *are moments of a real S-fraction of the form*

$$\cfrac{1}{z - \cfrac{(1-g_0)g_1}{z - \cfrac{(1-g_1)g_2}{z - \cdots}}}, \tag{69.1}$$

in which $0 \leq g_p \leq 1$, $p = 0, 1, 2, \cdots$. *When the condition is satisfied, then the solution is the function* $\theta(u)$ (*suitably normalized*) *such that*

$$\int_{-1}^{+1} \frac{d\theta(u)}{z - u} \tag{69.2}$$

is the value of the S-fraction (cf. (67.4)).

Proof. We suppose first that $d_0, d_1, d_2, \cdots$ are moments of the S-fraction (69.1). We have seen (cf. Theorems 27.3 and 27.4, and (67.4)) that this S-fraction converges uniformly over every finite closed region whose distance from the interval $-1 \leq x \leq +1$ is positive, to a function of the foım (69.2). By Theorem 54.1, the power series expansion in descending powers of z, namely, $\Sigma(d_p/z^{p+1})$, converges for $|z| > 1$ to the value of the S-fraction, i.e., to the integral (69.2). Consequently,

$$d_p = \int_{-1}^{+1} u^p d\theta(u), \quad p = 0, 1, 2, \cdots,$$

so that the moment problem (68.5) has a solution.

We now suppose, conversely, that $d_0 = 1, d_1, d_2, \cdots$ is a sequence of numbers such that the moment problem (68.5) has a solution $\theta(u)$ of the specified character, and shall prove that these numbers are the moments of an S-fraction (69.1).

If $\theta(u)$ is a simple step-function, so that there are (say) exactly m points u_0 such that $\theta(u_0 + t) > \theta(u_0 - t)$ for $t > 0$, then the quadratic forms

$$\int_{-1}^{+1} (X_0 + X_1 u + \cdots + X_n u^n)^2 d\theta(u), \quad n = 0, 1, 2, \cdots, m - 1,$$

are positive definite by virtue of Theorem 63.5. Therefore, the determinants Δ_n of (50.2), formed with the moments d_p, are positive for $n = 0, 1, 2, \cdots, m - 1$. We may therefore construct a terminating J-fraction (51.1) in which $a_0 = d_0 = 1$, and $a_1, a_2, \cdots, a_{m-1}$ are positive. This J-fraction has the value (69.2) since this integral and the J-fraction are rational functions of z in which numerator and denominator are of degree $m - 1$ and m, respectively, and since the power series expansion in descending powers of z of the J-fraction agrees term by term with the like expansion of the integral for the first $2m$ terms. Therefore,

$d_0, d_1, d_2, \cdots$ are the moments of this terminating J-fraction. Inasmuch as, by (68.4), the integral (69.2) is an odd function of z, it follows that the J-fraction has the form

$$\cfrac{1}{z - \cfrac{a_1}{z - \cfrac{a_2}{z - \cdot \; \cdot \; - \cfrac{a_{m-1}}{z}}}}. \tag{69.3}$$

The denominators $B_p(z)$ of (69.3) satisfy the conditions

$$\int_{-1-c}^{+1+c} u^r B_p(u) d\theta(u) = 0, \quad r = 0, 1, 2, \cdots, p - 1, \quad p \leq m,$$

for all $c > 0$, by (50.1). Hence, since the roots of $B_p(u)$ are the poles of (69.2) if $p = m$, and by virtue of Theorem 63.6, if $p < m$, $B_p(u)$, a polynomial of degree p, changes sign p times on the interior of the interval $-1 - c \leq u \leq +1 + c$. Since $B_p(u) > 0$ for sufficiently large values of u, it then follows that $B_p(u) > 0$ for $u > 1$.

From the fundamental recurrence formulas we now have

$$a_p = \frac{B_p(1 + c)}{B_{p-1}(1 + c)} \left(1 + c - \frac{B_{p+1}(1 + c)}{B_p(1 + c)}\right), \quad p = 1, 2, 3, \cdots, m - 1. \tag{69.4}$$

Since $a_p > 0$, it then follows that

$$0 < \frac{B_{p+1}(1 + c)}{B_p(1 + c)} < 1 + c, \quad p = 0, 1, 2, \cdots, m - 1, \quad (c > 0).$$

Hence, if we put

$$\lim_{c=0} \frac{B_{p+1}(1 + c)}{B_p(1 + c)} = 1 - g_p,$$

we have $0 \leq g_p \leq 1$, $p = 0, 1, 2, \cdots, m - 1$, and, by (69.4), $a_p = (1 - g_{p-1})g_p$, $p = 1, 2, 3, \cdots, m - 1$, so that (69.3) has the form (69.1).

In case $\theta(u)$ is not a simple step-function, the determinants $\Delta_n > 0$ for all values of n, and consequently we have, instead of (69.3), a nonterminating J-fraction, and $P(1/z) = \Sigma(d_p/z^{p+1})$ is its power series expansion. The above argument can be used to show that this J-fraction has the form (69.1).

This completes the proof of Theorem 69.1.

If we make an equivalence transformation and replace $-1/z^2$ by z in the S-fraction (69.1), and then make suitable changes in notation in accordance with the remarks in § 68, Theorem 69.1 may be stated as follows.

THEOREM 69.2. *The moment problem for the interval* (0, 1) (*cf. Definition* 68.1)

$$\mu_p = \int_0^1 u^p d\mu(u), \quad p = 0, 1, 2, \cdots, \tag{69.5}$$

has a solution if, and only if, the power series

$$\mu_0 - \mu_1 z + \mu_2 z^2 - \cdots \tag{69.6}$$

has a continued fraction expansion of the form

$$\cfrac{\mu_0}{1 + \cfrac{(1 - g_0)g_1 z}{1 + \cfrac{(1 - g_1)g_2 z}{1 + \cdot_{\cdot_{\cdot}}}}}, \tag{69.7}$$

where $\mu_0 \geq 0$ *and* $0 \leq g_p \leq 1$, $p = 0, 1, 2, \cdots$. [126.]

70. Some Geometry.[24] The moments μ_p of the continued fraction (69.7) may be characterized geometrically in an interesting way, if we regard $(\mu_0, \mu_1, \cdots, \mu_{n-1})$ as a point in n-dimensional Euclidean space.

We again take $\mu_0 = 1$. If we use the minimal parameters (cf. Theorem 19.2) of the chain sequence appearing in (69.7), that continued fraction becomes

$$\cfrac{1}{1 + \cfrac{m_1 z}{1 + \cfrac{(1 - m_1)m_2 z}{1 + \cfrac{(1 - m_2)m_3 z}{1 + \cdot_{\cdot_{\cdot}}}}}}, \tag{70.1}$$

[24] Cf. Dines [10] for further details on the ideas of this section.

where $0 \leq m_p \leq 1$, $p = 1, 2, 3, \cdots$. The coefficients μ_p in the power series expansion (69.6) of this continued fraction are polynomials in $m_1, m_2, m_3, \cdots$, the first three being given by

$$\mu_0 = 1,$$
$$\mu_1 = m_1, \tag{70.2}$$
$$\mu_2 = m_1 m_2 + m_1^2(1 - m_2).$$

These may be readily computed by means of the formulas of Stieltjes (Theorem 53.1).

Let us regard (μ_0, μ_1, μ_2) as the co-ordinates of a point in 3-space. Since $0 \leq m_p \leq 1$, $p = 1, 2, 3$, we see by (70.2) that this point is characterized by the fact that it is in the shaded

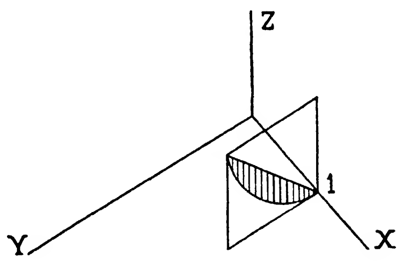

FIG. 12.

region shown in the accompanying figure. This region is bounded by the parabolic arc

$$z = y^2, \quad x = 1, \quad 0 \leq y \leq 1, \tag{70.3}$$

and by the straight line segment $z = y$, $x = 1$, $0 \leq y \leq 1$. Any point of this region must be of the form

$$[1, m_1, m_1 m_2 + m_1^2(1 - m_2)], \quad 0 \leq m_1 \leq 1, \quad 0 \leq m_2 \leq 1,$$

and, conversely, every such point is in the region.

This region, which we shall call E_3, has two characteristic properties, namely:

(a) If p and q are two points of E_3, then all points of the line segment $\overline{pq}$ are in E_3. This is the property called **convexity**.

(b) E_3 is the smallest convex set in 3-space which contains the parabolic arc (70.3), i.e., E_3 is the intersection of all the convex sets of 3-space which contain this arc; it is the so-called **convex extension** of the arc.

Since we have fixed $\mu_0 = 1$, the point (μ_0, μ_1, μ_2) is actually in a convex set in 2-space.

We shall establish the following theorem:

THEOREM 70.1. *The moment problem* (69.5) *for the interval* (0, 1), *where we assume* $\mu_0 = 1$, *has a solution if, and only if, for every* $n(= 1, 2, 3, \cdots)$ *the point* $(\mu_1, \mu_2, \mu_3, \cdots, \mu_n)$ *is in the convex extension of the arc whose parametric equations are*

$$\begin{aligned} x_1 &= u, \\ x_2 &= u^2, \quad 0 \le u \le 1. \quad [78.] \\ &\cdot \quad \cdot \quad \cdot \\ x_n &= u^n, \end{aligned} \tag{70.4}$$

Proof. Let C denote the arc (70.4). This is a bounded closed set of points in n-space. Its convex extension $E(C)$ consists of those points and only those points which may be centroids of distributions of positive masses of total mass unity, placed at a finite number of suitably placed points of C:

$$\begin{matrix} (x_{11}, & x_{12}, & \cdots, & x_{1n}), \\ (x_{21}, & x_{22}, & \cdots, & x_{2n}), \\ & & \cdot \quad \cdot \quad \cdot & \\ (x_{m1}, & x_{m2}, & \cdots, & x_{mn}). \end{matrix} \tag{70.5}$$

The points of $E(C)$ are then all those points $(\mu_1, \mu_2, \cdots, \mu_n)$ whose pth coordinates have the form

$$\mu_p = \sum_{r=1}^{m} \sigma_r x_{rp}, \quad \text{where} \quad \sigma_r \ge 0, \quad \sum_{r=1}^{m} \sigma_r = 1.$$

Since $x_{rp} = u_r{}^p$, where $0 \le u_r \le 1$, we then have:

$$\mu_p = \sum_{r=1}^{m} u_r{}^p \sigma_r = \int_0^1 u^p d\mu_n(u), \quad p = 1, 2, 3, \cdots, n,$$

where $\mu_n(u)$ is a simple step-function with the jump σ_r at the point u_r, and $\mu_n(1) - \mu_n(0) = 1$. For any fixed p, we may now let n increase over a suitable sequence of values for which

$$\lim_{n=\infty} \mu_n(u) = \mu(u),$$

where $\mu(u)$ is a bounded nondecreasing function, and $\mu(1) - \mu(0) = 1$. Then,

$$\mu_p = \int_0^1 u^p d\mu(u), \quad p = 0, 1, 2, \cdots, \tag{70.6}$$

so that the moment problem has a solution.

Let us suppose, conversely, that (70.6) holds, and we shall prove that $(\mu_1, \mu_2, \mu_3, \cdots, \mu_n)$ is a point of $E(C)$. The proof may be made by assuming the contrary and arriving at a contradiction. If $(\mu_1, \mu_2, \mu_3, \cdots, \mu_n)$ is not in $E(C)$, then there exists a plane

$$k_1(x_1 - \mu_1) + k_2(x_2 - \mu_2) + \cdots + k_n(x_n - \mu_n) = 0$$

passing through this point, separating n-space into two open half-spaces, one of which contains C (a so-called "bounding plane of C"). We may suppose that the notation has been so chosen that for all u in the interval $0 \leq u \leq 1$,

$$k_1(u - \mu_1) + k_2(u^2 - \mu_2) + \cdots + k_n(u^n - \mu_n) > 0. \tag{70.7}$$

Since the left-hand member of this inequality is a continuous function of u, it follows from Theorem 63.5 that

$$\int_0^1 \{k_1(u - \mu_1) + k_2(u^2 - \mu_2) + \cdots + k_n(u^n - \mu_n)\} d\mu(u) > 0.$$

But, by (70.6), this is impossible.

The proof of Theorem 70.1 is now complete.

Remark. This proof goes through in the same way if, instead of the functions (70.4), we use arbitrary continuous real functions $f_p(u)$, $p = 1, 2, 3, \cdots$. The corresponding "moment problem" is then

$$\mu_p = \int_0^1 f_p(u) d\phi(u), \quad p = 0, 1, 2, \cdots, \quad (f_0 \equiv 1).$$

71. Totally Monotone Sequences. A sequence of real numbers $\mu_0, \mu_1, \mu_2, \cdots$ is called **totally monotone** if

$$\mu_p \geq 0, \quad \Delta\mu_p = \mu_p - \mu_{p+1} \geq 0, \quad \cdots,$$
$$\Delta^n\mu_p = \Delta^{n-1}\mu_p - \Delta^{n-1}\mu_{p+1} \geq 0, \quad \cdots, \tag{71.1}$$
$$(p = 0, 1, 2, \cdots).$$

If the μ_p are given by (69.5), where $\mu(u)$ is nondecreasing, then

$$\Delta^k\mu_p = \int_0^1 (1-u)^k u^p d\mu(u) \geq 0, \quad k, \mathrm{p} = 0, 1, 2, \cdots,$$

where we define $\Delta^0\mu_p = \mu_p$, so that the sequence $\{\mu_p\}$ is totally monotone. The converse is also true.

THEOREM 71.1. *The moment problem* (69.5) *has a solution if, and only if, the given sequence* $\mu_0, \mu_1, \mu_2, \cdots$ *is a totally monotone sequence* [**28**].

Proof.[25] We have just seen that the condition is necessary. If we suppose the sequence is totally monotone, then, for any positive integer p,

$$\begin{aligned}
\mu_n &\geq 0, & n &= 0, 1, 2, \cdots, p,\\
\Delta\mu_n &\geq 0, & n &= 0, 1, 2, \cdots, p-1,\\
\Delta^2\mu_n &\geq 0, & n &= 0, 1, 2, \cdots, p-2, \\
&\cdot\quad\cdot\quad\cdot\\
\Delta^{p-1}\mu_n &\geq 0, & n &= 0, 1,\\
\Delta^p\mu_n &\geq 0, & n &= 0.
\end{aligned} \tag{71.2}$$

This system of inequalities is equivalent to the simpler system

$$\Delta^p\mu_0 \geq 0, \quad \Delta^{p-1}\mu_1 \geq 0, \cdots, \quad \Delta^0\mu_p \geq 0. \tag{71.3}$$

For, of course, (71.2) implies (71.3). Then, since

$$\Delta^r\mu_s = \Delta^{r-1}\mu_s - \Delta^{r-1}\mu_{s+1},$$

it readily follows that (71.3) implies (71.2).

[25] This proof is due to Schoenberg [82]. Hausdorff gave several proofs in [29].

The system (71.3) is, in turn, equivalent to the system of equations

$$
\begin{aligned}
\mu_0 - \binom{p}{1}\mu_1 + \binom{p}{2}\mu_2 - \cdots + (-1)^p\mu_p &= r_{p,0}, \\
\mu_1 - \binom{p-1}{1}\mu_2 + \cdots + (-1)^{p-1}\mu_p &= r_{p,1}, \\
&\cdots \\
\mu_{p-1} - \mu_p &= r_{p,p-1}, \\
\mu_p &= r_{p,p},
\end{aligned}
\tag{71.4}
$$

where the $r_{i,j}$ are arbitrary nonnegative numbers. This system may be readily solved for the μ_n in terms of the $r_{i,j}$. We find that

$$
\mu_n = \sum_{m=n}^{p} \binom{p-n}{m-n} r_{p,m} = \sum_{m=n}^{p} \frac{\binom{p-n}{m-n}}{\binom{p}{m}} L_{p,m}, \tag{71.5}
$$

where

$$
L_{p,m} = \binom{p}{m} r_{p,m} \geq 0.
$$

We may express the μ_n as Stieltjes integrals. Let $\mu_p(u)$ be defined as follows:

$$
\mu_p(u) = \begin{cases}
0, & u \leq 0, \\
L_{p,0}, & 0 < u \leq \frac{1}{p}, \\
L_{p,0} + L_{p,1}, & \frac{1}{p} < u \leq \frac{2}{p}, \\
\cdots & \\
L_{p,0} + L_{p,1} + \cdots + L_{p,p-1}, & \frac{p-1}{p} < u < \frac{p}{p}, \\
L_{p,0} + L_{p,1} + \cdots + L_{p,p}, & \frac{p}{p} \leq u.
\end{cases}
$$

The function $\mu_p(u)$ is nondecreasing, and is bounded inasmuch as

$\mu_p(1) = \mu_0$, by virtue of (71.5) with $n = 0$. In terms of the function $\mu_p(u)$, we may now write

$$\mu_n = \sum_{m=0}^{p} \frac{m(m-1)(m-2)\cdots(m-n+1)}{p(p-1)(p-2)\cdots(p-n+1)} L_{p,m}$$

$$= \int_0^1 \frac{u\left(u-\frac{1}{p}\right)\left(u-\frac{2}{p}\right)\cdots\left(u-\frac{n-1}{p}\right)}{1\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\cdots\left(1-\frac{n-1}{p}\right)} d\mu_p(u)$$

$$= \int_0^1 u^n d\mu_p(u) + 0\left(\frac{1}{p}\right),$$

where $0(1/p) \to 0$ as $p \to \infty$. On applying Theorems 64.1 and 64.2, we now conclude immediately that there exists a bounded nondecreasing function $\mu(u)$ such that

$$\mu_n = \int_0^1 u^n d\mu(u), \quad n = 0, 1, 2, \cdots.$$

This completes the proof of Theorem 71.1.

72. Composition of Moment Sequences. If $\{\lambda_p\}$ and $\{\mu_p\}$ are two totally monotone sequences, then it is evident that $\{\lambda_p + \mu_p\}$ is a totally monotone sequence. Also, since $\Delta(\lambda_p\mu_p) = \lambda_p\Delta\mu_p + \mu_{p+1}\Delta\lambda_p$, it follows immediately that $\{\lambda_p\mu_p\}$ is a totally monotone sequence. Let

$$\lambda_p = \int_0^1 u^p d\lambda(u), \quad \mu_p = \int_0^1 u^p d\mu(u), \quad p = 0, 1, 2, \cdots.$$

Then we must have

$$\lambda_p\mu_p = \int_0^1 u^p d\nu(u), \quad p = 0, 1, 2, \cdots, \tag{72.1}$$

where $\nu(u)$ is a bounded nondecreasing function. What is the relationship among $\lambda(u)$, $\mu(u)$ and $\nu(u)$? In order to answer this question, we consider the functions

$$f_\lambda(z) = \sum_{p=0}^{\infty} \lambda_p(-z)^p = \int_0^1 \frac{d\lambda(u)}{1+zu},$$

$$f_\mu(z) = \sum_{p=0}^{\infty} \mu_p(-z)^p = \int_0^1 \frac{d\mu(u)}{1+zu},$$

and

$$f_\nu(z) = \sum_{p=0}^{\infty} \nu_p(-z)^p = \int_0^1 \frac{d\nu(u)}{1+zu}.$$

Then we have,

$$f_\nu(z) = \int_0^1 \int_0^1 \frac{d\lambda(v)}{1+zuv}\, d\mu(u). \tag{72.2}$$

Supposing the functions $\lambda(u)$, $\mu(u)$ and $\nu(u)$ all normalized, we then find by the Stieltjes inversion formula, with $s < 0, 0 < t < 1$, that

$$-\frac{1}{\pi} \lim_{y=+0} \int_s^t \mathfrak{I}\left[\frac{1}{z} f_\nu\left(-\frac{1}{z}\right)\right] dz = \nu(t), \quad (z = x + iy).$$

Using (72.2), one may then show, as in the proof of the Stieltjes inversion formula, that

$$\nu(t) =$$

$$\lim_{y=0}\left[-\frac{1}{\pi}\int_0^1\int_0^1\left(\arc\tan\frac{uv-t}{y} - \arc\tan\frac{uv-s}{s}\right) d\lambda(v)d\mu(u)\right].$$

If we make the change of variable $w = uv$, and assume that t has only those values such that $\lambda(t/u)$ and $\mu(u)$ have no discontinuities in common,[26] $(0 \leq u \leq 1)$, we may readily evaluate this limit, and obtain the formula

$$\nu(t) = \int_0^1 \lambda\left(\frac{t}{u}\right) d\mu(u). \tag{72.3}$$

This is valid for all except at most a countable set of values of t.

Turning now to the difference, $\{\lambda_p - \mu_p\}$, of two totally monotone sequences, we are led to consider the following more general moment problem.

[26] Cf. Exercise 13.12. See [17] for further details concerning the formula (72.3).

To determine a function $\phi(u)$ *of bounded variation on the interval* $0 \leq u \leq 1$, *such that*

$$c_p = \int_0^1 u^p d\phi(u), \quad p = 0, 1, 2, \cdots, \tag{72.4}$$

where $\{c_p\}$ *is a given sequence of constants.*

If we permit the c_p to be complex numbers, then this problem clearly has a solution if, and only if, c_p is expressible in the form $c_p = \mu_p^{(1)} - \mu_p^{(2)} + i\mu_p^{(3)} - i\mu_p^{(4)}$, $p = 0, 1, 2, \cdots$, where $\{\mu_p^{(1)}\}, \cdots, \{\mu_p^{(4)}\}$ are totally monotone sequences.

We shall now prove the following theorem.

THEOREM 72.1. *The moment problem* (72.4) *has a solution if, and only if, the given constants* c_p *satisfy the inequalities*

$$M_n = \sum_{p=0}^{n} \binom{n}{p} |\Delta^{n-p} c_p| \leq M, \quad n = 0, 1, 2, \cdots, \tag{72.5}$$

where $\binom{n}{p} = n!/p!(n-p)!$, *and M is a constant independent of n.* **[78.]**

Proof.[27] If the moment problem (72.4) has a solution, then

$$\begin{aligned}
\sum_{p=0}^{n} \binom{n}{p} |\Delta^{n-p} c_p| \\
&= \sum_{p=0}^{n} \binom{n}{p} |\Delta^{n-p}(\mu_p^{(1)} - \mu_p^{(2)} + i\mu_p^{(3)} - i\mu_p^{(4)})| \\
&\leq \sum_{p=0}^{n} \binom{n}{p} \Delta^{n-p}\mu_p^{(1)} + \sum_{p=0}^{n} \binom{n}{p} \Delta^{n-p}\mu_p^{(2)} \\
&\quad + \sum_{p=0}^{n} \binom{n}{p} \Delta^{n-p}\mu_p^{(3)} + \sum_{p=0}^{n} \binom{n}{p} \Delta^{n-p}\mu_p^{(4)} \\
&= \mu_0^{(1)} + \mu_0^{(2)} + \mu_0^{(3)} + \mu_0^{(4)} = M,
\end{aligned}$$

so that the condition (72.5) is *necessary* for the moment problem (72.4) to have a solution.

We assume now that (72.5) holds, and shall prove that the moment problem (72.4) has a solution. We may evidently assume that the c_p are real.

[27] This proof is due to Hausdorff [28].

Using the recurrence relation $\Delta^m c_p = \Delta^{m-1} c_p - \Delta^{m-1} c_{p+1}$, we readily find by mathematical induction that

$$| \Delta^m c_p | \leq | \Delta^{m+1} c_p | + | \Delta^m c_{p+1} | \leq | \Delta^{m+2} c_p | + \binom{2}{1} | \Delta^{m+1} c_{p+1} | + | \Delta^m c_{p+2} | \leq \cdots \leq | \Delta^{m+k} c_p | + \binom{k}{1} | \Delta^{m+k-1} c_{p+1} | + \cdots + \binom{k}{k} | \Delta^m c_{p+k} | \leq \cdots.$$

Denote the general member of this sequence by $D_{m,p}{}^{(k)}$. In particular, we have $D_{0,0}{}^{(k)} = M_k$, so that $M_0 \leq M_1 \leq M_2 \leq \cdots \leq M$, and consequently the sequence $\{M_p\}$ is convergent. Now

$$D_{m,p}{}^{(k)} = \sum_{h=0}^{k} \binom{k}{h} | \Delta^{m+k-h} c_{p+h} | \leq \sum_{h=0}^{k} \binom{m+p+k}{p+h} | \Delta^{m+k-h} c_{p+h} |$$

$$= \sum_{r=p}^{p+k} \binom{m+p+k}{r} | \Delta^{m+p+k-r} c_r | \leq M_{m+p+k} \leq M.$$

Moreover,

$$D_{m,p}{}^{(k)} \leq D_{m+1,p}{}^{(k)} + D_{m,p+1}{}^{(k)} = D_{m,p}{}^{(k+1)}.$$

Consequently, we have

$$\lim_{k=\infty} D_{m,p}{}^{(k)} = \sigma_{m,p} \leq M;$$

$$\sigma_{m,p} = \sigma_{m+1,p} + \sigma_{m,p+1}, \quad m, p = 0, 1, 2, \cdots.$$

Therefore, if we put $\sigma_p = \sigma_{0,p}$, we see that $\sigma_{1,p} = \Delta\sigma_p$, $\sigma_{2,p} = \Delta^2\sigma_p$, $\cdots$, $\sigma_{m,p} = \Delta^m\sigma_p$, $\cdots$, so that $\sigma_0, \sigma_1, \sigma_2, \cdots$ is a totally monotone sequence. Inasmuch as

$$\sigma_{m,p} = \Delta^m \sigma_p \geq | \Delta^m c_p |,$$

it follows that the sequences

$$a_p = \tfrac{1}{2}(\sigma_p + c_p), \quad b_p = \tfrac{1}{2}(\sigma_p - c_p), \quad p = 0, 1, 2, \cdots,$$

are totally monotone sequences. Thus, $\{c_p\} = \{a_p - b_p\}$ is the difference of two totally monotone sequences, and consequently the moment problem (72.4) has a solution.

This completes the proof of Theorem 72.1.

Exercise 14

14.1. Let $\{a_p\}$ and $\{b_p\}$ be two totally monotone sequences, so that $\{a_p b_p\}$ is a totally monotone sequence. If $\lim_{n=\infty} \Delta^n a_0 = \lim_{n=\infty} \Delta^n b_0 = 0$, then $\lim_{n=\infty} \Delta^n(a_0 b_0) = 0$.

Chapter XV

BOUNDED ANALYTIC FUNCTIONS

In this chapter we shall be concerned primarily with the following problem.

Given: Two regions G and H in the complex plane, and the class of all functions $f(z)$ which, for z in G, are analytic and have their values in H.

To determine: A class C of continued fractions, convergent over G, such that each of the functions is equal to just one of the continued fractions, and each of the continued fractions is equal to just one of the functions.

We shall suppose that both G and H are simply connected domains having more than one boundary point. We may then suppose that these regions are taken, for example, to be circular regions, half-planes, or the portion of the plane exterior to some ray. We seek to determine G and H in such a way that the continued fractions will have as simple a form as possible, and, preferably, that they will be in some way related to the continued fractions already considered.

Our main result is as follows.

Let G be the portion of the complex plane exterior to the cut along the real axis from -1 to $-\infty$, and let H be the right half-plane, $R(z) > 0$. Then C is the class of continued fractions of the form

$$\cfrac{\mu\sqrt{1+z}}{1 + ir_0\sqrt{1+z} + \cfrac{(1-g_0)g_1z}{1 + ir_1\sqrt{1+z} + \cfrac{(1-g_1)g_2z}{1 + ir_2\sqrt{1+z} + \cdots}}},$$

where $\mu > 0$, $-\infty < r_p < +\infty$, $0 \leq g_p \leq 1$, $p = 0, 1, 2, \cdots$. When some partial quotient vanishes identically, we agree that the continued fraction shall terminate with that partial quotient. The branch of $\sqrt{1+z}$ is taken in G which is positive for real z.

When $r_p = 0$, $p = 0, 1, 2, \cdots$, the above continued fraction reduces to (27.8) (except for a factor). This appears also in § 11 and in Chapter XIV (cf. (69.7)). The class C thus contains the subclass C_0 of continued fractions of the form

$$\cfrac{\mu\sqrt{1+z}}{1+\cfrac{(1-g_0)g_1z}{1+\cfrac{(1-g_1)g_2z}{1+\cdots}}},$$

which have already been considered.

We develop in detail some properties of the continued fractions of the class C_0, and subsequently we proceed to the general class C.

73. Integral Formulas for Bounded Analytic Functions. We shall need the following theorem of F. Riesz [**78**] and Herglotz [**36**].

THEOREM 73.1. *A function $f(z)$ is analytic and has a positive real part for $|z| < 1$ if, and only if,*

$$f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\phi(t) + qi, \tag{73.1}$$

where $\phi(t)$ is a bounded nondecreasing function of t such that $\phi(0) = 0$, $\phi(2\pi) > 0$, and where q is a real constant. The function $\phi(t)$ is determined uniquely up to an additive constant at all its points of continuity by $f(z)$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{p=0}^{\infty} a_p z^p \\ &= b_0 + iq + \sum_{p=1}^{\infty} (b_p + ic_p) r^p (\cos p\theta + i \sin p\theta), \end{aligned} \tag{73.2}$$

where $z = re^{i\theta}, 0 < r < 1, b_p = \Re(a_p), q = \Im(a_0), c_{p+1} = \Im(a_{p+1})$, $p = 0, 1, 2, \cdots$. Then, by hypothesis,

$$u(r, \theta) = \Re[f(z)] = b_0 + \sum_{p=1}^{\infty} r^p(b_p \cos p\theta - c_p \sin p\theta) > 0,$$

so that the function

$$\phi_r(t) = \frac{1}{2\pi}\int_0^t u(r, \theta)d\theta, \quad 0 \leq t \leq 2\pi,$$

is a nondecreasing function of t, and $\phi_r(0) = 0$, $\phi_r(2\pi) = b_0 > 0$. Moreover,

$$\int_0^{2\pi} d\phi_r(t) = b_0, \quad \int_0^{2\pi} 2 \cos pt d\phi_r(t) = r^p b_p,$$

$$\int_0^{2\pi} -2 \sin pt d\phi_r(t) = r^p c_p, \quad p = 1, 2, 3, \cdots.$$

By Theorems 64.1 and 64.2, there exists a bounded nondecreasing function $\phi(t)$ such that $\phi(0) = 0$, $\phi(2\pi) = b_0$, and a sequence of values of r approaching unity, such that these equations go over into

$$\int_0^{2\pi} d\phi(t) = b_0, \quad \int_0^{2\pi} 2 \cos pt d\phi(t) = b_p, \quad \int_0^{2\pi} -2 \sin pt d\phi(t) = c_p,$$
$$p = 1, 2, 3, \cdots.$$

When these values are substituted into (73.2), we get

$$f(z) = iq + \int_0^{2\pi} \left[1 + 2\sum_{p=1}^{\infty}(ze^{-it})^p\right] d\phi(t)$$
$$= iq + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\phi(t),$$

so that (73.1) holds. The proof that $\phi(t)$ is determined uniquely up to an additive constant at all its points of continuity by $f(z)$ is similar to the proof of Theorem 68.1. One uses in this case the theorem of Weierstrass on the approximation to continuous functions by trigonometric sums.

Conversely, any function of the form (73.1) is easily seen to be analytic and to have a positive real part for $|z| < 1$.

If we multiply (73.1) by $-i$ and replace z by $(1 + iz)/(1 - iz)$ we obtain, after some simple transformations,

$$F(z) = -if\left(\frac{1+iz}{1-iz}\right) = \int_0^{2\pi} \frac{1 + z \tan \frac{t}{2}}{z - \tan \frac{t}{2}}\, d\phi(t) + q.$$

If we now put

$$\theta(u) = \begin{cases} \phi(2 \arc\tan u + 2\pi) & \text{for} \quad u < 0, \\ \phi(2 \arc\tan u) + \phi(2\pi) & \text{for} \quad u \geq 0, \end{cases}$$

we find that this goes over into

$$F(z) = \int_{-\infty}^{+\infty} \frac{1 + zu}{z - u}\, d\theta(u) - \sigma z + q, \tag{73.3}$$

where $\sigma = \phi(\pi + 0) - \phi(\pi - 0) \geq 0$. The function $\theta(u)$ is a bounded nondecreasing function of u which is uniquely determined up to an additive constant at all its points of continuity by $F(z)$. It is required that at least one of the numbers σ or $\theta(+\infty) - \theta(-\infty)$ be positive.

The expression (73.3) *is the most general expression for a function which is analytic and has a negative imaginary part for* $\mathfrak{J}(z) > 0$.

Let us suppose that the function $f(z)$ of Theorem 73.1 is real for real values of z. Then, instead of (73.2), we have

$$f(z) = b_0 + \sum_{p=1}^{\infty} b_p r^p (\cos p\theta + i \sin p\theta).$$

From the proof of Theorem 73.1, we then find that

$$f(z) = \int_0^{2\pi} \frac{1 - z^2}{1 - 2z \cos t + z^2}\, d\phi(t),$$

which can be written in the form

$$f(z) = \frac{1+z}{1-z} \int_0^{2\pi} \frac{d\phi(t)}{1 + \dfrac{4z}{(1-z)^2} \sin^2 \dfrac{t}{2}}. \tag{73.4}$$

This is the most general expression for a function $f(z)$ *which is analytic and has a positive real part for* $|z| < 1$, *and which is real for real values of* z.

Consider now the transformation

$$w = \frac{4z}{(1-z)^2}. \tag{73.5}$$

This maps the domain $|z| < 1$ conformally upon the portion of the w-plane exterior to the cut along the real axis from -1 to $-\infty$. We denote the latter domain by Ext $(-1, -\infty)$. We note that

$$\frac{1+z}{1-z} = \sqrt{1+w}, \tag{73.6}$$

where that branch of $\sqrt{1+w}$ is chosen in Ext $(-1, -\infty)$ which is positive for $w = 0$. Then

$$z = \frac{\sqrt{1+w}-1}{\sqrt{1+w}+1}. \tag{73.7}$$

We make this substitution in the formula (73.4). On putting

$$G(w) = f\left(\frac{\sqrt{1+w}-1}{\sqrt{1+w}-1}\right), \tag{73.8}$$

that formula becomes, if we write z for w,

$$G(z) = \sqrt{1+z}\int_0^{2\pi} \frac{d\phi(t)}{1 + z\sin^2\frac{t}{2}}. \tag{73.9}$$

This is the most general expression for a function which is analytic and has a positive real part in the domain Ext $(-1, -\infty)$, *and which is real for real values of* z.

74. Continued Fraction Expansions for Real Analytic Functions. The integral in (73.9) may be expanded into a power series in z, convergent for $|z| < 1$, of the form

$$\int_0^{2\pi} \frac{d\phi(t)}{1 + z\sin^2\frac{t}{2}} = \mu_0 - \mu_1 z + \mu_2 z^2 - \mu_3 z^3 + \cdots, \tag{74.1}$$

where

$$\mu_p = \int_0^{2\pi} \sin^{2p}\frac{t}{2}\,d\phi(t), \quad p = 0, 1, 2, \cdots.$$

Inasmuch as

$$\Delta^m \mu_n = \int_0^{2\pi} \cos^{2m} \frac{t}{2} \cdot \sin^{2n} \frac{t}{2} \, d\phi(t) > 0, \quad m, n = 0, 1, 2, \cdots,$$

we see that the sequence $\{\mu_p\}$ is *totally monotone.* Therefore, by Theorems 69.2 and 71.1, the integral (74.1) has a continued fraction expansion of the form (69.7). On substituting this into the formula (73.9), we then obtain the continued fraction expansion

$$G(z) = \cfrac{\mu_0 \sqrt{1+z}}{1 + \cfrac{(1-g_0)g_1 z}{1 + \cfrac{(1-g_1)g_2 z}{1 + \cdots}}} \tag{74.2}$$

Conversely, any continued fraction of this form has the integral representation

$$G(z) = \sqrt{1+z} \int_0^1 \frac{d\phi(u)}{1+zu},$$

where $\phi(u)$ is a bounded nondecreasing function and $\phi(1) - \phi(0) = \mu_0 > 0$. This function is analytic and has a positive real part in Ext $(-1, -\infty)$, and is real for real values of z. Hence, we have proved

THEOREM 74.1. *A necessary and sufficient condition for a function $G(z)$ to be analytic and have a positive real part in the domain Ext* $(-1, -\infty)$, *and be real for real z, is that it have a continued fraction expansion of the form* (74.2), *where* $\mu_0 > 0$, $0 \le g_p \le 1$, $p = 0, 1, 2, \cdots$. *We shall agree that in case some partial numerator of the continued fraction vanishes identically, then the continued fraction shall terminate with the first identically vanishing partial quotient. With this agreement, the continued fraction expansions are unique* [**129**].

Remark. We have seen that a function $G(z)$ is analytic and has a positive real part in Ext $(-1, -\infty)$, and is real for real z, if, and only if,

$$G(z) = \sqrt{1+z} \sum_{p=0}^{\infty} (-1)^p \mu_p z^p, \tag{74.3}$$

where $\{\mu_p\}$ is a totally monotone sequence with $\mu_0 > 0$. If we replace w by z in (73.8), we may then write (74.3) in the form

$$f\left(\frac{\sqrt{1+z}-1}{\sqrt{1+z}+1}\right) = \sqrt{1+z}\sum_{p=0}^{\infty}(-1)^p\mu_p z^p. \tag{74.4}$$

Here,

$$f(w) = \sum_{p=0}^{\infty}(-1)^p a_p w^p$$

is a real power series convergent for $|w| < 1$, and $\Re[f(w)] > 0$ in this domain. Thus, if we replace z by $-z$ in (74.4), we obtain

$$\sum_{p=0}^{\infty} a_p\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^p = \frac{\sum_{p=0}^{\infty} e_p V_p z^p}{\sum_{p=0}^{\infty} e_p z^p}, \tag{74.5}$$

where we have put

$$e_p = \frac{\binom{2p}{p}}{2^{2p}}, \quad V_p = \frac{\mu_p}{e_p},$$

e_p being the coefficient of z^p in the power series for $1/\sqrt{1-z}$. The formula (74.5) is precisely the formula which Gronwall [**24**] used to determine V_p as the sum

$$V_p = \sum_{r=0}^{p} \frac{(p!)^2}{(p-r)!(p+r)!} a_p,$$

which is known as the pth **de la Vallée Poussin mean** for the series Σa_p. This has the property that $\lim_{p=\infty} V_p = \Sigma a_p$ whenever the latter series converges. This limit may exist even when the series Σa_p diverges, and it thus assigns a sum to the divergent series. Gronwall generalized the formula (74.5) by using more general mapping functions and weight functions.

75. Continued Fraction Expansions for $1/G(z)$ and for $G[-z/(1+z)]$ in Terms of the Expansions for $G(z)$. If $G(z)$ is analytic and has a positive real part in Ext $(-1, -\infty)$, and is real for z real, then $1/G(z)$ and $G[-z/(1+z)]$ also have these properties. We shall prove that if $G(z)$ is given by (74.2), and if we take $g_0 = 0$, which is always permissible, then [**126**]

$$\frac{1}{G(z)} = \cfrac{\frac{1}{\mu_0}\sqrt{1+z}}{1 + \cfrac{(1-g_1)z}{1 + \cfrac{g_1(1-g_2)z}{1 + \cfrac{g_2(1-g_3)z}{1 + \cdots}}}}, \tag{75.1}$$

and [**18**]

$$G\left(\frac{-z}{1+z}\right) = \cfrac{\mu_0\sqrt{1+z}}{1 + \cfrac{(1-g_1)z}{1 + \cfrac{g_1 g_2 z}{1 + \cfrac{(1-g_2)(1-g_3)z}{1 + \cfrac{g_3 g_4 z}{1 + \cdots}}}}}. \tag{75.2}$$

The continued fraction (75.1) is obtained from (74.2) by replacing μ_0 by $1/\mu_0$ and g_p by $1 - g_p$ for $p = 1, 2, 3, \cdots$; while the continued fraction (75.2) is obtained from (74.2) by replacing g_p by $1 - g_p$ for only the *odd* values of p.

The relation (75.1) is an immediate consequence of the following identity [**126, 127**].

$$\cfrac{1}{1 + \cfrac{g_1 z}{1 + \cfrac{(1-g_1)g_2 z}{1 + \cfrac{(1-g_2)g_3 z}{1 + \cdots}}}} \cdot \cfrac{1}{1 + \cfrac{(1-g_1)z}{1 + \cfrac{g_1(1-g_2)z}{1 + \cfrac{g_2(1-g_3)z}{1 + \cdots}}}} = \frac{1}{1+z}. \tag{75.3}$$

To prove this, let $A_p(z)/B_p(z)$ and $C_p(z)/D_p(z)$ denote the pth approximants of the first and second continued fractions which appear as factors in the left-hand member. Then the following relations hold.

$$(1 + z)A_n(z) = g_{n-1}zD_{n-1}(z) + D_n(z), \quad n = 1, 2, 3, \cdots,$$
$$B_n(z) = g_{n-1}zC_{n-1}(z) + C_n(z), \tag{75.4}$$

where g_0 must be taken equal to unity, and $A_0 = C_0 = 0$, $B_0 = D_0 = 1$. These may readily be verified for $n = 1$, 2, and then, by mathematical induction, for all n. By means of (75.4) we obtain the formula

$$(1 + z)A_n(z)C_{n-1}(z) - B_n(z)D_{n-1}(z) = k_n z^{n-1},$$

where k_n is independent of z. Therefore,

$$\frac{A_n(z)}{B_n(z)} \cdot \frac{C_{n-1}(z)}{D_{n-1}(z)} - \frac{1}{1 + z} = \frac{k_n z^{n-1}}{(1 + z)B_n(z)D_{n-1}(z)}.$$

From this it follows that the expansion in ascending powers of z of the difference in the left-hand member begins with the term in z^{n-1}. This means that the product of the power series expansions of the factors in the left-hand member of (75.3) is equal to the power series for $1/(1 + z)$, so that the identity is established.

If we take the even part of (74.2) and then replace z by $-z/(1 + z)$, the resulting continued fraction is precisely the even part of (75.2). This establishes the validity of (75.2).

Remark. The above formulas hold in a formal sense when the g_p are arbitrary constants and $G(z)$ is the formal power series expansion for (74.2). The transformation represented by (75.1) can be formulated in several different ways. One of these is (75.3). Another is as follows. Let

$$f(z) = \cfrac{g_1}{1 + \cfrac{(1 - g_1)g_2 z}{1 + \cfrac{(1 - g_2)g_3 z}{1 + \cdot}}}. \tag{75.5}$$

Then

$$\frac{1 - f(z)}{1 + zf(z)} = \cfrac{1 - g_1}{1 + \cfrac{g_1(1 - g_2)z}{1 + \cfrac{g_2(1 - g_3)z}{1 + \cdots}}} \tag{75.6}$$

The transformation represented by (75.2) can be formulated as follows. If (75.5) holds, then

$$1 - f\left(\frac{-z}{1 + z}\right) = \cfrac{1 - g_1}{1 + \cfrac{g_1 g_2 z}{1 + \cfrac{(1 - g_2)(1 - g_3)z}{1 + \cdots}}} \tag{75.7}$$

These are to be regarded, in general, as formal power series identities.

76. Condition for $G(z)/\sqrt{1 + z}$ to Be Bounded in the Unit Circle. Let $G(z)$ be given by (74.2), and put

$$M_G = \underset{|z|<1}{\text{l.u.b.}} \frac{G(z)}{\sqrt{1 + z}}. \tag{76.1}$$

We shall prove the following theorem.

THEOREM 76.1. *Let $G(z)$ be analytic and have a positive real part in the domain Ext $(-1, -\infty)$, and be real for real z, so that $G(z)$ is given by* (74.2). *Then the number M_G defined by* (76.1) *does not exceed unity if, and only if, the parameters g_p of the chain sequence appearing in* (74.2) *can be so chosen that $g_0 = \mu_0$.* **[126.]**

Proof. Suppose first that the parameters can be chosen as specified. We may evidently assume that $0 \leq g_p < 1$ for $p = 0, 1, 2, \cdots$, or that $0 < g_p \leq 1$, $p = 0, 1, 2, \cdots$. Hence, it follows from Theorem 11.1 that $M_G \leq 1$.

If, conversely, $M_G \leq 1$, then the value M_G is taken on by the continued fraction for $z = -1$, i.e.,

$$M_G = \cfrac{\mu_0}{1 - \cfrac{m_1}{1 - \cfrac{(1 - m_1)m_2}{1 - \cfrac{(1 - m_2)m_3}{1 - \cdots}}}}$$

where the m_p are the minimal parameters of the chain sequence $\{(1 - g_{p-1})g_p\}$. Thus, by (19.6), $M_G = \mu_0/M_0 \leq 1$, where M_0 is the 0th maximal parameter of this chain sequence. We may therefore choose the parameters so that $g_0 = \mu_0$ (cf. Exercise 4.1).

The condition $M_G \leq 1$ can be formulated in terms of the moments μ_p as follows.

THEOREM 76.2. *Let* $G(z)$ *be given by* (74.2), *so that*

$$G(z) = \sqrt{1 + z}\sum_{p=0}(-1)^p\mu_p z^p \quad \textit{for} \quad |z| < 1,$$

where $\{\mu_p\}$ *is a totally monotone sequence. Then* $M_G \leq 1$ *if, and only if,* $\sum_{p=0}^{\infty}\mu_p \leq 1$.

Proof. If $\Sigma\mu_p \leq 1$, then $M_G = \Sigma\mu_p \leq 1$. If $\Sigma\mu_p > 1$, then one may readily verify that $\lim_{z=-1}[G(z)/\sqrt{1 + z}] > 1$, ($z$ real and greater than -1).

THEOREM 76.3. *Let* $G(z)$ *be given by* (74.2). *Then* $M_G \leq 1$ *if, and only if,*

$$G(z) = \sqrt{1 + z}\int_0^1 \frac{(1 - u)d\phi(u)}{1 + zu}, \tag{76.2}$$

where $\phi(u)$ *is a bounded nondecreasing function such that* $\phi(1) - \phi(0) \leq 1$. [**126.**]

Proof. By Theorem 76.2, $M_G \leq 1$ if, and only if, $\Sigma\mu_p \leq 1$. Put $\lambda_p = \sum_{r=p}^{\infty}\mu_p$. Then $\{\lambda_p\}$ is a totally monotone sequence, so that

$$\mu_p = \int_0^1 u^p d\phi(u), \quad p = 0, 1, 2, \cdots,$$

where $\phi(u)$ is a bounded nondecreasing function, and $\phi(1) - \phi(0) = \lambda_0 \leq 1$. Then

$$\mu_p = \Delta\lambda_p = \int_0^1 (1 - u)u^p d\phi(u), \quad p = 0, 1, 2, \cdots,$$

and consequently (76.2) holds. Conversely, if (76.2) holds, then

$$\mu_p = \int_0^1 (1 - u)u^p d\phi(u), \quad p = 0, 1, 2, \cdots,$$

$$\Sigma\mu_p = \phi(1) - \phi(0) - \lim_{p=\infty} \int_0^1 u^p d\phi(u) \leq 1,$$

so that $M_G \leq 1$ by virtue of Theorem 76.2.

77. Analytic Functions Bounded in the Unit Circle. Following the method of Schur [**84**], we shall now obtain expansions for *arbitrary* functions $f(z)$ analytic for $|z| < 1$, such that

$$M(f) = \operatorname*{l.u.b.}_{|z|<1} |f(z)| \leq 1.$$

THEOREM 77.1. *Let* $\{\sigma_p\}$ *be an infinite sequence of complex numbers with moduli less than unity,*

$$|\sigma_p| < 1, \quad p = 0, 1, 2, \cdots, \tag{77.1}$$

and let

$$S_p(z; t) = \sigma_0 + \cfrac{(1 - |\sigma_0|^2)z}{\bar{\sigma}_0 z - \cfrac{1}{\sigma_1 + \cfrac{(1 - |\sigma_1|^2)z}{\bar{\sigma}_1 z - \cdot \; \cdot \; \cdot - \cfrac{1}{\sigma_p + \cfrac{(1 - |\sigma_p|^2)z}{\bar{\sigma}_p z - \cfrac{1}{t}}}}}} \tag{77.2}$$

Then,

$$|S_p(z; t)| < 1 \quad \textit{for} \quad |t| \leq 1, \quad |z| < 1, \quad p = 0, 1, 2, \cdots. \tag{77.3}$$

Let $\{t_p\}$ *be a sequence of independent variables. There exists a function* $f(z)$, *analytic for* $|z| < 1$, *such that* $M(f) \leq 1$, *and such that, for every positive number* $r < 1$,

$$\lim_{p=\infty} S_p(z; t_p) = f(z), \tag{77.4}$$

uniformly for $|z| \leq r$, $|t_p| \leq 1$, $p = 0, 1, 2, \cdots$.

Conversely, if $f(z)$ *is a given function, analytic for* $|z| < 1$, *such that* $M(f) \leq 1$, *then one of the following three statements holds.*

(a) $f(z) \equiv \sigma_0$, *a constant with modulus unity.*

(b) *There exists uniquely a finite sequence* $\sigma_0, \sigma_1, \cdots, \sigma_{n+1}, n \geq 0$, *such that*

$$|\sigma_p| < 1, \quad p = 0, 1, 2, \cdots, n, \quad |\sigma_{n+1}| = 1, \tag{77.5}$$

and such that

$$f(z) = S_n(z; \sigma_{n+1}). \tag{77.6}$$

(c) *There exists uniquely an infinite sequence* $\{\sigma_p\}$ *satisfying* (77.1), *such that* (77.4) *holds* **[84, 128, 129]**.

Proof. Consider the linear transformation

$$s = s(z; t) = \frac{\sigma - zt}{1 - \bar{\sigma}zt}, \quad |\sigma| < 1, \tag{77.7}$$

of the t-plane into the s-plane, the transformation depending upon the parameter z. We observe that $|s| < 1$ if, and only if, $|\sigma - zt| < |1 - \bar{\sigma}zt|$, or $(|zt|^2 - 1)(1 - |\sigma|^2) < 0$. Hence, if $|z| < 1$, then $|t| \leq 1$ implies that $|s| < 1$. The same property must hold for the product of two or more such transformations. Let

$$s = s_p(z; t) = \frac{\sigma_p - zt}{1 - \bar{\sigma}_p zt}, \quad |\sigma_p| < 1, \quad p = 0, 1, 2, \cdots,$$

be an infinite sequence of transformations of this form. We may write

$$s = s_p(z; t) = \sigma_p + \frac{(1 - |\sigma_p|^2)z}{\bar{\sigma}_p z - (1/t)}. \tag{77.8}$$

Then the product $s_0 s_1 \cdots s_p(z; t)$ is given by (77.2); and (77.3) holds.

For $k \leq 2p + 2$, let $A_k(z)/B_k(z)$ be the kth approximant of the continued fraction (77.2). By means of the fundamental recurrence formulas one may readily verify that

$$A_{2k+2}(z)B_{2k}(z) - A_{2k}(z)B_{2k+2}(z) = (-1)^k \pi_k \sigma_{k+1} z^{k+1}, \quad k \leq p,$$
$$(\sigma_{p+1} = t),$$

where we have put

$$\pi_k = (1 - |\sigma_0|^2)(1 - |\sigma_1|^2) \cdots (1 - |\sigma_k|^2).$$

From this it follows, by a familiar argument (cf. (42.9) and (42.10)), that if we put

$$S_p(z; t) = a_0^{(p)} + a_1^{(p)}z + a_2^{(p)}z^2 + \cdots,$$

then $a_0^{(p)}, \cdots, a_p^{(p)}$ are independent of p and of t:

$$a_m^{(p)} = c_m, \quad m = 0, 1, 2, \cdots, p.$$

As p is increased, we thus see that there is determined a power series $f(z) = \Sigma c_p z^p$ with which the power series for $S_p(z; t)$ agrees term by term for the first $p + 1$ terms. From (77.3) it follows that $| a_m^{(p)} | \leq 1, m = 0, 1, 2, \cdots$, so that $| c_m | \leq 1, m = 0, 1, 2, \cdots$. The function $f(z)$ is analytic for $| z | < 1$, and $|f(0)| = | \sigma_0 | < 1$. Now, if $| z | \leq r < 1, | t_k | \leq 1, k = 0, 1, 2, \cdots$, then

$$|f(z) - S_p(z; t_p)| = \left| \sum_{m=0}^{\infty} (c_m - a_m^{(p)})z^m \right| \leq 2 \sum_{m=p+1}^{\infty} r^m = \frac{2r^{p+1}}{1 - r},$$

and therefore (77.4) holds uniformly for $| z | \leq r$, $| t_p | \leq 1$, $p = 0, 1, 2, \cdots$. From (77.3) and the fact that $|f(0)| < 1$, it follows that $|f(z)| < 1$ for $| z | < 1$, and therefore $M(f) \leq 1$.

Turning now to the converse, let $f(z) = \Sigma c_p z^p$ be any function which is analytic and has modulus not greater than unity for $| z | < 1$. Put $c_0 = \sigma_0$. If $| \sigma_0 | = 1$, then $f(z) \equiv \sigma_0$, i.e., (a) holds. If $| \sigma_0 | < 1$, let

$$f_1(z) = \frac{1}{z} \frac{\sigma_0 - f(z)}{1 - \bar{\sigma}_0 f(z)}.$$

In the notation previously introduced, this may be written

$$f_1(z) = \frac{1}{z} s_0(f(z); 1).$$

Inasmuch as $s_0(f(0); 1) = 0$ and $| s_0(f(z); 1) | < 1$ for $| z | < 1$, it follows by Schwarz's lemma that $f_1(z)$ is analytic and has modulus not greater than unity for $| z | < 1$. Put $f_1(0) = \sigma_1$. If $| \sigma_1 | = 1$, then $f_1(z) \equiv \sigma_1$, and in this case

$$f(z) = \sigma_0 + \frac{(1 - | \sigma_0 |^2)z}{\bar{\sigma}_0 z - (1/\sigma_1)} = S_0(z; \sigma_1).$$

If, on the other hand, $|\sigma_1| < 1$, we put

$$f_2(z) = \frac{1}{z} s_1(f_1(z); 1),$$

and conclude as before that $f_2(z)$ is analytic and has modulus not greater than unity for $|z| < 1$. If $f_2(0) = \sigma_2$ has modulus unity, then $f_2(z) \equiv \sigma_2$, and in this case

$$f(z) = S_1(z; \sigma_2).$$

If $|\sigma_2| < 1$, we put

$$f_3(z) = \frac{1}{z} s_2(f_2(z); 1),$$

and so on.

These considerations show that there are just two possibilities. Either there is an index $n \geq 0$ such that (77.5) holds, in which case $f(z)$ is given by (77.6); or else (77.1) holds. In the latter event we have, for every p,

$$f(z) = S_p[z; f_{p+1}(z)], \tag{77.9}$$

where

$$f_{p+1}(z) = \frac{1}{z} s_p[f_p(z); 1], \quad p = 0, 1, 2, \cdots, \quad f_0 = f.$$

Hence, since $|f_{p+1}(z)| < 1$, we conclude immediately that (77.4) holds.

The constant $\sigma_0 = f(0)$ is uniquely determined by $f(z)$. One may readily show by mathematical induction that $\sigma_1, \sigma_2, \sigma_3, \cdots$ are uniquely determined.

This completes the proof of Theorem 77.1.

78. Continued Fraction Expansions for Arbitrary Functions Which Are Analytic and Have Positive Real Parts in Ext $(-1, -\infty)$. We come now to the main theorem of the present chapter, which extends to arbitrary analytic functions the expansion theorem for real analytic functions given in § 74.

THEOREM 78.1. *A necessary and sufficient condition for a function $H(z)$ to be analytic and have a positive real part in the domain Ext $(-1, -\infty)$ is that it have a continued fraction expansion of the form*

$$H(z)=\cfrac{\mu\sqrt{1+z}}{1+ir_0\sqrt{1+z}+\cfrac{g_1 z}{1+ir_1\sqrt{1+z}+\cfrac{(1-g_1)g_2 z}{1+ir_2\sqrt{1+z}+\cfrac{(1-g_2)g_3 z}{1+ir_3\sqrt{1+z}+\cdots}}}}, \tag{78.1}$$

where $\mu > 0$, $-\infty < r_p < +\infty$, $0 \leq g_{p+1} \leq 1$, $p = 0, 1, 2, \cdots$. *We shall agree that in case some partial numerator of the continued fraction vanishes identically, then the continued fraction shall terminate with the first identically vanishing partial quotient. With this agreement, the continued fraction expansions are unique.*

If $0 < g_p < 1$, $p = 1, 2, 3, 4, \cdots$, *then the continued fraction converges uniformly over every finite closed domain in Ext* $(-1, -\infty)$. [**129.**]

Proof. The proof consists in transforming the continued fraction (77.2) in a suitable way.

(i) It will be convenient to denote by U the class of analytic functions $f(z)$ of Theorem 77.1, which map the unit circle $|z| < 1$ into all or part of itself. Consider the transformation

$$k(z) = \frac{1 + zf(z)}{1 - zf(z)}. \tag{78.2}$$

Since

$$\Re[k(z)] = \frac{1 - |zf(z)|^2}{|1 - zf(z)|^2},$$

it follows that $\Re[k(z)] > 0$ for $|z| < 1$ provided $f(z)$ is in U. Conversely, if $\Re[k(z)] > 0$ for $|z| < 1$, then $|zf(z)| < 1$ for $|z| < 1$, so that, by Schwarz's lemma, $f(z)$ is in U. We denote the class of these functions $k(z)$ by K. The formula (78.2) sets up a one-to-one correspondence between the classes U and K. We note that $k(0) = 1$.

(ii) Let $\sigma_0, \sigma_1, \cdots, \sigma_p$ be numbers with moduli less than unity, and let $S_p(z; t)$ be defined by (77.2). We now put

$$f_0(z; t) = S_p(z; t), \quad f_{k+1}(z; t) = \frac{1}{z}\frac{\sigma_k - f_k(z; t)}{1 - \bar{\sigma}_k f_k(z; t)}, \qquad k = 0, 1, 2, \cdots, p. \tag{78.3}$$

Then

$$f_k(z;t) = \frac{\sigma_k - zf_{k+1}(z;t)}{1 - \bar{\sigma}_k z f_{k+1}(z;t)} = \sigma_k + \cfrac{(1 - |\sigma_k|^2)z}{\bar{\sigma}_k z - \cfrac{1}{f_{k+1}(z;t)}},$$

so that we must have $f_{p+1}(z;t) = t$.

We now define new functions $h_k(z;t)$ by means of the equations

$$h_n(z;t) = \frac{1 - \delta_n f_n(z;t)}{1 + z\delta_n f_n(z;t)}, \quad n = 0, 1, 2, \cdots, p + 1, \quad (78.4)$$

where the δ_n are numbers different from zero to be determined. If we substitute the values of $f_k(z;t)$ and $f_{k+1}(z;t)$ obtained from (78.4) for $n = k$ and $n = k + 1$ into (78.3), we find that $h_k = h_k(z;t)$ and $h_{k+1} = h_{k+1}(z;t)$ are related by means of the equation

$$h_k = \frac{h_{k+1}(\delta_k - \delta_{k+1} - \bar{\sigma}_k + \delta_{k+1}\delta_k\sigma_k)z + (\delta_{k+1}\sigma_k\delta_k - \delta_{k+1} - z\delta_k + z\bar{\sigma}_k)}{(z\bar{\sigma}_k + z^2\delta_k - \delta_{k+1} - z\delta_{k+1}\sigma_k\delta_k) - h_{k+1}(\bar{\sigma}_k + z\delta_k + \delta_{k+1} + z\delta_{k+1}\sigma_k\delta_k)},$$

$$k = 0, 1, 2, \cdots, p.$$

We now determine the δ_k so that the factor multiplying h_{k+1} in the numerator of this expression is zero, namely, we take

$$\delta_0 = 1, \quad \delta_k = \frac{\delta_{k-1} - \bar{\sigma}_{k-1}}{1 - \sigma_{k-1}\delta_{k-1}}, \quad k = 1, 2, 3, \cdots, p + 1. \quad (78.5)$$

We note that $|\delta_n| = 1$, $n = 0, 1, 2, \cdots, p + 1$. The expression connecting h_k and h_{k+1} may now be written

$$h_k(z;t) = \frac{|1 - \sigma_k\delta_k|^2}{(1 - \bar{\sigma}_k\bar{\delta}_k) - (1 - \sigma_k\delta_k)z + (1 - |\sigma_k\delta_k|^2)zh_{k+1}(z;t)},$$

$$k = 0, 1, 2, \cdots, p. \quad (78.6)$$

In particular, since $\delta_0 = 1$, $f_0(z;t) = S_p(z;t)$, we have

$$\frac{1 - S_p(z;t)}{1 + zS_p(z;t)} = \frac{|1 - \sigma_0\delta_0|^2}{(1 - \bar{\delta}_0\bar{\sigma}_0) - (1 - \delta_0\sigma_0)z + (1 - |\sigma_0\delta_0|^2)zh_1(z;t)}.$$

On multiplying both members of this equation by $2z/(1-z)$, adding 1 to both members, and then taking reciprocals, we obtain

$$\begin{aligned}\frac{1+zS_p(z;t)}{1-zS_p(z;t)} & \\ = \cfrac{1+z}{1-z+\cfrac{2z|1-\sigma_0\delta_0|^2}{(1-\bar{\sigma}_0\delta_0)-(1-\sigma_0\delta_0)z+(1-|\sigma_0\delta_0|^2)zh_1(z;t)}}. &\end{aligned} \tag{78.7}$$

If t is a constant with modulus not greater than unity, this is a function in the class K introduced before (cf. (78.2)). If we substitute in succession for $k = 1, 2, 3, \cdots, p$ from (78.6) into (78.7), we get a continued fraction expansion for this function. Our next step is to introduce new parameters in place of the σ_n in this continued fraction.

(iii) Let

$$\sigma_{k-1}\delta_{k-1} = 1 - 2u_k, \quad k = 1, 2, 3, \cdots, p+1. \tag{78.8}$$

Since $|\delta_{k-1}| = 1$, $|\sigma_{k-1}| < 1$, then

$$|u_k - \tfrac{1}{2}| < \tfrac{1}{2}, \quad k = 1, 2, 3, \cdots, p+1, \tag{78.9}$$

so that $\Re(u_k) > 0$. We now put

$$g_k = \frac{|u_k|^2}{\Re(u_k)}, \quad r_k = -\frac{\Im(u_k)}{\Re(u_k)}, \quad k = 1, 2, 3, \cdots, p+1, \tag{78.10}$$

so that

$$\begin{gathered}\arg u_k = -\arc\tan r_k = -\phi_k, \quad u_k = g_k \cos\phi_k, \\ k = 1, 2, 3, \cdots, p+1.\end{gathered} \tag{78.11}$$

The numbers g_k and r_k satisfy the inequalities

$$0 < g_k < 1, \quad -\infty < r_k < +\infty, \quad k = 1, 2, 3, \cdots, p+1. \tag{78.12}$$

If, conversely, g_k and r_k are any numbers satisfying (78.12), then numbers u_k satisfying (78.9) are uniquely determined by (78.11). The u_k, in turn, determine uniquely numbers σ_k such that (78.8) and (78.5) hold. In fact,

$$\sigma_0 = 1 - 2u_1, \quad \sigma_k = \frac{u_1u_2\cdots u_k}{\bar{u}_1\bar{u}_2\cdots\bar{u}_k}(1-2u_{k+1}), \quad k = 1, 2, 3, \cdots, p.$$

If we introduce the new parameters g_k and r_k, the expressions (78.7) and (78.6) become

$$\frac{1+zS_p(z;t)}{1-zS_p(z;t)} = \cfrac{1+z}{1-z+\cfrac{4g_1}{(1+ir_1)-(1-ir_1)z+2(1-g_1)zh_1(z;t)}}, \tag{78.13}$$

and

$$h_k(z;t) = \frac{2g_{k+1}}{(1+ir_{k+1})-(1-ir_{k+1})z+2(1-g_{k+1})zh_{k+1}(z;t)},$$

$$k = 1, 2, 3, \cdots, p. \tag{78.14}$$

(iv) We now suppose that $\sigma_0, \sigma_1, \sigma_2, \cdots$ is an infinite sequence of numbers with moduli less than unity, so that the preceding formulas hold for arbitrarily large values of p. By (78.4), with $n = p + 1$, we have, remembering that $f_{p+1}(z; t) = t$,

$$h_{p+1}(z;t) = \frac{1-\delta_{p+1}t}{1+z\delta_{p+1}t}. \tag{78.15}$$

We take $t = \bar{\delta}_{p+1}$, so that $|t| = 1$ and $h_{p+1}(z; t) = 0$. From (78.13) and (78.14) we then conclude that

$$\frac{1+zS_p(z;\bar{\delta}_{p+1})}{1-zS_p(z;\bar{\delta}_{p+1})} = \cfrac{1+z}{1-z+\cfrac{4g_1z}{(1+ir_1)-(1-ir_1)z+\cfrac{4(1-g_1)g_2z}{(1+ir_2)-(1-ir_2)z+\cdots+\cfrac{4(1-g_p)g_{p+1}z}{(1+ir_{p+1})-(1-ir_{p+1})z}}}}. \tag{78.16}$$

By (77.4), *this converges uniformly for* $|z| \leq r < 1$, *as* $p \to \infty$, *to a function* $k(z) = [1 + zf(z)]/[1 - zf(z)]$, *where* $f(z)$ *is in* U *so that* $k(z)$ *is in* K (cf. (i)).

(v) Let us now suppose that $\sigma_0, \sigma_1, \cdots, \sigma_p$ have moduli less than unity, and let σ_{p+1} be any number with modulus unity

$(p \geq 0)$. In case $1 - \delta_{p+1}\sigma_{p+1} = 0$, we see by (78.15) that $h_{p+1}(z; \sigma_{p+1}) = 0$, so that

$$\frac{1 + zS_p(z; \sigma_{p+1})}{1 - zS_p(z; \sigma_{p+1})} \tag{78.17}$$

is given by the continued fraction in the right-hand member of (78.16). If, on the other hand, $1 - \delta_{p+1}\sigma_{p+1} \neq 0$, then

$$h_{p+1}(z; \sigma_{p+1}) = \frac{1 - \delta_{p+1}\sigma_{p+1}}{1 + z\delta_{p+1}\sigma_{p+1}}$$
$$= \frac{|1 - \sigma_{p+1}\delta_{p+1}|^2}{(1 - \bar{\sigma}_{p+1}\delta_{p+1}) - (1 - \sigma_{p+1}\delta_{p+1})z}.$$

Put $\sigma_{p+1}\delta_{p+1} = 1 - u_{p+2}$. Since $|\sigma_{p+1}\delta_{p+1}| = 1$, it follows that $|u_{p+2} - \frac{1}{2}| = \frac{1}{2}$. Since $\sigma_{p+1}\delta_{p+1} \neq 1$, we must therefore have $\Re(u_{p+2}) > 0$. Hence, if

$$g_{p+2} = \frac{|u_{p+2}|^2}{\Re(u_{p+2})}, \quad r_{p+2} = -\frac{\Im(u_{p+2})}{\Re(u_{p+2})},$$

then

$$h_{p+1}(z; \sigma_{p+1}) = \frac{2g_{p+2}}{(1 + ir_{p+1}) - (1 - ir_{p+1})z}.$$

Here, $g_{p+2} = 1$. Clearly, if $g_{p+2} = 1$ and r_{p+2} is real, then the preceding relations determine σ_{p+1} uniquely. We now see at once that in this case the function (78.17) is given by the continued fraction in the right-hand member of (78.16) with p replaced by $p + 1$.

(vi) On referring to Theorem 77.1 and (i), we find that we now have continued fraction expansions for all the functions of the class K with the exception of those of the form

$$\frac{1 + z\sigma_0}{1 - z\sigma_0}, \quad \text{where} \quad |\sigma_0| = 1.$$

If $\sigma_0 = 1$, this is the first approximant of the continued fraction in the right-hand member of (78.16). If $\sigma_0 \neq 1$, and we put $\sigma_0 = e^{i\phi}$, then this function is the second approximant of the continued fraction in the right-hand member of (78.16) with $g_1 = 1$ and $r_1 = \cot(\phi/2)$.

This accounts for all the functions of the class K. Every function of the class K has a unique infinite or terminating continued fraction expansion of the form

$$\cfrac{1+z}{1-z+\cfrac{4g_1z}{(1+ir_1)-(1-ir_1)z+\cfrac{4(1-g_1)g_2z}{(1+ir_2)-(1-ir_2)z+\cdots}}}, \tag{78.18}$$

and, conversely, every such continued fraction represents a function of the class K.

In case $0 < g_p < 1$, $p = 1, 2, 3, \cdots$, the continued fraction converges uniformly for $|z| \leq r$, for every positive constant r less than unity.

(vii) The functions $k(z)$ of the class K have the property $k(0) = 1$. If $\mu > 0$, $-\infty < r_0 < +\infty$, then the function

$$k_1(z) = \cfrac{\mu}{ir_0 + \cfrac{1}{k(z)}} \tag{78.19}$$

is evidently analytic and has a positive real part for $|z| < 1$, provided $k(z)$ is in K. Conversely, if $k_1(z)$ is analytic and has a positive real part for $|z| < 1$, and if $k_1(0) = p + iq, p > 0, q$ real, then the relation (78.19) determines uniquely a function $k(z)$ in K if we there take $r_0 = -q/p, \mu = (p^2 + q^2)/p$. On substituting for $k(z)$ its continued fraction expansion of the form (78.18), we then obtain from this relation the expansion

$$k_1(z) = \cfrac{\mu(1+z)}{(1+ir_0)-(1-ir_0)z+\cfrac{4g_1z}{(1+ir_1)-(1-ir_1)z+\cfrac{4(1-g_1)g_2z}{(1+ir_2)-(1-ir_2)z+\cdots}}}. \tag{78.20}$$

The function $k_1(z)$ is analytic and has a positive real part for $|z| < 1$ if, and only if, it has an infinite or terminating continued fraction expansion of this form. On making the substitution

(73.7) in (78.20) and then writing z for w, and putting $H(z) = k_1[(\sqrt{1+z}-1)/(\sqrt{1+z}+1)]$, we obtain at once the expansion (78.1) as the most general expression for a function which is analytic and has a positive real part in the domain Ext $(-1, -\infty)$. The facts concerning convergence and uniqueness of the expansion are contained in the preceding discussion.

The proof of Theorem 78.1 is now complete.

EXERCISE 15

15.1. Let

$$f(z) = \cfrac{g_1}{1 + \cfrac{(1-g_1)g_2z}{1 + \cfrac{(1-g_2)g_3z}{1 + \cdots}}}$$

where $0 < g_p < 1$, $p = 1, 2, 3, \cdots$. Put $f_0 = f$,

$$f_{p+1} = \frac{1}{z}\frac{t_p - f_p}{1 - t_pf_p}, \quad t_p = f_p(0), \quad p = 0, 1, 2, \cdots.$$

Show that for $p = 1, 2, 3, \cdots$,

$$f_p(z) = \cfrac{g_1^{(p)}}{1 + \cfrac{(1-g_1^{(p)})g_2^{(p)}z}{1 + \cfrac{(1-g_2^{(p)})g_3^{(p)}z}{1 + \cdots}}}$$

where $0 < g_k^{(p)} < 1$, $k = 1, 2, 3, \cdots$. **[126.]**

15.2. Let

$$\int_0^1 \frac{d\phi(u)}{1+zu} = \cfrac{1}{1 + \cfrac{g_1z}{1 + \cfrac{(1-g_1)g_2z}{1 + \cdots}}},$$

where $0 < g_p < 1$, $p = 1, 2, 3, \cdots$. Prove that $\phi(u)$ is continuous at $u = 0$ if, and only if, the series

$$\sum \frac{g_1g_3\cdots g_{2p+1}(1-g_2)(1-g_4)\cdots(1-g_{2p})}{g_2g_4\cdots g_{2p+2}(1-g_1)(1-g_3)\cdots(1-g_{2p+1})}$$

diverges, and is continuous at $u = 1$ if, and only if, the series

$$\sum \frac{(1-g_1)(1-g_2)\cdots(1-g_{2p-1})}{g_1g_2\cdots g_{2p}}$$

diverges. **[18.]**

15.3. Let $f(z)$ denote the function of 15.2. Show that the modulus of $(1+z)f(z)$ is bounded for $\Re(z) > -\frac{1}{2}$ if, and only if, the series

$$1 + \frac{(1-g_1)}{g_1} + \frac{(1-g_1)g_2}{g_1(1-g_2)} + \frac{(1-g_1)g_2(1-g_3)}{g_1(1-g_2)g_3} + \cdots$$

is convergent. [**19.**]

15.4. Show that $f(z)$ is bounded for $|z| < 1$ and $(1+z)f(z)$ is bounded for $\Re(z) > -\frac{1}{2}$ if, and only if,

$$f(z) = \int_0^1 \frac{u(1-u)d\theta(u)}{1+zu},$$

i.e., $\phi(u) = \int_0^u t(1-t)d\theta(t)$. [**18.**]

15.5. Let

$$f(x) = 1 + \cfrac{rx}{1 + \cfrac{(1-r)sx}{1 + \cfrac{(1-s)rsx}{1 + \cfrac{(1-rs)s^2x}{1 + \cfrac{(1-s^2)rs^2x}{1 + \cfrac{(1-rs^2)s^3x}{1 + \cdots}}}}}},$$

where $0 < r < 1$, $0 < s < 1$. Show that

(a) $f(x)$ is a meromorphic function of x whose poles are all on the negative half of the real axis.

(b) $f(x)$ satisfies the functional equation

$$f(x) = 1 + \frac{rx}{1+sx} f(sx).$$

(c) $f(x) = 1 + rx - rs(1-r)x^2 + rs^2(1-r)(1-rs)x^3 - rs^3(1-r)(1-rs)(1-rs^2)x^4 + \cdots.$

(d) $f(-1) = \prod_{p=0}^{\infty} (1 - rs^p).$

(e) $f(x) = 1 + x\sum_{p=1}^{\infty} \frac{M_p}{1+xs^p},$

where $M_1 = rf(-1)$, and

$$M_p = \frac{r^p f(-1)}{(1-s)(1-s^2)\cdots(1-s^{p-1})}, \quad p = 2, 3, 4, \cdots. \quad [\textbf{131.}]$$

15.6. Let

$$y_0 + \cfrac{x_1}{y_1 + \cfrac{x_2}{y_2 + \cdots}}$$

be any continued fraction with partial numerators $x_p \neq 0$, and denote the nth numerator and denominator by X_n and Y_n, respectively. Put

$$D_n = p_{n-1}(p_n + q_n y_{n+1}) - q_n q_{n-1} x_{n+1}, \quad n = 0, 1, 2, \cdots,$$

where $p_{-1} = 1$, $q_{-1} = 0$, and the other p_n and q_n are any numbers such that $D_n \neq 0$, $n = 0, 1, 2, \cdots$. Put

$$E_n = p_n(p_{n-2}y_n - q_{n-2}x_n) + q_n[p_{n-2}(y_ny_{n+1} + x_{n+1}) - q_{n-2}x_ny_{n+1}],$$
$$n = 1, 2, 3, \cdots;$$

$$b_0 = y_0 + \frac{q_0x_1}{D_0}, \quad a_1 = \frac{x_1D_1}{D_0^2},$$

$$a_n = \frac{x_nD_n}{D_{n-1}}, \quad n = 2, 3, 4, \cdots,$$

$$b_n = \frac{E_n}{D_{n-1}}, \quad n = 1, 2, 3, \cdots.$$

Let A_n and B_n denote the nth numerator and denominator of the continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}$$

Then,

$$\begin{aligned} D_0A_n &= p_nX_n + q_nX_{n+1}, \\ D_0B_n &= p_nY_n + q_nY_{n+1}, \end{aligned} \qquad n = 0, 1, 2, \cdots. \quad \textbf{[58.]}$$

15.7. Let p_n and q_n be regarded as parameters, put

$$y_n = b_n = 1, \quad n = 1, 2, 3, \cdots,$$

and show that

$$x_1 = \frac{(b_0 - y_0)(p_0 + q_0)}{q_0}, \quad x_2 = \frac{p_0 + q_0 - p_1 - q_1}{q_1},$$

$$x_{n+1} = \frac{t_1t_2 \cdots t_{n-1}}{s_1s_2 \cdots s_n}\left(r_1 + r_2\frac{s_1}{t_1} + r_3\frac{s_1s_2}{t_1t_2} + \cdots + r_n\frac{s_1s_2 \cdots s_{n-1}}{t_1t_2 \cdots t_{n-1}}\right),$$

where

$$r_{n+1} = p_{n-1}(p_n + q_n - p_{n+1} - q_{n+1}),$$

$$s_{n+1} = p_{n-1}q_{n+1},$$

$$t_{n+1} = q_n(p_{n+2} + q_{n+2} - q_{n+1}). \quad \textbf{[19.]}$$

15.8. Obtain the identity (75.3) by taking $q_n = 1$, $p_n = g_nz$, $g_0 = 1$, $g_n \neq 0$, $n = 0, 1, 2, \cdots$; $y_0 = 0$, $b_0 = 1/(1 + z)$. **[19.]**

15.9. If $\{\mu_p\}$ is a totally monotone sequence in which $\mu_0 > 0$, and $f(z) = \Sigma(-1)^p\mu_pz^p$, $1/f(z) = b + b_0z - b_1z^2 + b_2z^3 - \cdots$, then the sequence $\{b_p\}$ is totally monotone, and the function $F(z) = \Sigma(-1)^pb_pz^p$ has modulus not greater than unity for $|z| < 1$. **[70a.]**

15.10. If $\{b_p\}$ is a totally monotone sequence such that the function $F(z) = \Sigma(-1)^pb_pz^p$ has modulus not greater than unity for $|z| < 1$, and if we put $f(z) = 1/(zF(z) + 1) = \Sigma(-1)^p\mu_pz^p$, then $\{\mu_p\}$ is a totally monotone sequence.

15.11. If (77.1) holds and the series $\Sigma|\sigma_p|$ is convergent, then $\lim_{p=\infty} S_p(z;\sigma_{p+1}) = f(z)$ uniformly for $|z| \leq 1$, and

$$|f(z)|^2 \leq 1 - \prod_{p=0}^{\infty} \frac{1 - |\sigma_p|}{1 + |\sigma_p|} < 1 \quad \text{for} \quad |z| \leq 1. \quad [84.]$$

15.12. In the continued fraction of 15.1, let the g_p be such that the series $\Sigma| g_p - \frac{1}{2} |$ is convergent. Then the function $f(z)$ is bounded over the domain Ext $(-1, -\infty)$. Moreover, as $z \to s$, where $-\infty < s \leq -1$, from the upper half-plane, then $f(z) \to u(s)$, where $u(s)$ is a continuous function of s which is real if, and only if, $s = -1$. As $z \to s$ from the lower half-plane, $f(z) \to \overline{(us)}$. [130.]

15.13. The function $H(z)$ represented by (78.1) is bounded over Ext $(-1, -\infty)$ if the series

$$\Sigma \sqrt{\frac{1}{4} - \frac{(1 - g_p)g_p}{1 + r_p^2}}$$

is convergent.

15.14. The power series $f(z) = \Sigma c_p z^p$ is convergent and $|f(z)|$ is bounded for $|z| < 1$ if, and only if, the bilinear form

$$\sum_{p \geq q} c_{p-q} x_p y_q$$

is bounded. If N is the norm of the form, then

$$N = \underset{|z|<1}{\text{l.u.b.}} |f(z)|. \quad [84.]$$

15.15. The power series $f(z)$ of 15.14 is convergent and has a nonnegative real part for $|z| < 1$ if, and only if, the Hermitian form

$$\sum_{p \geq q} (c_{p-q} \bar{x}_p x_q + \bar{c}_{p-q} x_p \bar{x}_q) \geq 0. \quad [84.]$$

15.16. The zeros of the polynomial $c_0 + c_1 z + \cdots + c_n z^n$, $c_n \neq 0$, are all in the interior of the unit circle if, and only if, the Hermitian form

$$\sum_{p=0}^{n-1} (| \bar{c}_n x_p + \bar{c}_{n-1} x_{p+1} + \cdots + \bar{c}_{p+1} x_{n-1} |^2 \\ - | c_0 x_p + c_1 x_{p+1} + \cdots + c_{n-p-1} x_{n-1} |^2)$$

is positive definite. [84.]

15.17. If (77.1) holds with the σ_p real, and if the series

$$S = \Sigma \frac{(1 - \sigma_0)(1 - \sigma_1) \cdots (1 - \sigma_p)}{(1 + \sigma_0)(1 + \sigma_1) \cdots (1 + \sigma_p)}$$

is convergent, then the function $f(z)$ of (77.4) has the property $M(f) = 1$. The same conclusion holds if one of the series obtained from S by replacing σ_p by $-\sigma_p$, by replacing σ_{2p+1} by $-\sigma_{2p+1}$, or by replacing σ_{2p} by $-\sigma_{2p}$, $(p = 0, 1, 2, \cdots)$, is convergent [132].

15.18. If (77.1) holds with the σ_p real, and if $\lim_{p=\infty} (1 + \sigma_{p-1})(1 - \sigma_p) = 0$, then the function $f(z)$ of (77.4) has an essential singularity at $z = 1$, and no other singularities in the extended plane except poles [**132**].

15.19. A necessary and sufficient condition for a function $f(z)$ to be analytic and have a negative imaginary part for $\mathfrak{J}(z) > 0$ is that it have a continued fraction expansion of the form

$$\cfrac{c}{z - r_0 - \cfrac{g_1(1 + z^2)}{z - r_1 - \cfrac{(1 - g_1)g_2(1 + z^2)}{z - r_2 - \cfrac{(1 - g_2)g_3(1 + z^2)}{z - r_3 - \cdots}}}},$$

where $0 \leq g_p \leq 1$, $-\infty < r_{p-1} < +\infty$, $p = 1, 2, 3, \cdots$, $c > 0$, with the agreement made in Theorem 78.1 in case some g_p is equal to zero or to unity. If $0 < g_p < 1$, $p = 1, 2, 3, \cdots$, then the continued fraction converges uniformly over every finite closed domain in the upper half-plane $\mathfrak{J}(z) > 0$. [**137a.**]

15.20. To indicate the dependence of the function $f(z)$ of (77.4) upon the σ_p, let us write

$$f(z) = (z; \sigma_0, \sigma_1, \sigma_2, \cdots).$$

Then,

$$-f(z) = (z; -\sigma_0, -\sigma_1, -\sigma_2, \cdots),$$

$$f(-z) = (z; \sigma_0, -\sigma_1, \sigma_2, -\sigma_3, \sigma_4, \cdots).$$

Show that the identities of § 75 are easy consequences of these formulas [**128**].

15.21. If $a > 0$, $b \geq 0$, then

$$-ai - bz = \cfrac{a + b}{\cfrac{(a + 2b)(1 + z^2)}{(a + b)2}} \atop {z - \cfrac{\cfrac{a(a + 3b)(1 + z^2)}{(a + b)(a + 2b)4}}{z - \cfrac{\cfrac{(a + b)(a + 4b)(1 + z^2)}{(a + 2b)(a + 3b)4}}{z - \cfrac{\cfrac{(a + 2b)(a + 5b)(1 + z^2)}{(a + 3b)(a + 4b)4}}{z - \cdots}}}}$$

Chapter XVI

HAUSDORFF SUMMABILITY

The theory of summability is concerned primarily with the transformation of one sequence into another. For example, the sequence

$$t_p = \frac{s_p + s_{p+1}}{2}, \quad p = 0, 1, 2, \cdots,$$

is obtained by transformation of the sequence $\{s_p\}$. We observe that if $\lim_{p=\infty} s_p = s$, then $\lim_{p=\infty} t_p = s$. A transformation having this property is said to be a **regular** transformation. We also observe that $\lim_{p=\infty} t_p$ may exist even though $\lim_{p=\infty} s_p$ fails to exist, e.g., if $s_p = [1 + (-1)^p]/2$. If we write a sequence as a one-column matrix, the above transformation may be expressed by the single matrix equation

$$\begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \cdots \\ 0, & \frac{1}{2}, & \frac{1}{2}, & 0, & \cdots \\ 0, & 0, & \frac{1}{2}, & \frac{1}{2}, & \cdots \\ & & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

or simply $\mathbf{t} = \mathbf{As}$, where $\mathbf{A}$ denotes the matrix of the transformation. We shall use one and the same letter to denote a transformation and the matrix of the transformation. A sequence $\{s_p\}$ is said to be **A-summable** to the value v if $\lim_{p=\infty} t_p = v$, where $\mathbf{t} = \mathbf{As}$.

Two transformations **A** and **B** are said to be **mutually consistent** if they assign one and the same value to an arbitrary sequence $\{s_p\}$ which they both sum. *If two regular transformations are permutable, then they are mutually consistent.* For, let **A** and **B** denote two transformations which are permutable, i.e., $\mathbf{AB} = \mathbf{BA}$, and let $\{s_p\}$ be a sequence which they both sum. If $\mathbf{t} = \mathbf{As}$ and $\mathbf{u} = \mathbf{Bs}$, then $\{t_p\}$ and $\{u_p\}$ are both convergent sequences. Let $\mathbf{v} = \mathbf{Bt} = \mathbf{BAs}$, $\mathbf{w} = \mathbf{Au} = \mathbf{ABs}$. Since **A** and **B** are regular, $\lim_{p=\infty} v_p = \lim_{p=\infty} t_p$ and $\lim_{p=\infty} w_p = \lim_{p=\infty} u_p$. But, since $\mathbf{AB} = \mathbf{BA}$, it follows that $\mathbf{v} = \mathbf{w}$, and therefore **A** and **B** are mutually consistent transformations.

Our main object here is to show how to construct classes of permutable transformations and, in particular, the class of all transformations with triangular matrices $\mathbf{A} = (a_{pq})$, $a_{pq} = 0$ for $q > p$, which are permutable with the transformation

$$t_p = \frac{s_0 + s_1 + \cdots + s_p}{p + 1}, \quad p = 0, 1, 2, \cdots,$$

with matrix

$$\mathbf{C} = \begin{bmatrix} 1, & 0, & 0, & \cdots \\ \frac{1}{2}, & \frac{1}{2}, & 0, & \cdots \\ \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \cdots \\ & \cdot & \cdot & \cdot \end{bmatrix}.$$

A triangular matrix **A** which is permutable with **C** is called a **Hausdorff matrix,** and the corresponding transformation is called a **Hausdorff transformation** or **Hausdorff mean.** Since **C** is a triangular matrix which is necessarily permutable with itself, the corresponding transformation is a Hausdorff mean. This is the familiar arithmetic mean, or **Cesàro mean of order 1,** and is usually denoted by $(C, 1)$.

With the aid of the formula

$$\int_0^1 (1 - u)^{n-p} u^p du = \frac{p!(n - p)!}{(n + 1)!},$$

we may write

$$\frac{s_0 + s_1 + \cdots + s_n}{n + 1} = \int_0^1 \sum_{p=0}^{n} \binom{n}{p} (1 - u)^{n-p} u^p s_p du.$$

If in the integral in the right-hand member of this formula we replace du by $d\phi(u)$, where $\phi(u)$ is any function of bounded variation on the interval $0 \leq u \leq 1$, we obtain a natural generalization of the Cesàro mean of order 1. These transformations

$$t_n = \int_0^1 \sum_{p=0}^{n} \binom{n}{p} (1-u)^{n-p} u^p s_p d\phi(u), \quad n = 0, 1, 2, \cdots,$$

are precisely the convergence preserving Hausdorff means, i.e., the Hausdorff means which transform convergent sequences into convergent sequences. It is therefore at once apparent that the theory of Hausdorff summability is closely connected with the moment problem for the interval (0, 1).

We shall now proceed with the development of some of these connections.

79. Hausdorff Matrices. Let $\mathbf{A} = (a_{pq})$, $a_{pq} = 0$ for $q > p$, be any triangular matrix in which the diagonal elements a_{pp} are different from zero. Then $\mathbf{A}$ has a unique inverse $\mathbf{A}^{-1}$ such that if $\mathbf{t} = \mathbf{As}$ then $\mathbf{s} = \mathbf{A}^{-1}\mathbf{t}$. Let $\mathbf{D} = (\delta_{pq} d_q)$ denote any diagonal matrix. Then the transformation

$$\mathbf{t} = \mathbf{A}^{-1}\mathbf{DAs} \tag{79.1}$$

is of the form $\mathbf{t} = \mathbf{Bs}$, where $\mathbf{B}$ is a triangular matrix. The transformation depends upon the matrix $\mathbf{A}$ and upon the sequence $\{d_p\}$. If $\lim_{p=\infty} t_p = s$, we shall say that the sequence $\{s_p\}$ is **(A, d_p)-summable** to the value s. A sequence $\{d_p\}$ such that (A, d_p) is regular, will be called an **A-regular sequence.** We shall denote by (A) the class of all matrices of the form $\mathbf{A}^{-1}\mathbf{DA}$, where $\mathbf{A}$ is a fixed matrix and $\mathbf{D}$ an arbitrary diagonal matrix.

THEOREM 79.1. *A matrix* $\mathbf{B}$ *each of whose rows has but a finite number of nonvanishing coefficients is permutable with all the matrices of the class* (A) *if, and only if, it is a member of the class* (A).

Proof. Suppose first that $\mathbf{B}$ is a member of the class (A), so that $\mathbf{ABA}^{-1} = \mathbf{D}$, a diagonal matrix. Let $\mathbf{B}_1 = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}$ be any member of the class (A). Then, since diagonal matrices are permutable,

$$\mathbf{B}_1\mathbf{B} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{AA}^{-1}\mathbf{DA} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{DA} = \mathbf{A}^{-1}\mathbf{DD}_1\mathbf{A} = \mathbf{BB}_1.$$

If, conversely, $\mathbf{B}$ is permutable with all the matrices of the class (A), it is permutable with a matrix $\mathbf{B}_1 = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}$, where $\mathbf{D}_1$ is a diagonal matrix whose diagonal elements are distinct. Put $\mathbf{D} = \mathbf{ABA}^{-1}$. Then we are to show that $\mathbf{D}$ is a diagonal matrix. We may write the equation $\mathbf{BB}_1 = \mathbf{B}_1\mathbf{B}$ as $\mathbf{ABA}^{-1}\mathbf{AB}_1\mathbf{A}^{-1} = \mathbf{AB}_1\mathbf{A}^{-1}\mathbf{ABA}^{-1}$, or $\mathbf{DD}_1 = \mathbf{D}_1\mathbf{D}$. Therefore, if $\mathbf{D} = (d_{pq})$, $\mathbf{D}_1 = (\delta_{pq}d_q')$, then $d_{pq}d_q' = d_p'd_{pq}$. Since $d_q' \neq d_p'$ if $q \neq p$, it follows that $d_{pq} = 0$ if $q \neq p$, so that $\mathbf{D}$ is a diagonal matrix, as was to be proved.

We have proved the stronger statement, that if $\mathbf{B}$ is permutable with a *single member* of the class (A) of the form $\mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}$, where $\mathbf{D}_1$ is a diagonal matrix whose diagonal elements are distinct, then $\mathbf{B}$ is in the class (A). Hence, if we can find a triangular matrix $\mathbf{H}$ such that the Cesàro matrix $\mathbf{C}$ can be written as $\mathbf{C} = \mathbf{H}^{-1}\mathbf{DH}$, where $\mathbf{D}$ is a diagonal matrix whose diagonal elements are distinct, then the class (H) will consist of all the triangular matrices which are permutable with $\mathbf{C}$. Moreover, all members of the class will be permutable with one another.

Accordingly, we seek a matrix $\mathbf{H}$ such that

$$\mathbf{HC} = \mathbf{DH},$$

where $\mathbf{D}$ is a diagonal matrix whose diagonal elements are distinct. That is, if $\mathbf{H} = (h_{pq})$, $\mathbf{C} = (c_{pq})$, $\mathbf{D} = (\delta_{pq}d_q)$, then

$$\sum_{(k)} h_{pk}c_{kq} = \sum_{(k)} \delta_{pk}d_kh_{kq},$$

or, if $q \leq p$,

$$\sum_{k=q}^{p} \frac{h_{pk}}{k+1} = d_ph_{pq}.$$

For $p = q$, this gives

$$\frac{h_{pp}}{p+1} = d_ph_{pp},$$

so that if $h_{pp} \neq 0$, then $d_p = 1/(p+1)$. The condition for determining the h_{pq} then becomes

$$\sum_{k=q}^{p} h_{pk}\frac{p+1}{k+1} = h_{pq}, \quad q \leq p.$$

On replacing q by $q + 1$ and subtracting the resulting equation from this, we obtain the recurrence formula

$$h_{pq} = -\frac{q+1}{p-q} h_{p,p+1}, \quad q < p.$$

Therefore,

$$h_{pq} = (-1)^{q-p} \binom{p}{q} h_{pp},$$

the choice of h_{pp} being at our disposal subject only to the condition $h_{pp} \neq 0$. We shall take $h_{pp} = (-1)^p$, and thus have for the general coefficient in our matrix **H** the formula

$$h_{pq} = (-1)^q \binom{p}{q}, \quad q \leq p;$$

and $\mathbf{H} = (h_{pq})$ clearly has the desired property, namely: $\mathbf{HCH}^{-1}$ is a diagonal matrix whose diagonal elements are distinct. We state this result as:

THEOREM 79.2. *A triangular matrix* **A** *is permutable with the Cesàro matrix* **C** *if, and only if,* $\mathbf{A} = \mathbf{H}^{-1}\mathbf{DH}$, *where* **D** *is a diagonal matrix, and* **H** *is the matrix*

$$\mathbf{H} = \begin{bmatrix} \binom{0}{0}, & 0, & 0, & 0, & \cdots \\ \binom{1}{0}, & -\binom{1}{1}, & 0, & 0, & \cdots \\ \binom{2}{0}, & -\binom{2}{1}, & \binom{2}{2}, & 0, & \cdots \\ & & \cdots & & \end{bmatrix}.$$

Moreover,

$$\mathbf{C} = \mathbf{H}^{-1}\left(\delta_{pq} \cdot \frac{1}{q+1}\right)\mathbf{H}. \quad [28.]$$

The matrices $\mathbf{H}^{-1}\mathbf{DH}$ are the **Hausdorff matrices,** and the corresponding transformations are the **Hausdorff means.**

80. A Theorem on (A, d_p)-Transformations. We shall now return to the general (A, d_p)-transformation, and shall prove the following theorem.

THEOREM 80.1. *The transformations* (A, d_p) *have the following properties:*

(a) *If* (A, d_p) *transforms the sequence* $\{s_p\}$ *into the sequence* $\{t_p\}$, *and* (A, d_p') *transforms* $\{t_p\}$ *into* $\{u_p\}$, *then* $(A, d_p d_p')$ *transforms* $\{s_p\}$ *into* $\{u_p\}$.

(b) *If* $\{d_p\}$ *and* $\{d_p'\}$ *are A-regular sequences, then* $\{d_p d_p'\}$ *is an A-regular sequence.*

(c) *If* (A, d_p) *sums the sequence* $\{s_p\}$ *to the value s, and if k is any number, then* (A, kd_p) *sums* $\{s_p\}$ *to the value ks.*

(d) *If the sequences* $\{d_p\}$ *and* $\{d_p'\}$ *are A-regular, and* $g + h = 1$, *then* $\{gd_p + hd_p'\}$ *is A-regular.*

(e) *If* $d_p \neq 0$, *then every sequence* (A, d_p)*-summable is* (A, d_p')*-summable, and to the same value, if, and only if,* $\{d_p'/d_p\}$ *is an A-regular sequence* [28, 22].

Proof. (a) Let $\mathbf{D} = (\delta_{pq} d_q)$, $\mathbf{D}_1 = (\delta_{pq} d_q')$. By hypothesis, $\mathbf{t} = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}\mathbf{s}$, $\mathbf{u} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}\mathbf{t}$, and therefore $\mathbf{u} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}\mathbf{A}^{-1}\mathbf{D}\mathbf{A}\mathbf{s} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{D}\mathbf{A}\mathbf{s}$. Since $\mathbf{D}_1\mathbf{D} = (\delta_{pq} d_q' d_q)$, part (a) is proved.

(b) This follows immediately from the preceding.

(c) If $\mathbf{t} = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}\mathbf{s}$, then $k\mathbf{t} = k\mathbf{A}^{-1}\mathbf{D}\mathbf{A}\mathbf{s} = \mathbf{A}^{-1}(k\mathbf{D})\mathbf{A}\mathbf{s}$.

(d) If $\mathbf{t} = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}\mathbf{s}$, $\mathbf{u} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}\mathbf{s}$, then

$$g\mathbf{t} + h\mathbf{u} = \mathbf{A}^{-1}(g\mathbf{D} + h\mathbf{D}_1)\mathbf{A}\mathbf{s}.$$

If $\{s_p\}$ is convergent, then $\lim_{p=\infty} t_p = \lim_{p=\infty} u_p = \lim_{p=\infty} s_p$. Therefore, since $g + h = 1$, $\lim_{p=\infty} (gt_p + hu_p) = \lim_{p=\infty} s_p$.

(e) Suppose first that every sequence (A, d_p)-summable is (A, d_p') summable, and to the same value. We shall indicate this symbolically by writing

$$(A, d_p') \supset (A, d_p), \tag{80.1}$$

read "(A, d_p') includes (A, d_p)." Let $\{s_p\}$ be a convergent sequence with the limit v. Determine the sequence $\{t_p\}$ by the equation $\mathbf{s} = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}\mathbf{t}$. This is possible since $d_p \neq 0$. Put $\mathbf{u} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}\mathbf{t}$. Since $\{t_p\}$ is (A, d_p)-summable to the value v, it follows by (80.1) that $\lim_{p=\infty} u_p = v$. Now

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}\mathbf{t} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{A}\mathbf{A}^{-1}\mathbf{D}^{-1}\mathbf{A}\mathbf{s} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{D}^{-1}\mathbf{A}\mathbf{s},$$

and we therefore conclude that the convergent sequence $\{s_p\}$ is $(A, d_p'/d_p)$-summable to the value v. That is, $\{d_p'/d_p\}$ is an A-regular sequence.

Suppose, conversely, that $\{d_p'/d_p\}$ is A-regular, and let $\{t_p\}$ be an (A, d_p)-summable sequence. That is, if $\mathbf{s} = \mathbf{A}^{-1}\mathbf{DAt}$, then $\lim_{p=\infty} s_p = v$ exists. Since $(A, d_p'/d_p)$ is regular, it then follows that if $\mathbf{u} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{D}^{-1}\mathbf{As}$, then $\lim_{p=\infty} u_p = v$. But $\mathbf{u} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{D}^{-1}\mathbf{AA}^{-1}\mathbf{DAt} = \mathbf{A}^{-1}\mathbf{D}_1\mathbf{At}$, and hence we conclude that $\{t_p\}$ is (A, d_p')-summable to the value v, i.e., (80.1) holds.

81. Hausdorff Means. In this section we shall establish the fundamental formulas of the Hausdorff theory.

THEOREM 81.1. *If* **H** *is the matrix of Theorem* 79.2, *then the transformation* (H, d_p) *can be written in the form*

$$t_n = \sum_{p=0}^{n} \binom{n}{p} (\Delta^{n-p} d_p) s_p, \quad n = 0, 1, 2, \cdots, \tag{81.1}$$

where

$$\Delta^r d_s = \sum_{p=0}^{r} (-1)^p \binom{r}{p} d_{s+p}. \quad [28.]$$

Proof. It is required to prove that the matrix

$$\mathbf{H}^{-1}\mathbf{DH} = \left(\binom{p}{q} \Delta^{p-q} d_p\right). \tag{81.2}$$

Let $\mathbf{H} = (h_{pq})$. Then $\mathbf{H}^2 = \left(\sum_{(k)} h_{pk}h_{kq}\right)$. Now

$$\sum_{(k)} h_{pk}h_{kq} = \begin{cases} 0 & \text{if} \quad q > p, \\ \sum_{k=q}^{p} (-1)^{k+q} \binom{p}{k}\binom{k}{q} & \text{if} \quad q \leq p. \end{cases}$$

This is obviously equal to 1 if $q = p$. If $q < p$, let $r = k - q$, and we have

$$\sum_{k=q}^{p} (-1)^{k+q} \binom{p}{k}\binom{k}{q} = \sum_{r=0}^{p-q} (-1)^r \binom{p}{q}\binom{p-q}{r} = \binom{p}{q}(1-1)^{p-q}$$
$$= 0.$$

Therefore, $\mathbf{H}^2 = (\delta_{pq})$, so that $\mathbf{H} = \mathbf{H}^{-1}$. Consequently

$$\mathbf{H}^{-1}\mathbf{D}\mathbf{H} = \mathbf{H}\mathbf{D}\mathbf{H} = \Big(\sum_{(k,r)} h_{pk}\delta_{kr}d_r h_{rq}\Big) = \Big(\sum_{(k)} h_{pk}h_{kq}d_k\Big).$$

Now,

$$\sum_{(k)} h_{pk}h_{kq}d_k = \begin{cases} 0 & \text{if } q > p, \\ \displaystyle\sum_{k=q}^{p}(-1)^{k+q}\binom{p}{k}\binom{k}{q}d_k & \text{if } q \le p. \end{cases}$$

For $q \le p$ we put $k = q + r$ and find that

$$\sum_{k=q}^{p}(-1)^{k+q}\binom{p}{k}\binom{k}{q}d_k = \binom{p}{q}\sum_{r=0}^{p-q}(-1)^r\binom{p-q}{r}d_{q+r}$$
$$= \binom{p}{q}\Delta^{p-q}d_q,$$

as was to be proved.

THEOREM 81.2. *The Hausdorff mean* (81.1) *is convergence preserving, i.e., transforms any convergent sequence into a convergent sequence, if, and only if, there exists a function* $\phi(u)$ *of bounded variation on the interval* $0 \le u \le 1$, *such that*

$$d_p = \int_0^1 u^p d\phi(u), \quad p = 0, 1, 2, \cdots. \quad [28.]$$

Proof. A transformation of the form $t_n = \sum_{p=0}^{n} b_{np}s_p$ is convergence preserving if, and only if, the following three conditions are satisfied.

(a) $\displaystyle\sum_{p=0}^{n} | b_{np} | \le M, \quad n = 0, 1, 2, \cdots, \quad M$ a constant.

(b) $\displaystyle\lim_{n=\infty} b_{np} = L_p, \quad p = 0, 1, 2, \cdots, \quad L_p$ finite. (81.3)

(c) $\displaystyle\lim_{n=\infty}\sum_{p=0}^{n} b_{np} = L, \quad L$ finite.

The transformation is regular if, and only if, (81.3) holds with $L_p = 0$, $p = 0, 1, 2, \cdots$, and $L = 1$. This is a fundamental theorem in the theory of summability [**91, 104**].

The condition (c) is automatically satisfied by any transformation of the form (81.1), and in this case $L = d_0$, inasmuch as

$$\sum_{p=0}^{n}\binom{n}{p}\Delta^{n-p}d_p = d_0, \quad n = 0, 1, 2, \cdots. \tag{81.4}$$

The conditions (a) and (b), applied to (81.1), give

$$\text{(a}'\text{)} \qquad \sum_{p=0}^{n}\binom{n}{p}\left|\Delta^{n-p}d_p\right| \leq M, \quad n = 0, 1, 2, \cdots.$$

$$\text{(b}'\text{)} \qquad \lim_{n=\infty}\binom{n}{p}\Delta^{n-p}d_p = L_p, \quad p = 0, 1, 2, \cdots. \tag{81.5}$$

Since, by Theorem 72.1, the condition (a′) is both necessary and sufficient for the existence of a function $\phi(u)$ of the specified character, it remains only to be proved that

$$\lim_{n=\infty}\binom{n}{p}\int_0^1(1 - u)^{n-p}u^p d\phi(u) = L_p, \quad p = 0, 1, 2, \cdots, \tag{81.6}$$

when $\phi(u)$ is a function of bounded variation on the interval $0 \leq u \leq 1$. To that end, we write the expression under the limit sign in the form

$$\binom{n}{p}\int_0^t(1 - u)^{n-p}u^p d\phi(u) + \binom{n}{p}\int_t^1(1 - u)^{n-p}u^p d\phi(u),$$

where $0 < t < 1$. We may evidently suppose that $\phi(u)$ is real and nondecreasing, for the general case can readily be reduced tc this. Assume, for the moment, that $\phi(u)$ is continuous at $u = 0$ Then, for the first of the above integrals we have

$$0 \leq \binom{n}{p}\int_0^t(1 - u)^{n-p}u^p d\phi(u) \leq \int_0^t[(1 - u) + u]^n d\phi(u)$$
$$= \phi(t) - \phi(0).$$

This will be less than an arbitrarily small assigned $\epsilon > 0$ if $0 < t < \delta$, where $\delta = \delta(\epsilon) > 0$ is sufficiently small. For a fixed t, the second integral does not exceed

$$\binom{n}{p}(1 - t)^{n-p}[\phi(1) - \phi(t)] < \epsilon,$$

for all sufficiently large values of n. We therefore conclude that (81.6) holds with $L_p = 0$, $p = 0, 1, 2, \cdots$, in the case where $\phi(u)$ is continuous at $u = 0$.

If $\phi(u)$ is discontinuous at $u = 0$, put $c = \phi(+0) - \phi(0)$, and define the function $\theta(u) = \phi(u) - c$ for $u > 0$, $\theta(0) = \phi(0)$. Then $\theta(u)$ is nondecreasing and is continuous at $u = 0$. Moreover,

$$\binom{n}{p}\int_0^1 (1-u)^{n-p}u^p d\phi(u) = \begin{cases} \binom{n}{p}\int_0^1 (1-u)^{n-p}u^p d\theta(u) & \text{for } p > 0, \\ \binom{n}{p}\int_0^1 (1-u)^{n-p}u^p d\theta(u) + c & \text{for } p = 0. \end{cases}$$

Hence, it follows from the preceding that (81.6) holds with

$$L_p = 0 \quad \text{for} \quad p > 0, \quad L_0 = \phi(+0) - \phi(0). \tag{81.7}$$

The preceding proof contains the following theorem.

THEOREM 81.3. *The Hausdorff mean (H, d_p) is regular if, and only if, the moment problem*

$$d_p = \int_0^1 u^p d\phi(u), \quad p = 0, 1, 2, \cdots,$$

has a solution $\phi(u)$ of bounded variation on the interval $0 \leq u \leq 1$, such that $\phi(1) - \phi(0) = 1$ and $\phi(+0) = \phi(0)$. [28.]

82. Examples of Hausdorff Means. We shall now give several examples of Hausdorff means.

Example 1. The Cesàro mean of order 1, $(C, 1)$, is the Hausdorff mean $(H, 1/(p+1))$. Here $\phi(u) \equiv u$, so that the mean is regular by Theorem 81.3.

Example 2. The Hausdorff mean (H, r^p) is called the **Euler-Knopp mean.** If $0 < r < 1$, then $\phi(u)$ is the step-function

$$\phi(u) = \begin{cases} 0 & \text{for} \quad u < r, \\ \frac{1}{2} & \text{for} \quad u = r, \\ 1 & \text{for} \quad u > r. \end{cases}$$

This is a regular mean.

Example 3. The **Hölder mean, H_c,** is the Hausdorff mean for which

$$\phi(u) = \frac{1}{\Gamma(c)} \int_0^u \left(\log \frac{1}{t}\right)^{c-1} dt, \quad \Re(c) > 0.$$

It will be noted that H_1 is the same as $(C, 1)$.

Example 4. The **Cesàro mean of order c, (C, c),** is the Hausdorff mean for which

$$\phi(u) = 1 - (1 - u)^c, \quad \Re(c) > 0.$$

The means H_c and (C, c) are equivalent in the sense that any sequence summable by the one is summable by the other also. They are, of course, mutually consistent.

Example 5. Let

$$a_p = \int_0^\infty \frac{d\mu(u)}{1 + pu}, \quad p = 0, 1, 2, \cdots,$$

where $\mu(u)$ is a bounded nondecreasing function and $\mu(+\infty) - \mu(0) = 1$. Then the Hausdorff mean (H, a_p) is a regular mean. In fact,

$$\Delta^m a_n = \int_0^\infty \frac{u^m m! d\mu(u)}{(1 + nu)(1 + [n + 1]u) \cdots (1 + [n + m]u)} > 0,$$

so that the sequence $\{a_p\}$ is totally monotone. Since $a_0 = \mu(-\infty) - \mu(0) = 1$, and since, as may be easily seen, $\lim_{p=\infty} \Delta^p a_0 = 0$, it follows that (H, a_p) is regular [**88, 22**].

83. The Hausdorff Inclusion Problem. Let (H, a_p) and (H, b_p) be two Hausdorff means, and suppose that $b_p \neq 0, p = 0, 1, 2, \cdots$. By (e) of Theorem 80.1,

$$(H, a_p) \supset (H, b_p) \tag{83.1}$$

if, and only if, the sequence $\{a_p/b_p\}$ is H-regular, i.e., if, and only if, there exists a function $\phi(u)$ of bounded variation on the interval $0 \leq u \leq 1$, such that $\phi(1) - \phi(0) = 1$, $\phi(+0) = \phi(0)$, and such that

$$\frac{a_p}{b_p} = \int_0^1 u^p d\phi(u), \quad p = 0, 1, 2, \cdots. \tag{83.2}$$

The **Hausdorff inclusion problem** is the problem of establishing whether or not the inclusion relation (83.1) holds between two given Hausdorff means. Necessary and sufficient conditions for (83.1) to hold between two given regular Hausdorff means have been formulated in various ways (cf. [**17**]). We shall limit the discussion here to an example, in which use is made of the continued fraction theory [**88**, **22**].

Consider the regular Hausdorff mean defined in Example 5 of § 82. We shall establish the inclusion relation [**22**]

$$(C, 1) \supset (H, a_p). \tag{83.3}$$

Let

$$f(z) = \int_0^{+\infty} \frac{d\mu(u)}{1 + zu},$$

so that $a_p = f(p)$. Then

$$f(1 + z) = \int_0^1 \frac{(1 - v)d\lambda(v)}{1 + zv}, \quad \text{where} \quad \lambda(v) = \mu\left(\frac{v}{1 - v}\right).$$

The function $\lambda(v)$ is a bounded nondecreasing function of v, and $\lambda(1) - \lambda(0) = 1$. By Theorems 76.3 and 76.1, we therefore have

$$f(1 + z) = \cfrac{g_0}{1 + \cfrac{(1 - g_0)g_1 z}{1 + \cfrac{(1 - g_1)g_2 z}{1 + \cdots}}}, \tag{83.4}$$

where $0 < g_0 \leq 1$, $0 \leq g_p \leq 1$, $p = 1, 2, 3, \cdots$.

If $g_0 = 1$, so that $f(z) \equiv 1$, then (H, a_p) is the identity transformation, i.e., *convergence*, and is of course included in $(C, 1)$.

By (83.4),

$$\frac{g_0}{f(z + 1)} = 1 + (1 - g_0)z \cdot \cfrac{g_1}{1 + \cfrac{(1 - g_1)g_2 z}{1 + \cfrac{(1 - g_2)g_3 z}{1 + \cdots}}}$$

$$= 1 + (1 - g_0)z \cdot f_1(z + 1),$$

where $f_1(z + 1)$ has the form

$$f_1(z+1) = \int_0^1 \frac{(1-v)d\lambda_1(v)}{1+zv} = \int_0^\infty \frac{d\mu_1(u)}{1+(z+1)u},$$

$$\lambda_1(v) = \mu_1\left(\frac{v}{1-v}\right).$$

The function $\lambda_1(v)$ is a bounded nondecreasing function. If $g_0 = f(1) = a_1 < 1$, then, since $f(0) = 1$, it is easy to see that $f_1(0) = 1$, and therefore $\lambda_1(1) - \lambda_1(0) = 1$. Thus, the sequence $b_p = f_1(p)$, $p = 0, 1, 2, \cdots$, is H-regular, and we have

$$\frac{a_1}{a_p} = 1 + (1 - a_1)(p - 1)b_p, \quad p = 0, 1, 2, \cdots, \quad (a_1 < 1). \tag{83.5}$$

Since $(C, 1)$ is the Hausdorff mean $[H, 1/(p + 1)]$, it is required to prove that

$$\frac{1}{p+1} : a_p = \frac{1}{(p+1)a_p}, \quad p = 0, 1, 2, \cdots, \tag{83.6}$$

is an H-regular sequence. We have, by (83.5),

$$\frac{1}{(p+1)a_p} = \frac{1}{a_1}\left(\frac{1}{p+1} + (1-a_1)b_p - 2(1-a_1)\frac{b_p}{p+1}\right).$$

Now, the sequences $\{1/(p + 1)\}$ and $\{b_p\}$ are H-regular. Hence, by (b) of Theorem 80.1, $\{b_p/(p + 1)\}$ is H-regular. We then conclude from (d) of Theorem 80.1 that (83.6) is H-regular, and (83.3) is thereby established.

Exercise 16

16.1. The Hausdorff mean of Example 5, § 82, is equivalent to convergence, i.e., will sum only convergent sequences, if, and only if, the function $\mu(u)$ is discontinuous at $u = 0$; and is equivalent to $(C, 1)$ if, and only if, the integral $\int_0^\infty d\mu(u)/u$ is finite [**88, 22**].

16.2. Let g be a number between 0 and 1, $0 < g < 1$, and let

$$c_0 - c_1z + c_2z^2 - \cdots = \cfrac{1}{1 + \cfrac{g(1-g)z}{1 + \cfrac{g(1-g)z}{1 + \cdots}}}$$

Show that the Hausdorff mean (H, c_p) includes $(C, 1)$. [**19.**]

16.3. Let (H, a_p) and (H, b_p) be regular Hausdorff means, and suppose that $b_p \neq 0$, $p = 0, 1, 2, \cdots$. Then $(H, a_p) \supset (H, b_p)$ if, and only if, there exists a function $\phi_c(u)$ of bounded variation on the interval $0 \leq u \leq 1$ such that $\phi_c(+0) = \phi_c(0) = 0$, $\phi_c(1) = 1$, and such that, for $|t| < 1$,

$$f_a(t) = \int_0^1 f_b(tu)d\phi_c(u)$$

where $f_x(t) = \Sigma(-1)^p x_p t^p$. [17.]

16.4. Let (H, a_p) be a regular Hausdorff mean, and let $a_p = \int_0^1 u^p d\phi(u)$, $p = 0, 1, 2, \cdots$, $\phi(0) = 0$. Then $(H, a_p) \supset (C, m)$, m a positive integer, if, and only if, the following five conditions are all satisfied.

(a) $\phi(u)$ is absolutely continuous and has absolutely continuous derivatives of orders $1, 2, \cdots, m-1$ for $0 < u \leq 1$.

(b) $\phi^{(m-1)}(u)$ has a finite right-hand derivative $\phi_r^{(m)}(u)$ for $0 < u < 1$, and a finite left-hand derivative $\phi_l^{(m)}(u)$ for $0 < u \leq 1$.

(c) $\phi(u), u\phi'(u), \cdots, u^{m-1}\phi^{(m-1)}(u), u^m\phi_r^{(m)}(u)$ and $u^m\phi_l^{(m)}(u)$ are of bounded variation in the interval $0 \leq u \leq 1$.

(d) $1 - \phi(u), \phi'(u), \cdots, \phi^{(m-1)}(u)$ tend to zero as $u \to 1 - 0$.

(e) $\phi(u), u\phi'(u), \cdots, u^{m-1}\phi^{(m-1)}(u), u^m\phi_r^{(m)}(u)$ and $u^m\phi_l^{(m)}(u)$ tend to zero as $u \to +0$. [17.]

16.5. If (H, s_p) and (H, c_p) are convergence preserving Hausdorff means, and

$$t_p = \sum_{q=0}^{p} s_q \binom{p}{q} \Delta^{p-q} c_q, \quad p = 0, 1, 2, \cdots,$$

then (H, t_p) is a convergence preserving Hausdorff mean [22].

16.6. Let

$$\sum_{p=0}^{\infty}(-1)^p \mu_p z^p = \cfrac{\mu_0}{1 + \cfrac{g_1 z}{1 + \cfrac{(1-g_1)g_2 z}{1 + \cfrac{(1-g_2)g_3 z}{1 + \cdots}}}} \qquad \mu_0 \neq 0.$$

Then,

$$\sum_{p=0}^{\infty}(-1)^p \Delta\mu_p z^p = \cfrac{\Delta\mu_0}{1 + \cfrac{(1-g_1)z}{1 + \cfrac{g_1(1-g_2)z}{1 + \cfrac{g_2(1-g_3)z}{1 + \cdots}}}},$$

and

$$\sum_{p=0}^{\infty}(-1)^p \Delta^p \mu_0 z^p = \cfrac{\mu_0}{1 + \cfrac{(1-g_1)z}{1 + \cfrac{g_1 g_2 z}{1 + \cfrac{(1-g_2)(1-g_3)z}{1 + \cfrac{g_3 g_4 z}{1 + \cdots}}}}}. \quad [18.]$$

16.7. Let $f(w)$ be any function which is analytic and has modulus not exceeding unity for $|w| < 1$, and which is real when w is real. Then there exists a function $F(z)$ of the form

$$F(z) = \int_0^1 \frac{d\phi(u)}{1 + zu}$$

where $\phi(u)$ is a bounded nondecreasing function of u on the interval $0 \leq u \leq 1$, of modulus not greater than unity for $|z| < 1$, such that

$$\frac{1}{2}(1 - w)\frac{1 - f(w)}{1 + wf(w)} = F(z), \quad \text{where} \quad z = \frac{4w}{(1 - w)^2} \quad \text{for} \quad |w| < 1.$$

Conversely, if $F(z)$ is given, then $f(w)$ is determined [**132**].

16.8. Put $f(w) = \Sigma(-1)^p c_p w^p$, $F(z) = \Sigma(-1)^p C_p z^p$ in the preceding identity. Then

$$C_n = \sum_{p=0}^{n} T_{n,p} c_p \quad \text{where} \quad T_{n,p} = (\tfrac{1}{2})^{2n+1}(2p + 1)(n + p + 1)^{-1}\binom{2n}{n - p}.$$

Put $S_n = C_0 + C_1 + \cdots + C_n$, $s_n = c_0 + c_1 + \cdots + c_n$. Then

$$S_n = \sum_{p=0}^{n} (\tfrac{1}{2})^{2n+1}\binom{2n + 2}{n - p} s_p.$$

This is a regular transformation having the following properties.

(a) The transformation sums the geometrical series Σx^p to the sum $1/(1 - x)$ inside the curve whose polar equation is $r = 2 - \cos\theta + [(1 - \cos\theta)(3 - \cos\theta)]^{1/2}$, but does not sum the geometrical series outside or upon this curve.

(b) The method of summation defined by this transformation includes de la Vallée Poussin summability but is not included by de la Vallée Poussin summability (cf. § 74). [**88.**]

Chapter XVII

THE MOMENT PROBLEM FOR AN INFINITE INTERVAL

By the **moment problem for the interval** $(-\infty, +\infty)$ we shall understand the problem of determining a bounded nondecreasing function $\phi(u)$ on this interval, such that

$$c_p = \int_{-\infty}^{+\infty} u^p d\phi(u), \quad p = 0, 1, 2, \cdots,$$

where $\{c_p\}$ is a given sequence of constants. If we require $\phi(u)$ to be constant for $u > 1$ and for $u < 0$, this problem reduces to the moment problem of Chapter XIV. We shall put to one side the case where there is a solution $\phi(u)$ which is a simple step-function having a finite number of jumps, and shall characterize those sequences $\{c_p\}$ for which the moment problem has a solution $\phi(u)$ which takes on infinitely many different values. We shall find that a solution of this kind exists if, and only if, the formal power series $\Sigma(c_p/z^{p+1})$ is the power series expansion of an infinite real J-fraction. Moreover, the solutions are just those functions $\phi(u)$ such that the functions

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u}$$

are equivalent functions of the J-fraction.

84. Asymptotic Expressions for J-fractions. We consider in this chapter infinite J-fractions

$$\cfrac{1}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cfrac{a_2^2}{b_3 + z - \cdots}}}, \tag{84.1}$$

in which $a_p \neq 0$, $p = 1, 2, 3, \cdots$. We shall make use of the ideas and notations introduced in Chapter IV and in §§ 60–61.

Definition 84.1. *Let $f(z)$ be a function defined for $I(z) > 0$. We shall say that $f(z)$ is* **asymptotically equal** *to the J-fraction* (84.1) *in the half-plane $\mathfrak{J}(z) > 0$, if for every positive number δ, and every index $p = 0, 1, 2, \cdots$,*

$$\lim_{\substack{z=\infty \\ \mathfrak{J}(z)\geq\delta}} z^{2p}\left(f(z) - \frac{X_{p+1}(z)}{Y_{p+1}(z)}\right) = 0, \tag{84.2}$$

where $X_{p+1}(z)/Y_{p+1}(z)$ is the pth *approximant of the J-fraction* (*cf.* (17.9)).

Definition 84.2. *Let $f(z)$ be a function defined for $\mathfrak{J}(z) > 0$. We shall say that $f(z)$ is* **asymptotically equal** *to the power series $\Sigma(c_p/z^{p+1})$ in the half-plane $\mathfrak{J}(z) > 0$, if for every positive number δ, and every index $p = 1, 2, 3, \cdots$,*

$$\lim_{\substack{z=\infty \\ \mathfrak{J}(z)\geq\delta}} z^{p}\left(f(z) - \sum_{k=0}^{p-1} \frac{c_k}{z^{k+1}}\right) = 0. \tag{84.3}$$

Theorem 84.1. *A function $f(z)$ is asymptotically equal to the J-fraction* (84.1) *if, and only if, it is asymptotically equal to the power series expansion of the J-fraction.*

Proof. By (51.4) and (17.9),

$$\frac{X_{p+1}(z)}{Y_{p+1}(z)} = \sum_{k=0}^{2p-1} \frac{c_k}{z^{k+1}} + \frac{G(z)}{z^{2p+1}},$$

where $|G(z)| < C$, C a positive constant, for all z with modulus sufficiently large. If $n \leq 2p$, we therefore have

$$z^n\left(f(z) - \sum_{k=0}^{n-1} \frac{c_k}{z^{k+1}}\right) = \frac{1}{z^{2p-n}} \cdot z^{2p}\left(f(z) - \frac{X_{p+1}(z)}{Y_{p+1}(z)}\right) + \frac{h(z)}{z},$$

where $h(z)$ remains bounded as $z \to \infty$. Therefore, if $f(z)$ is asymptotically equal to the J-fraction in the half-plane $\mathfrak{J}(z) > 0$, this expression must tend to zero as $z \to \infty$ along any path in the half-plane $\mathfrak{J}(z) \geq \delta$, for every positive constant δ. Hence, $f(z)$ is asymptotically equal to the power series expansion of the J-fraction.

Suppose, conversely, that $f(z)$ is asymptotically equal to the power series expansion of the J-fraction. Then,

$$z^{2p}\left(f(z) - \frac{X_{p+1}(z)}{Y_{p+1}(z)}\right) = z^{2p}\left(f(z) - \sum_{k=0}^{2p-1} \frac{c_k}{z^{k+1}}\right) - \frac{G(z)}{z},$$

and, since the right-hand member tends to zero as $z \to \infty$ in $\mathfrak{J}(z) \geq \delta$, we conclude that $f(z)$ is asymptotically equal to the J-fraction.

THEOREM 84.2. *If $f(z)$ is asymptotically equal to the J-fraction* (84.1), *and*

$$f(z) = \frac{1}{b_1 + z - a_1^2 g(z)},$$

then $g(z)$ is asymptotically equal to the J-fraction obtained from (84.1) *by advancing the subscripts of all the a_p and b_p by unity.*

Proof. Let $f_n(z)$ and $g_n(z)$ denote the nth approximants of the J-fraction (84.1) and of the J-fraction obtained from (84.1) by advancing the subscripts of all the a_p and b_p by unity, respectively. Then, one may readily verify the identity

$$z^{2r}[g(z) - g_r(z)] = \frac{z^{2r+2}[f(z) - f_{r+1}(z)]}{a_1^2 \cdot zf(z) \cdot zf_{r+1}(z)}.$$

On letting $z \to \infty$ along any path in $\mathfrak{J}(z) \geq \delta$, we see that

$$zf(z) \to 1, \quad zf_{r+1}(z) \to 1, \quad z^{2r+2}[f(z) - f_{r+1}(z)] \to 0,$$

so that

$$z^{2r}[g(z) - g_r(z)] \to 0,$$

and the theorem is established.

THEOREM 84.3. *Let* (84.1) *be a positive definite J-fraction, and let* $f(z)$ *be an equivalent function* (*cf.* § 61). *Then* $f(z)$ *is asymptotically equal to the J-fraction* [**138**].

Proof. By (60.3), with $w_q(z) = f(z)$, $q = 1, 2, 3, \cdots$, we get:

$$z^{2p}\left(f(z) - \frac{X_{p+1}(z)}{Y_{p+1}(z)}\right) = \frac{z^{2p}\rho_{p+1,p}(z)}{Y_p(z)Y_{p+1}(z)}.$$

Therefore, by (62.5),

$$z^{2p}\left(f(z) - \frac{X_{p+1}(z)}{Y_{p+1}(z)}\right) = \frac{z^{2p-1}}{Y_p(z)Y_{p+1}(z)} \cdot \left(\frac{0(1)}{z} + \frac{0(1)}{yz}\right),$$

where $y = \mathfrak{J}(z) \geq \delta$. Since $Y_p(z)Y_{p+1}(z)$ is a polynomial in z of degree $2p - 1$, it follows that the left-hand member of this equality tends to 0 as $z \to \infty$ in $\mathfrak{J}(z) \geq \delta$. This establishes the theorem.

For *real* J-fractions we have the following stronger theorem.

THEOREM 84.4. *A function* $f(z)$ *which for* $\mathfrak{J}(z) > 0$ *is analytic and has a negative imaginary part,* $\mathfrak{J}[f(z)] < 0$, *is asymptotically equal to a real J-fraction* (84.1) *if, and only if, it is an equivalent function of the J-fraction* [**62**].

Remark. For real J-fractions, the determinate case and the indeterminate case correspond exactly to the limit-point case and the limit-circle case, respectively. This is not in general true for positive definite J-fractions. We venture the conjecture that Theorem 84.4 is not true in general for positive definite J-fractions. It would be desirable to have a definition of "equivalent function" for which the theorem would hold. We offer the conjecture that the required function is: "Any function $f(z)$ which for $\mathfrak{J}(z) > 0$ is analytic and satisfies the inequality (61.7)." It should be noted that Theorem 84.3 holds for these functions.

Proof of Theorem 84.4. The sufficiency of the condition is contained in Theorem 84.3. We suppose, then, that for $\mathfrak{J}(z) > 0$, the function $f(z)$ is analytic, has negative imaginary part, and is asymptotically equal to a *real* J-fraction (84.1), and shall prove that $f(z)$ is an equivalent function of the J-fraction. We are to show that for every z with $\mathfrak{J}(z) > 0$, the values of $f(z)$ are in all the circles $K_p(z)$ of § 17. In the case of a real J-fraction, $K_p(z)$

is the image of the half-plane $\mathfrak{J}(w) \geq 0$ under the transformation (cf. (17.4))

$$t = t_0 t_1 \cdots t_p(w).$$

For each $p = 1, 2, 3, \cdots$, let the function $f_p(z)$ be determined such that

$$f(z) = t_0 t_1 \cdots t_p\left(\frac{1}{f_p(z)}\right).$$

The theorem will be established when we have shown that

$$\mathfrak{J}\left(\frac{1}{f_p(z)}\right) > 0, \quad \text{i.e.,} \quad \mathfrak{J}[f_p(z)] < 0, \quad \text{for} \quad \mathfrak{J}(z) > 0.$$

In view of Theorem 84.2, it will evidently suffice to show that for $\mathfrak{J}(z) > 0, f_1(z)$ is analytic and $\mathfrak{J}[f_1(z)] < 0$. We have:

$$f_1(z) = \frac{b_1 + z}{a_1^2} - \frac{1}{a_1^2 f(z)} = \frac{b_1 + z}{a_1^2} - \frac{\overline{f(z)}}{a_1^2 |f(z)|^2},$$

and, therefore, since a_1 and b_1 are real,

$$\mathfrak{J}[f_1(z)] < \frac{y}{a_1^2}, \quad \text{where} \quad y = \mathfrak{J}(z) > 0.$$

The function $\mathfrak{J}[f_1(z)]$ is clearly harmonic for $\mathfrak{J}(z) > 0$. If possible, let $\mathfrak{J}[f_1(z_0)] = d_1 > 0$, $\mathfrak{J}(z_0) > 0$. Let $d > 0$ be so chosen that $(y/a_1^2) < d_1/2$ for $0 < y < d$; and let r be so chosen that $|z_0| < r$, and $|f_1(z)| < d_1/2$ for $|z| \geq r$, $\mathfrak{J}(z) \geq d$. This is possible by virtue of Theorem 84.2 from which it follows that $\lim_{z=\infty} f_1(z) = 0$ for $\mathfrak{J}(z) \geq d$. Then, in the region $|z| < r, \mathfrak{J}(z) > d$, the function $\mathfrak{J}[f_1(z)]$ is harmonic, is not greater than $d_1/2$ on the boundary, and takes on the value d_1 at an interior point z_0. Since this is impossible, we conclude that $\mathfrak{J}[f_1(z)] \leq 0$ for $\mathfrak{J}(z) > 0$. Since $f_1(z)$ is not a constant, we then have $\mathfrak{J}[f_1(z)] < 0$ for $\mathfrak{J}(z) > 0$.

Inasmuch as $f_1(z)$ is asymptotically equal to the real J-fraction obtained from (84.1) by advancing the subscripts of all the a_p and b_p by unity, it now follows that $f_2(z)$ is analytic and has a negative imaginary part for $\mathfrak{J}(z) > 0$, and so on. This completes the proof of Theorem 84.4.

Since there is just one equivalent function of the J-fraction in the determinate case, we have

THEOREM 84.5. *In the determinate case for a real J-fraction, there is one and only one function $f(z)$, analytic and with negative imaginary part for $\mathfrak{J}(z) > 0$, which is asymptotically equal to the J-fraction* [62].

In the indeterminate case, which, for real J-fractions, coincides with the limit-circle case, we may represent all the equivalent functions parametrically in terms of the entire functions $p(z)$, $u(z)$, $q(z)$, $v(z)$ of Theorem 23.3.

In order to indicate dependence upon the parameter z, we shall now write (cf. (17.4)),

$$t_0 t_1 \cdots t_p(w) = T_p(w; z).$$

By means of (23.1) and (17.4) we readily find that

$$t = T_p(w; z) = \frac{U_p(z) T_p(w; 0) - P_p(z)}{V_p(z) T_p(w; 0) - Q_p(z)}. \tag{84.4}$$

Now, for every p, the transformation $t = T_p(w; 0)$ maps the half-plane $\mathfrak{J}(w) \geq 0$ upon the half-plane $\mathfrak{J}(t) \leq 0$. Consequently, if in (84.4) we replace $T_p(w; 0)$ by any function $\theta(z)$ such that for $\mathfrak{J}(z) > 0$, $\theta(z)$ is analytic and $\mathfrak{J}[\theta(z)] \leq 0$, or by the constant ∞, the resulting function will have its values in $K_p(z)$ for all p and for $\mathfrak{J}(z) > 0$. On letting p tend to ∞, we then conclude from (23.5) that

$$\frac{u(z)\theta(z) - p(z)}{v(z)\theta(z) - q(z)} \tag{84.5}$$

is an equivalent function of the J-fraction. Conversely, if $f(z)$ is any equivalent function of the J-fraction, we may determine, for each value of p, a function $\theta_p(z)$ such that

$$f(z) = \frac{U_p(z)\theta_p(z) - P_p(z)}{V_p(z)\theta_p(z) - Q_p(z)}, \tag{84.6}$$

and $\theta_p(z)$ is necessarily analytic for $\mathfrak{J}(z) > 0$ and $\mathfrak{J}[\theta_p(z)] \leq 0$ in this domain, or $\theta_p(z) \equiv \infty$. On letting $p \to \infty$, the function $\theta_p(z)$ must have the constant limit ∞, or else must converge to a limit $\theta(z)$ which, for $\mathfrak{J}(z) > 0$, is analytic and has imaginary part not greater than zero. Hence, we conclude that $f(z)$ has the form

(84.5). We have completed the proof of the following theorem of R. Nevanlinna [**62**].

THEOREM 84.6. *In the indeterminate case for a real J-fraction, a necessary and sufficient condition for $f(z)$ to be an equivalent function is that $f(z)$ have the form* (84.5), *where $u(z)$, $p(z)$, $v(z)$, $q(z)$ are the entire functions introduced in Theorem* 23.3, *and $\theta(z)$ is an arbitrary function which for $\mathfrak{I}(z) > 0$ is analytic and has $\mathfrak{I}[\theta(z)] \leq 0$, or else $\theta(z) \equiv \infty$, in which case* (84.5) *is the function $u(z)/v(z)$.*

85. A Theorem of Hamburger. We have seen in Theorem 66.1 that an equivalent function $f(z)$ of a positive definite J-fraction can be represented as a Stieltjes transform

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z-u}, \tag{85.1}$$

where $\phi(u)$ is a bounded nondecreasing function, and $\phi(+\infty) - \phi(-\infty) = 1$. The following theorem of Hamburger [**26**] furnishes a necessary and sufficient condition for the integrals $\int_{-\infty}^{+\infty} u^p d\phi(u)$, $p = 0, 1, 2, \cdots$, to exist.

THEOREM 85.1. *Let*

$$P\left(\frac{1}{z}\right) = \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots$$

be a power series with real coefficients. The bounded nondecreasing function $\phi(u)$ of (85.1) *satisfies the equations*

$$c_p = \int_{-\infty}^{+\infty} u^p d\phi(u), \quad p = 0, 1, 2, \cdots, \tag{85.2}$$

if, and only if, the function $f(z)$ of (85.1) *satisfies the equations*

$$\lim_{z=\infty} z^p \left(f(z) - \frac{c_0}{z} - \frac{c_1}{z^2} - \cdots - \frac{c_{p-1}}{z^p}\right) = 0, \quad p = 1, 2, 3, \cdots, \tag{85.3}$$

where $z \to \infty$ along the positive imaginary axis [**26**].

Proof. We prove first that (85.3) follows from (85.2). We have

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u}$$

$$= \int_{-\infty}^{+\infty} \left(\frac{1}{z} + \frac{u}{z^2} + \cdots + \frac{u^{n-1}}{z^n} + \frac{u^n}{z^n(z - u)} \right) d\phi(u)$$

$$= \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_{n-1}}{z^n} + \int_{-\infty}^{+\infty} \frac{u^n d\phi(u)}{z^n(z - u)},$$

and therefore

$$z^n \left(f(z) - \frac{c_0}{z} - \frac{c_1}{z^2} - \cdots - \frac{c_{n-1}}{z^n} \right) = \frac{1}{z} \int_{-\infty}^{+\infty} \frac{z}{z - u} u^n d\phi(u).$$

Now if $z = iy, y > 0$, then $|z/(z - u)| \leq 1$. Hence, if n is *even*, the modulus of the last expression does not exceed

$$\frac{1}{y} \int_{-\infty}^{+\infty} u^n d\phi(u) = \frac{c_n}{y},$$

which approaches 0 as $y \to +\infty$. If n is *odd*, we may write

$$z^n \left(f(z) - \frac{c_0}{z} - \frac{c_1}{z^2} - \cdots - \frac{c_{n-1}}{z^n} \right) = \frac{c_n}{z} + \frac{1}{z^2} \int_{-\infty}^{+\infty} \frac{z}{z - u} u^{n+1} d\phi(u),$$

so that (85.3) holds in this case also.

We suppose now, conversely, that (85.3) holds as $z \to \infty$ along the positive imaginary axis, and shall prove that (85.2) holds. By hypothesis,

$$\lim_{y=+\infty} \int_{-\infty}^{+\infty} \frac{iy d\phi(u)}{iy - u} = c_0,$$

or, since c_0 is real,

$$\lim_{y=+\infty} \int_{-\infty}^{+\infty} \frac{d\phi(u)}{1 + (u/y)^2} = c_0,$$

and the integral increases as y increases. Consequently, if $a < b, 0 < t < y$:

$$\int_a^b \frac{d\phi(u)}{1 + (u/t)^2} \leq \int_a^b \frac{d\phi(u)}{1 + (u/y)^2} \leq \int_{-\infty}^{+\infty} \frac{d\phi(u)}{1 + (u/y)^2} \leq c_0.$$

If we let $y \to +\infty$, this gives

$$\int_a^b \frac{d\phi(u)}{1 + (u/t)^2} \leq \int_a^b d\phi(u) \leq c_0.$$

On letting $a \to -\infty$ and $b \to +\infty$, and subsequently letting $t \to +\infty$, we obtain (85.2) for the case $p = 0$.

Using induction, let us assume that (85.2) holds for $n = 2k$, $k \geq 0$, and we shall prove that it holds for $n = 2k + 1$ and $2k + 2$, and hence for all n. According to our assumption, we may write

$$f(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_{2k}}{z^{2k+1}} + \frac{1}{z^{2k+1}} \int_{-\infty}^{+\infty} \frac{u^{2k+1} d\phi(u)}{z - u},$$

or

$$z^{2k+1}\left(f(z) - \frac{c_0}{z} - \frac{c_1}{z^2} - \cdots - \frac{c_{2k}}{z^{2k+1}}\right) = \int_{-\infty}^{+\infty} \frac{u^{2k+1} d\phi(u)}{z - u}.$$

We denote this function by $f_1(z)$. Then it is evident that

$$\lim_{z = \infty} z^p \left(f_1(z) - \frac{c_{2k+1}}{z} - \frac{c_{2k+2}}{z^2} - \cdots - \frac{c_{2k+p}}{z^p}\right) = 0,$$

$$p = 1, 2, 3, \cdots,$$

where $z \to \infty$ along the positive imaginary axis. In particular,

$$\lim_{y = +\infty} iy[iyf_1(iy) - c_{2k+1}] = c_{2k+2},$$

or, since c_{2k+1} is real,

$$c_{2k+2} = \lim_{y = +\infty} - y^2 f_1(iy) = \lim_{y = +\infty} \int_{-\infty}^{+\infty} \frac{u^{2k+2} d\phi(u)}{1 + (u/y)^2}.$$

Using exactly the same argument as before, we then conclude that (85.2) holds for $p = 2k + 2$.

Inasmuch as $|u^{2k+1}| < u^{2k+2}$ for $|u| > 1$, we now see that the integral

$$\int_{-\infty}^{+\infty} u^{2k+1} d\phi(u)$$

is convergent. Then, using the preceding argument, we find that

$$\lim_{y=+\infty} \int_0^{+\infty} \frac{u^{2k+1}d\phi(u)}{1+(u/y)^2} = \int_0^{+\infty} u^{2k+1}d\phi(u),$$

$$\lim_{y=+\infty} \int_{-\infty}^0 \frac{-u^{2k+1}d\phi(u)}{1+(u/y)^2} = -\int_{-\infty}^0 u^{2k+1}d\phi(u),$$

so that, finally,

$$c_{2k+1} = \lim_{y=+\infty} iyf_1(iy) = \lim_{y=+\infty} \int_{-\infty}^{+\infty} \frac{u^{2k+1}d\phi(u)}{1+(u/y)^2} = \int_{-\infty}^{+\infty} u^{2k+1}d\phi(u),$$

and the theorem is proved.

We shall now supplement Theorem 85.1 as follows:

THEOREM 85.2. *If the function $f(z)$ of* (85.1) *satisfies* (85.3), *as $z \to \infty$ along the positive imaginary axis, then $f(z)$ satisfies* (85.3) *as $z \to \infty$ along any path in $\Im(z) \geq \delta$, for every positive δ.*

Proof. Under the hypothesis of the theorem, we know that (85.2) holds. Then, if $n \geq 2$ is *even*, we have, as in the proof of Theorem 85.1,

$$z^{n-1}\left(f(z) - \frac{c_0}{z} - \frac{c_1}{z^2} - \cdots - \frac{c_{n-2}}{z^{n-1}}\right) = \frac{c_{n-1}}{z} + \frac{1}{z}\int_{-\infty}^{+\infty} \frac{u^n}{z-u}\, d\phi(u),$$

so that if $\Im(z) \geq \delta > 0$, the modulus of this expression does not exceed

$$\frac{|c_{n-1}| + (c_n/\delta)}{|z|},$$

and must therefore approach 0 as $z \to \infty$ in the half-plane $\Im(z) \geq \delta$. It then readily follows that (85.3) holds as $z \to \infty$ in any such half-plane.

From Theorems 84.1, 84.4 and 85.1 we now have immediately:

THEOREM 85.3. *If* (85.2) *holds, where $P(1/z)$ is the power series expansion of a real J-fraction, then the function $f(z)$ given by* (85.1) *is an equivalent function of the J-fraction.*

Remark. We have defined "asymptotically equal" with reference to the half-planes $\Im(z) \geq \delta > 0$. Nevanlinna used instead the angular domains

$$V_c\colon c \leq \arg z \leq \pi - c, \quad 0 < c < \frac{\pi}{2},$$

and began by proving the following theorem. Let $f(z)$ be a function which for $\mathfrak{I}(z) > 0$ is analytic and has imaginary part $\mathfrak{I}(f(z)) \leq 0$. Let $f(z)$ be asymptotically equal to the real power series $P(1/z) = \Sigma(c_p/z^{p+1})$ in the sense that

$$\lim z^n \left(f(z) - \sum_{p=0}^{n-1} \frac{c_p}{z^{p+1}}\right) = 0, \quad n = 1, 2, 3, \cdots,$$

as $z \to \infty$ in every angular domain V_c. Then $P(1/z)$ is either equal to a terminating real J-fraction, or else is the power series expansion of an infinite real J-fraction. He then showed that the function $f(z)$ is necessarily asymptotically equal to $P(1/z)$ in every half-plane $\mathfrak{I}(z) \geq \delta > 0$. In our somewhat simpler approach to these problems, we have missed getting the important result that, for the functions and power series under consideration, asymptotic equality in the angular domains V_c is entirely equivalent to asymptotic equality in the half-planes $\mathfrak{I}(z) \geq \delta > 0$.

86. The Moment Problem for the Interval $(-\infty, +\infty)$. We are now in a position to prove the following theorem.

THEOREM 86.1. *Let $c_0 = 1, c_1, c_2, \cdots$ be a given sequence of real constants. There exists a bounded nondecreasing function $\phi(u)$, taking on infinitely many different values in the interval $-\infty < u < +\infty$, such that*

$$c_p = \int_{-\infty}^{+\infty} u^p d\phi(u), \quad p = 0, 1, 2, \cdots, \tag{86.1}$$

if, and only if, the determinants

$$\Delta_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ \cdot & \cdot & \cdot & \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix} > 0, \quad p = 0, 1, 2, \cdots. \quad [26.] \tag{86.2}$$

Proof. If a function $\phi(u)$ of the specified character exists, then we find, on applying Theorem 63.5 to the quadratic forms

$$\sum_{p,q=0}^{n} c_{p+q}x_px_q = \int_{-\infty}^{+\infty} (x_0 + x_1u + \cdots + x_nu^n)^2 d\phi(u), \quad n = 0, 1, 2, \cdots, \tag{86.3}$$

that (86.2) necessarily holds.

If, conversely, (86.2) holds, then we find by means of the formula (51.5) that the formal power series $P(1/z) = \Sigma(c_p/z^{p+1})$ has a *real* J-fraction expansion (84.1). Let $f(z)$ be any equivalent function of this J-fraction, and choose $\phi(u)$ in accordance with Theorem 66.1, so that (85.1) holds for this equivalent function. By Theorems 84.4 and 84.1, $f(z)$ is asymptotically equal to $P(1/z)$ in the half-plane $\mathfrak{J}(z) > 0$. Therefore, by Theorem 85.1, the function $\phi(u)$ satisfies (86.1). That $\phi(u)$ takes on infinitely many different values now follows from (86.1) and (86.2). In fact, if this were not the case, then the quadratic forms (86.3) would not be positive definite for large values of n, contrary to (86.2).

This completes the proof of Theorem 86.1.

We have shown that every equivalent function $f(z)$ of the J-fraction expansion of $P(1/z)$ yields a solution $\phi(u)$ of the moment problem (86.1), which is determined if we write $f(z)$ in the form (85.1). The function $\phi(u)$ can then be expressed in terms of $f(z)$ by means of the Stieltjes inversion formula (65.4). *There are no other solutions of the moment problem* (86.1). In fact, if $\phi(u)$ satisfies (86.1), then the function $f(z)$ given by (85.1) is asymptotically equal to $P(1/z)$, by Theorems 85.1 and 85.2, and therefore $f(z)$ is an equivalent function of the J-fraction expansion of $f(z)$, by virtue of Theorems 84.1 and 84.4. Since the J-fraction has just one equivalent function in the determinate case, and infinitely many in the indeterminate case, we therefore have the following theorem.

THEOREM 86.2. *Let $c_0 = 1, c_1, c_2, \cdots$ be a given sequence of real constants for which* (86.2) *holds, and let* (84.1) *be the J-fraction expansion of the formal power series $\Sigma(c_p/z^{p+1})$. The moment problem* (86.1) *has just one solution $\phi(u)$ such that $\phi(-\infty) = 0$, $\phi(u) = [\phi(u + 0) + \phi(u - 0)]/2$ for $-\infty < u < +\infty$, if the determinate case holds for this J-fraction, and has infinitely many different solutions satisfying these conditions if the indeterminate case holds* **[62]**.

The moment problem (86.1) is said to be **determinate** or **indeterminate** according as it has one or infinitely many solutions $\phi(u)$, respectively, normalized by the conditions $\phi(-\infty) = 0$, $\phi(u) = [\phi(u + 0) + \phi(u - 0)]/2$, $-\infty < u < +\infty$.

87. The Stieltjes Moment Problem. This is the problem of determining a function $\phi(u)$ which takes on infinitely many values in the interval $0 \leq u < +\infty$, is constant for $u < 0$, and satisfies the equations (86.1), where $c_0, c_1, c_2, \cdots$ are given real constants. Here the integrals have to be extended over only the positive half of the real axis. We shall prove the following theorem.

THEOREM 87.1. *Let $c_0, c_1, c_2, \cdots$ be a given sequence of real numbers. There exists a bounded nondecreasing function $\phi(u)$, taking on infinitely many values in the interval $0 \leq u < +\infty$, such that*

$$c_p = \int_0^{+\infty} u^p d\phi(u), \quad p = 0, 1, 2, \cdots, \tag{87.1}$$

if, and only if, the numbers c_p satisfy the condition (86.2) *and, in addition, the condition*

$$\Omega_p = \begin{vmatrix} c_1, & c_2, & \cdots, & c_{p+1} \\ c_2, & c_3, & \cdots, & c_{p+2} \\ & \cdot & \cdot & \cdot \\ c_{p+1}, & c_{p+2}, & \cdots, & c_{2p+1} \end{vmatrix} > 0, \quad p = 0, 1, 2, \cdots. \quad [95.] \tag{87.2}$$

Proof. Since the moment problem (87.1) is a special case of the moment problem (86.1), it is clear that the condition (86.2) is necessary in order for a solution to exist. On applying Theorem 63.5 to the quadratic forms

$$\sum_{p,q=0}^{n} c_{p+q+1} x_p x_q = \int_0^{\infty} u(x_0 + x_1 u + \cdots + x_n u^n)^2 d\phi(u),$$

$$n = 0, 1, 2, \cdots,$$

we conclude that the condition (87.2) is likewise necessary.

On referring to Theorem 52.1 and the remark following that theorem, we see that the necessary conditions (86.2) and (87.2) are together equivalent to the condition that the formal power series $\Sigma(c_p/z^{p+1})$ have an S-fraction expansion (52.4) in which the a_p are all positive. We shall prove that the latter condition is sufficient for the Stieltjes moment problem to have a solution.

We suppose, then, that $P(1/z) = \Sigma(c_p/z^{p+1})$ has an S-fraction expansion

$$\cfrac{a_0}{z - \cfrac{a_1}{1 - \cfrac{a_2}{z - \cdot_{\cdot_{\cdot}}}}}, \tag{87.3}$$

in which $a_p > 0$, $p = 0, 1, 2, \cdots$. On replacing z by ζ^2 and then multiplying by ζ, we find by means of an equivalence transformation that this goes over into the real J-fraction

$$\cfrac{a_0}{\zeta - \cfrac{a_1}{\zeta - \cfrac{a_2}{\zeta - \cdot_{\cdot_{\cdot}}}}}, \tag{87.4}$$

and the power series expansion goes over into $\zeta P(1/\zeta^2)$. Thus, the moments of the J-fraction (87.4) are $c_0, 0, c_1, 0, c_2, 0, \cdots$. By what we have proved in § 86, a function $\theta(u)$ which is bounded, nondecreasing, and which takes on infinitely many different values, satisfies the equations

$$0 = \int_{-\infty}^{+\infty} u^{2p+1} d\theta(u), \quad c_p = \int_{-\infty}^{+\infty} u^{2p} d\theta(u), \qquad p = 0, 1, 2, \cdots, \tag{87.5}$$

if, and only if,

$$f(\zeta) = \int_{-\infty}^{+\infty} \frac{d\theta(u)}{\zeta - u}$$

is an equivalent function of the J-fraction (87.4). Now,

$$\begin{aligned} c_p &= \int_{-\infty}^{+\infty} u^{2p} d\theta(u) = \int_{-\infty}^{0} u^{2p} d\theta(u) + \int_{0}^{+\infty} u^{2p} d\theta(u) \\ &= \int_{0}^{+\infty} u^{2p} d[-\theta(-u)] + \int_{0}^{+\infty} u^{2p} d\theta(u) \\ &= \int_{0}^{+\infty} u^{p} d\phi(u), \quad p = 0, 1, 2, \cdots, \end{aligned}$$

where

$$\phi(u) = \theta(\sqrt{u}) - \theta(-\sqrt{u}), \tag{87.6}$$

and therefore the Stieltjes moment problem has a solution.

This completes the proof of Theorem 87.1.

It is at once evident that *all* solutions of the Stieltjes moment problem are of the form (87.6), where $\theta(u)$ is a solution of the moment problem (87.5). Therefore, the Stieltjes moment problem is determinate or indeterminate, i.e., has just one or infinitely many solutions $\phi(u)$, respectively, normalized by the conditions $\phi(0) = 0$, $\phi(u) = [\phi(u+0) + \phi(u-0)]/2$, $u > 0$, according as the determinate case or the indeterminate case, respectively, holds for the J-fraction (87.4). On applying Definition 22.1 we then find, after making the necessary calculations, that the following theorem is true.

THEOREM 87.2. *Let $c_0, c_1, c_2, \cdots$ be numbers satisfying the conditions* (86.2) *and* (87.2). *Let the S-fraction expansion of the power series $\Sigma(c_p/z^{p+1})$ be written in the form*

$$\cfrac{1}{k_1 z - \cfrac{1}{k_2 - \cfrac{1}{k_3 z - \cfrac{1}{k_4 - \ddots}}}} \tag{87.7}$$

The Stieltjes moment problem (87.1) *is determinate if, and only if, the positive term series Σk_p is divergent* **[95]**.

We now consider the following question.

Let $c_0, c_1, c_2, \cdots$ be numbers satisfying the conditions (86.2) and (87.2), so that the Stieltjes moment problem (87.1) has a solution. Does there exist a function $\psi(u)$ which is bounded, nondecreasing, and which takes on infinitely many different values, but which is *not* constant for $u < 0$, such that

$$c_p = \int_{-\infty}^{+\infty} u^p d\psi(u), \quad p = 0, 1, 2, \cdots ? \tag{87.8}$$

The answer to this question is affirmative in case both the series Σk_{2p+1} and $\Sigma k_{2p+1}(k_2 + k_4 + \cdots + k_{2p})^2$ are convergent and the series Σk_{2p} is divergent, in (87.7). In fact, the Stieltjes moment problem is then determinate by Theorem 87.2, whereas the indeterminate case holds for the J-fraction which is the even part of (87.7) (cf. (29.13)). Therefore, the moment problem (87.8) has infinitely many different normalized solutions. Only one of these is constant for $u < 0$. **[26.]**

88. A Theorem of Carleman.[28] A number of criteria have been found which furnish sufficient conditions, in terms of the given constants c_p, for the moment problems (86.1) and (87.1) to be determinate. One of the simplest and most comprehensive of these is given by the following theorem of T. Carleman [6, 7].

THEOREM 88.1. (a) *Let* $c_0, c_1, c_2, \cdots$ *be a sequence of real constants for which the inequalities* (86.2) *hold, so that the moment problem* (86.1) *has a solution. This moment problem is determinate if the series*

$$\Sigma\left(\frac{1}{c_{2p}}\right)^{1/2p} \tag{88.1}$$

is divergent.

(b) *Let* $c_0, c_1, c_2, \cdots$ *be a sequence of real constants for which the inequalities* (86.2) *and* (87.2) *hold, so that the moment problem* (87.1) *has a solution. This moment problem is determinate if the series*

$$\Sigma\left(\frac{1}{c_p}\right)^{1/2p} \tag{88.2}$$

is divergent.

Remark. Since the moment problem (87.1) is a special case of the moment problem (86.1), namely (87.5), we conclude immediately that (b) is a consequence of (a). Carleman proved this theorem by means of methods of function theory. He also gave an independent algebraic proof of (b). We shall give here an algebraic proof of (a), based upon two lemmas and upon Theorems 53.1 and 25.1.

Lemma 88.1 (Carleman's inequality). Let $\sum_{p=1}^{\infty} u_p$ be a convergent series of nonnegative numbers whose sum is positive.

[28] Carleman [7] used the theorem of this section to establish fundamental theorems on quasi-analytic functions.

Then

$$\sum_{p=1}^{\infty} \sqrt[p]{u_1 u_2 \cdots u_p} < e \sum_{p=1}^{\infty} u_p, \tag{88.3}$$

where e, the base of natural logarithms, cannot be replaced by a smaller constant if Σu_p is arbitrary [7].

Proof. Let $v_1, v_2, v_3, \cdots$ be a sequence of positive numbers to be determined. Then

$$\sum_{p=1}^{n} \sqrt[p]{u_1 u_2 \cdots u_p} = \sum_{p=1}^{n} \sqrt[p]{\frac{u_1 v_1 \cdot u_2 v_2 \cdot \cdots \cdot u_p v_p}{v_1 v_2 \cdots v_p}}$$

$$\leq \sum_{p=1}^{n} \frac{1}{\sqrt[p]{v_1 v_2 \cdots v_p}} \cdot \frac{u_1 v_1 + u_2 v_2 + \cdots + u_p v_p}{p}$$

$$= \sum_{r=1}^{n} u_r v_r \sum_{p=r}^{n} \frac{1}{(p \sqrt[p]{v_1 v_2 \cdots v_p})}.$$

We now take $\sqrt[p]{v_1 v_2 \cdots v_p} = p + 1$, or $v_p = p\left(1 + \frac{1}{p}\right)^p$, so that

$$\sum_{p=r}^{n} \frac{1}{p \cdot \sqrt[p]{v_1 v_2 \cdots v_p}} = \sum_{p=r}^{n} \frac{1}{p(p+1)} = \frac{1}{r} - \frac{1}{n+1} < \frac{1}{r}.$$

We therefore have

$$\sum_{p=1}^{n} \sqrt[p]{u_1 u_2 \cdots u_p} \leq \sum_{r=1}^{n} u_r \left(1 + \frac{1}{r}\right)^r.$$

Since $[1 + (1/r)]^r < e$, we then obtain (88.3) on letting $n \to \infty$.

The proof that e is the best constant may be accomplished by taking $u_p = 1/p$ for $p \leq n$, $u_p = 0$ for $p > n$. On taking n sufficiently large, one may then make the quotient

$$\frac{\Sigma \sqrt[p]{u_1 u_2 \cdots u_p}}{\Sigma u_p}$$

as nearly equal to e as desired.

Lemma 88.2. For $n = 1, 2, 3, \cdots$, let $F_n = F_n(x_1, x_2, \cdots, x_n)$ denote the nth segment of the real quadratic form

$$\sum_{p,q=1}^{\infty} a_{pq}x_px_q, \quad a_{pq} = a_{qp}.$$

Then $F_n(x_1, x_2, \cdots, x_n) \geq 0$ for all real values of $x_1, x_2, \cdots, x_n$ and for $n = 1, 2, 3, \cdots$, if, and only if, there exists a transformation $F_n = \sum_{p=1}^{n} X_p^2$,

$$\begin{aligned}
X_1 &= b_{11}\beta_1x_1 + b_{12}\beta_2x_2 + b_{13}\beta_3x_3 + \cdots + b_{1n}\beta_nx_n, \\
X_2 &= \qquad\qquad\quad b_{22}\beta_2x_2 + b_{23}\beta_3x_3 + \cdots + b_{2n}\beta_nx_n, \\
X_3 &= \qquad\qquad\qquad\qquad\quad b_{33}\beta_3x_3 + \cdots + b_{3n}\beta_nx_n, \\
&\qquad \cdot \quad \cdot \quad \cdot \\
X_n &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad b_{nn}\beta_nx_n,
\end{aligned}$$

where $\beta_q^2 = a_{qq} \geq 0$, and b_{pq}, $1 \leq p \leq q \leq n$, are real numbers independent of n such that $\sum_{p=1}^{q} b_{pq}^2 = 1$. If the segments F_n are all positive definite, then $b_{qq}^2\beta_q^2 > 0$, $q = 1, 2, 3, \cdots$.

Proof. Since $F_n(0, 0, \cdots, 0, x_q, 0, 0, \cdots, 0) = a_{qq}x_q^2 \geq 0$, it follows that $a_{qq} = \beta_q^2 \geq 0$. Since $F_n(x_1, 0, 0, \cdots, 0, x_p, 0, 0, \cdots, 0) \geq 0$, it follows that $a_{1p}^2 \leq a_{11}a_{pp} = \beta_1^2\beta_p^2$. Hence, there exists a number s_{1p} such that $0 \leq s_{1p}^2 \leq 1$, $s_{1p} = 0$ if $\beta_1 = 0$, and such that $a_{1p} = s_{1p}\beta_1\beta_p$, $(p = 2, 3, 4, \cdots)$. We now put $c_{1p}^2 = 1 - s_{1p}^2$,

$$X_1 = \beta_1x_1 + s_{12}\beta_2x_2 + s_{13}\beta_3x_3 + \cdots + s_{1n}\beta_nx_n,$$

so that

$$\begin{aligned}
F_n - X_1^2 = c_{12}^2\beta_2^2x_2^2 &+ 2a_{23}'x_2x_3 + 2a_{24}'x_2x_4 + \cdots + 2a_{2n}'x_2x_n \\
&+ c_{13}^2\beta_3^2x_3^2 + 2a_{34}'x_3x_4 + \cdots + 2a_{3n}'x_3x_n \\
&+ c_{14}^2\beta_4^2x_4^2 + \cdots + 2a_{4n}'x_4x_n \\
&+ \cdots \\
&+ c_{1n}^2\beta_n^2x_n^2,
\end{aligned}$$

where the a_{pq}' are certain constants. Since the right-hand member is a quadratic form in $x_2, x_3, \cdots, x_n$, which may be regarded as a value of $F_n(x_1, x_2, \cdots, x_n)$, it is nonnegative. Consequently, we conclude as before that

$$a_{2p}'^2 \leq c_{12}^2\beta_2^2 c_{1p}^2\beta_p^2 \quad \text{or} \quad a_{2p}' = s_{2p}c_{12}\beta_2 c_{1p}\beta_p,$$

$$0 \leq s_{2p}^2 \leq 1, \quad s_{2p} = 0 \quad \text{if} \quad c_{12}\beta_2 = 0, \quad (p = 3, 4, \cdots, n).$$

On putting $c_{2p}^2 = 1 - s_{2p}^2$,

$$X_2 = c_{12}\beta_2 x_2 + s_{23}c_{13}\beta_3 x_3 + s_{24}c_{14}\beta_4 x_4 + \cdots + s_{2n}c_{1n}\beta_n x_n,$$

we then find that $F_n - X_1^2 - X_2^2$ is a quadratic form in $x_3, x_4, \cdots, x_n$, which may be regarded as a value of $F_n(x_1, x_2, \cdots, x_n)$, in which the coefficients of $x_3^2, x_4^2, \cdots, x_n^2$ are $c_{13}^2c_{23}^2\beta_3^2$, $c_{14}^2c_{24}^2\beta_4^2, \cdots, c_{1n}^2c_{2n}^2\beta_n^2$, respectively. Continuing in this manner, we finally obtain $F_n - X_1^2 - X_2^2 - \cdots - X_n^2 \equiv 0$, where $X_1, X_2, \cdots, X_n$ have the properties specified in the lemma.

Remark. Theorem 16.2 is the special case of Lemma 88.2 in which it is required that $a_{pq} = 0$ for $|p - q| > 1$.

Proof of Theorem 88.1. It has already been noted that part (b) of the theorem is a consequence of part (a). Under the hypothesis of part (a), all the segments of the quadratic form $F = \Sigma c_{p+q}x_p x_q$, $p, q = 0, 1, 2, \cdots$, are positive definite, and therefore, by Lemma 88.2,

$$\begin{aligned} F = {} & (b_{11}\sqrt{c_0}x_0 + b_{12}\sqrt{c_2}x_1 + b_{13}\sqrt{c_4}x_2 + \cdots)^2 \\ & + (b_{22}\sqrt{c_2}x_1 + b_{23}\sqrt{c_4}x_2 + \cdots)^2 + (b_{33}\sqrt{c_4}x_2 + \cdots)^2 \\ & + \cdots, \end{aligned}$$

where $b_{11}^2 = 1, b_{12}^2 + b_{22}^2 = 1, b_{13}^2 + b_{23}^2 + b_{33}^2 = 1, \cdots$. By Theorem 53.1 we then conclude that

$$a_0a_1 \cdots a_p = b_{p+1,p+1}^2 c_{2p} < c_{2p}, \quad p = 0, 1, 2, \cdots,$$

where $a_0, -a_1, -a_2, \cdots$ are the partial numerators of the J-fraction expansion of the power series $\Sigma(c_p/z^{p+1})$. By Lemma 88.1, it then follows that

$$e\sum_{p=1}^{n}\frac{1}{\sqrt{a_p}} > \sum_{p=1}^{n}\left(\frac{1}{a_1a_2\cdots a_p}\right)^{1/2p} > \sum_{p=1}^{n}\left(\frac{c_0}{c_{2p}}\right)^{1/2p},$$

so that the divergence of the series (88.1) implies the divergence of the series $\Sigma(1/a_p)^{1/2}$, and therefore, by Theorem 25.1, the determinacy of the moment problem.

Exercise 17

17.1. Let $c_0, c_1, c_2, \cdots$ be the moments of an infinite positive definite J-fraction, and let $f(z) = \int_{-\infty}^{+\infty} d\phi(u)/(z - u)$ be an equivalent function of the J-fraction. Then

$$c_p = \lim_{r=+0} \int_{-\infty}^{+\infty} \left(\frac{u}{1 + iru}\right)^p d\phi(u), \quad p = 0, 1, 2, \cdots.$$

Suggestion. Use a theorem of F. Nevanlinna [**61**] on differentiation of asymptotic series.

17.2. Let $\Sigma(-1)^p c_p z^p$ be the power series expansion of the S-fraction

$$\cfrac{1}{k_1 + \cfrac{z}{k_2 + \cfrac{z}{k_3 + \cdots}}},$$

where the k_p are positive. Show that $(c_{n+1}/c_n) > (c_n/c_{n-1})$, $n = 1, 2, 3, \cdots$, and that $\lim (c_{n+1}/c_n)$ is finite if, and only if, $1/k_n k_{n+1} < M$, where M is a finite number independent of n [**95**].

17.3. Let $\phi(u)$ be any solution of the Stieltjes moment problem. If x is real and positive, then

$$\int_0^\infty \frac{d\phi(u)}{x + u}$$

is greater than every even approximant and less than every odd approximant of the corresponding S-fraction [**95**].

17.4. Let $F = \Sigma(x_p^2 + 2a_p^{(1)}x_p x_{p+1} + 2a_p^{(2)}x_p x_{p+2})$ be a real J-form with 5 diagonals, which is never negative. Show that if $|a_p^{(2)}| \geq \frac{1}{2}$, $p = 1, 2, 3, \cdots$, then $\lim_{p=\infty} |a_p^{(2)}| = \frac{1}{2}$, and $\lim_{p=\infty} a_p^{(1)} = 0$.

Suggestion. Apply Lemma 88.2 and the theory of chain sequences.

Chapter XVIII

THE CONTINUED FRACTION OF GAUSS

This is the S-fraction expansion for the quotient $F(a, b+1, c+1; z)/F(a, b, c; z)$ of two hypergeometric series. We derive here the expansion, establish its convergence to the function expanded, and treat in considerable detail a number of special cases and limiting cases.

89. General Properties. We consider the **hypergeometric series**

$$F(a, b, c; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} z^3 + \cdots, \quad (89.1)$$

where a and b are any complex constants, and c is a complex constant different from $0, -1, -2, -3, \cdots$. If a or b is 0 or a negative integer, $F(a, b, c; z)$ reduces to a polynomial. Otherwise, it is an infinite series with radius of convergence equal to unity. For special values of the parameters, $F(a, b, c; z)$ reduces to elementary functions in a number of cases. For example,

$$\begin{aligned} F(1, 1, 2; -z) &= \frac{1}{z} \log (1 + z), \\ F(-k, 1, 1; -z) &= (1 + z)^k, \\ zF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; z^2) &= \text{arc sin } z, \\ zF(\tfrac{1}{2}, 1, \tfrac{3}{2}; -z^2) &= \text{arc tan } z. \end{aligned} \quad (89.2)$$

On replacing z by z/a and then letting a tend to ∞, the hypergeometric series becomes:

$$\Phi(b, c; z) = 1 + \frac{b}{c}z + \frac{b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots. \quad (89.3)$$

Similarly, on replacing z by z/b and letting b tend to ∞ in (89.3) we get:

$$\Psi(c; z) = 1 + \frac{1}{c}z + \frac{1}{c(c+1)}\frac{z^2}{2!} + \frac{1}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots; \quad (89.4)$$

and if we replace z by cz in (89.1) and let c tend to ∞ we obtain:

$$\Omega(a, b; z) = 1 + abz + a(a+1)b(b+1)\frac{z^2}{2!} + a(a+1)(a+2)b(b+1)(b+2)\frac{z^3}{3!} + \cdots. \quad (89.5)$$

Excepting when a or b is a negative integer or 0, the last series has zero radius of convergence.

To obtain the continued fraction of Gauss, we start with the relation

$$F(a, b, c; z) = F(a, b+1, c+1; z) - \frac{a(c-b)}{c(c+1)}zF(a+1, b+1, c+2; z). \quad (89.6)$$

By comparing coefficients of corresponding powers of z in the two members, one may readily verify that this is a power series identity. The relation (89.6) may be written in the form

$$\frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} = \cfrac{1}{1 - \cfrac{a(c-b)}{c(c+1)}z\cfrac{F(a+1, b+1, c+2; z)}{F(a, b+1, c+1; z)}}. \quad (89.7)$$

We now interchange a and b in (89.7), and afterwards replace b by $b + 1$ and c by $c + 1$. This gives

$$\frac{F(a+1, b+1, c+2; z)}{F(a, b+1, c+1; z)} = \frac{1}{1 - \dfrac{(b+1)(c-a+1)}{(c+1)(c+2)} z \dfrac{F(a+1, b+2, c+3; z)}{F(a+1, b+1, c+2; z)}}. \tag{89.8}$$

The quotient in the left-hand member of (89.8) is the same as the quotient of hypergeometric series appearing in the denominator of the right-hand member of (89.7). Also, if a, b, c are replaced by $a + 1$, $b + 1$, $c + 2$, respectively, in (89.7), the quotient in the left-hand member becomes equal to the quotient of hypergeometric series appearing in the denominator of the right-hand member of (89.8). On applying first one identity and then the other, we obtain by successive substitution the **continued fraction of Gauss** [20]:

$$\frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} = \cfrac{1}{1 - \cfrac{\dfrac{a(c-b)}{c(c+1)} z}{1 - \cfrac{\dfrac{(b+1)(c-a+1)}{(c+1)(c+2)} z}{1 - \cfrac{\dfrac{(a+1)(c-b+1)}{(c+2)(c+3)} z}{1 - \cfrac{\dfrac{(b+2)(c-a+2)}{(c+3)(c+4)} z}{1 - \cfrac{\dfrac{(a+2)(c-b+2)}{(c+4)(c+5)} z}{1 - \cdots}}}}}} \tag{89.9}$$

This is to be regarded, for the present, as a formal expansion. If we let $a_1, a_2, a_3, \cdots$ denote the coefficients of $-z$ in the partial numerators, so that

$$a_{2p+1} = \frac{(a+p)(c-b+p)}{(c+2p)(c+2p+1)},$$
$$a_{2p+2} = \frac{(b+p+1)(c-a+p+1)}{(c+2p+1)(c+2p+2)}, \tag{89.10}$$
$$p = 0, 1, 2, \cdots,$$

and if we put

$$P_{2n}(z) = \frac{F(a+n, b+n+1, c+2n+1; z)}{F(z+n, b+n, c+2n; z)},$$
$$P_{2n+1}(z) = \frac{F(a+n+1, b+n+1, c+2n+2; z)}{F(a+n, b+n+1, c+2n+1; z)}, \tag{89.11}$$
$$n = 0, 1, 2, \cdots,$$

then we have the identities

$$P_{n-1}(z) = \frac{1}{1 - a_n z P_n(z)}, \quad n = 1, 2, 3, \cdots. \tag{89.12}$$

Hence, for every n,

$$\frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} = \cfrac{1}{1 - \cfrac{a_1 z}{1 - \cfrac{a_2 z}{1 - \cdots - \cfrac{a_{n-1} z}{1 - a_n z P_n(z)}}}}. \tag{89.13}$$

If $a_1, a_2, \cdots, a_{n-1}$ are different from zero, while $a_n = 0$, then the continued fraction of Gauss terminates, and the quotient $F(a, b+1, c+1; z)/F(a, b, c; z)$ is a rational function of z, which is equal to the terminating continued fraction. If, on the other hand, $a_p \neq 0$, $p = 1, 2, 3, \cdots$, we may write (89.13) in the form

$$\frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} = \frac{A_n(z) - a_n z P_n(z) A_{n-1}(z)}{B_n(z) - a_n z P_n(z) B_{n-1}(z)},$$

where $A_p(z)$ and $B_p(z)$ are the pth numerator and denominator of the continued fraction of Gauss, and therefore

$$\frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} - \frac{A_{n-1}(z)}{B_{n-1}(z)} = \frac{a_1 a_2 \cdots a_{n-1} z^{n-1}}{B_{n-1}(z)[B_n(z) - a_n z B_{n-1}(z) P_n(z)]}.$$

This shows immediately that the power series in ascending powers of z for $A_{n-1}(z)/B_{n-1}(z)$ agrees with the power series $F(a, b+1, c+1; z)/F(a, b, c; z)$ term by term for the first n terms. Therefore, the power series for the hypergeometric quotient is the power series expansion of the continued fraction of Gauss.

THEOREM 89.1. *The continued fraction of Gauss converges throughout the z-plane exterior to the cut along the real axis from* 1 *to* $+\infty$, *excepting possibly at certain isolated points, is equal to the function* $F(a, b+1, c+1; z)/F(a, b, c; z)$ *in the neighborhood of the origin, and furnishes the analytic continuation of this function into the interior of the cut plane. The continued fraction converges uniformly over every bounded closed region exterior to the cut, which contains none of the above-mentioned isolated points. These points, if they exist, are poles of the function represented by the continued fraction* [**101, 77, 111**].

Proof. Since

$$\lim_{p=\infty} a_p = +\tfrac{1}{4},$$

the theorem follows immediately by Theorems 54.1 and 54.2.

We note that if

$$g_{2p} = \frac{c-a+p}{c+2p}, \quad g_{2p+1} = \frac{c-b+p}{c+2p+1}, \qquad p = 0, 1, 2, \cdots, \tag{89.14}$$

then the continued fraction of Gauss can be written in the form

$$\frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} = \cfrac{1}{1 - \cfrac{(1-g_0)g_1 z}{1 - \cfrac{(1-g_1)g_2 z}{1 - \cfrac{(1-g_2)g_3 z}{1 - \cdots}}}} \tag{89.15}$$

If we put $b = 0$ and replace c by $c - 1$, then the continued fraction of Gauss reduces to

$$F(a, 1, c; z) = \cfrac{1}{1 - \cfrac{\frac{a}{c} z}{1 - \cfrac{\frac{(c-a)}{c(c+1)} z}{1 - \cfrac{\frac{c(a+1)}{(c+1)(c+2)} z}{1 - \cfrac{\frac{2(c-a+1)}{(c+2)(c+3)} z}{1 - \cfrac{\frac{(c+1)(a+2)}{(c+3)(c+4)} z}{1 - \cdots}}}}}} \tag{89.16}$$

If we denote by $b_1, b_2, b_3, \cdots$ the coefficients of $-z$ in the partial numerators, then

$$b_{2p+1} = \frac{(a+p)(c+p-1)}{(c+2p-1)(c+2p)}, \quad b_{2p+2} = \frac{(p+1)(c-a+p)}{(c+2p)(c+2p+1)},$$
$$p = 0, 1, 2, \cdots. \tag{89.17}$$

If

$$h_{2p-1} = \frac{a+p-1}{c+2p-2}, \quad h_{2p} = \frac{p}{c+2p-1},$$
$$p = 1, 2, 3, \cdots, \tag{89.18}$$

then (89.16) may be written in the form

$$F(a, 1, c; z) = \cfrac{1}{1 - \cfrac{h_1 z}{1 - \cfrac{(1-h_1)h_2 z}{1 - \cfrac{(1-h_2)h_3 z}{1 - \cdots}}}} \tag{89.19}$$

One may verify that the denominators of (89.16) are the hypergeometric polynomials

$$\begin{aligned} B_{2n}(z) &= F(-n, 1 - a - n, 2 - c - 2n; z), \\ B_{2n+1} &= F(-n, -a - n, 1 - c - 2n; z). \end{aligned} \tag{89.20}$$

In case the parameters a, b and c in the continued fraction of Gauss are *real*, we note that the numbers (89.14) will satisfy the inequalities $0 < g_p < 1$, for $p \geq N$, provided N is sufficiently large. Hence, it follows from Theorems 27.5 and 69.1 that if $n > N$, the continued fraction

$$\cfrac{1}{1 - \cfrac{a_n z}{1 - \cfrac{a_{n+1} z}{1 - \cfrac{a_{n+2} z}{1 - \cdots}}}}, \tag{89.21}$$

where the a_p are given by (89.10), converges uniformly over every bounded closed region of the complex plane exterior to the cut along the real axis from $+1$ to $+\infty$, and is equal to a function of the form

$$\int_0^1 \frac{d\phi(u)}{1 - zu},$$

where $\phi(u)$ is bounded and nondecreasing. The number N depends only upon a, b and c, which is in contrast with the assertion made in Theorem 33.1, where the number N depends upon the region G.

If $0 < g_p < 1, p = 0, 1, 2, \cdots$, we may apply Theorem 11.1 and obtain the inequality:

$$\left| \frac{c - a}{c} \cdot \frac{F(a, b + 1, c + 1; z)}{F(z, b, c; z)} - \frac{c}{c + a} \right| \leq \frac{a}{c + a} \tag{89.22}$$

for

$$|z| < 1.$$

In particular, if $a = 1$, $b = 1$, $c = 2$, we have, by (89.2),

$$|\log(1+z) - \tfrac{4}{3}z| \leq \tfrac{2}{3}|z| \quad \text{for} \quad |z| \leq 1. \tag{89.23}$$

With the aid of (75.6) and (75.7), other inequalities can be obtained. We shall list some of these without proof.

$$\frac{|z|}{1+|z|} \leq |\log(1+z)| \leq |z|\frac{|1+z|}{1+|z|}, \quad \text{if} \quad |z| \leq 1; \tag{89.24}$$

$$\left|F(a, 1, 1+c; -z) - \frac{c}{2c-a}\right| \leq \frac{c}{2c-a}$$
if
$$\Re(z) \geq -\tfrac{1}{2}; \tag{89.25}$$

$$\left|\frac{1}{F(a, 1, c; -z)} - \frac{2c-a+cz}{2c-a}\right| \leq \frac{c-a}{2c-a}|z|$$
if
$$|z| \leq 1; \tag{89.26}$$

$$\left|\frac{1}{F(a, 1, c; -z)} - \frac{c+a+az}{c+a}\right| \leq \frac{a}{c+a}|z|$$
if
$$\Re(z) \geq -\tfrac{1}{2}. \tag{89.27}$$

These hold provided a, b and c are real, and the numbers (89.14) or (89.18), as the case may be, are between 0 and 1. [**127.**]

90. Elementary Functions. From (89.2) and (89.16) we obtain the following three expansions.

$$\log(1+z) = \cfrac{z}{1 + \cfrac{1^2z}{2 + \cfrac{1^2z}{3 + \cfrac{2^2z}{4 + \cfrac{2^2z}{5 + \cfrac{3^2z}{6 + \cdots}}}}}}, \quad [\mathbf{46, 42}], \tag{90.1}$$

$$(1+z)^k = \cfrac{1}{1 - \cfrac{kz}{1 + \cfrac{\frac{1\cdot(1+k)}{1\cdot 2}z}{1 + \cfrac{\frac{1\cdot(1-k)}{2\cdot 3}z}{1 + \cfrac{\frac{2(2+k)}{3\cdot 4}z}{1 + \cfrac{\frac{2(2-k)}{4\cdot 5}}{1 + \cfrac{\frac{3(3+k)}{5\cdot 6}z}{1 + \cdots}}}}}}}, \quad [42], \tag{90.2}$$

$$\arctan z = \cfrac{z}{1 + \cfrac{1\cdot z^2}{3 + \cfrac{4z^2}{5 + \cfrac{9z^2}{7 + \cfrac{16z^2}{9 + \cdots}}}}}. \quad [46, 42]. \tag{90.3}$$

The first two are valid exterior to the cut along the real axis from -1 to $-\infty$. The third is valid exterior to the cuts along the imaginary axis from i to $i\cdot\infty$ and from $-i$ to $-i\cdot\infty$.

Since

$$2zF\left(\frac{1}{2}, 1, \frac{3}{2}; z^2\right) = \log\frac{1+z}{1-z},$$

we have

$$\log\frac{1+z}{1-z} = \cfrac{2z}{1 - \cfrac{1\cdot z^2}{3 - \cfrac{4z^2}{5 - \cfrac{9z^2}{7 - \cdots}}}}, \quad [12], \tag{90.4}$$

valid exterior to the cuts along the real axis from -1 to $-\infty$ and from $+1$ to $+\infty$.

On replacing z by $1/z$ in (90.4), and making an equivalence transformation, we obtain

$$\log \frac{z+1}{z-1} = \cfrac{2}{z - \cfrac{\frac{1}{2}}{\frac{3}{2}z - \cfrac{\frac{2}{3}}{\frac{5}{3}z - \cfrac{\frac{3}{4}}{\frac{7}{4}z - \cfrac{\frac{4}{5}}{\frac{9}{5}z - \cdots}}}}} \tag{90.5}$$

This is valid exterior to the real interval $(-1, +1)$. This is the J-fraction whose denominators are the **Legendre polynomials.** In fact, we have:

$$\log \frac{z+1}{z-1} = \int_{-1}^{+1} \frac{du}{z-u}, \tag{90.6}$$

so that the formal integral introduced in § 50 is the actual integral

$$\int_{-1}^{+1} u^p du.$$

The Legendre polynomials are, by definition, the polynomials, suitably normalized, which are orthogonal on the interval $(-1, +1)$, and must therefore be, except possibly for a constant factor, the denominators of the J-fraction for the function (90.6). They are normalized so that they satisfy the recurrence formulas

$$L_0 = 1, \quad L_1 = z, \quad L_p = \frac{2p-1}{p} z L_{p-1} - \frac{p-1}{p} L_{p-2}, \qquad p = 2, 3, 4, \cdots. \tag{90.7}$$

As a generalization of (90.1) and (90.3) we have

$$\int_0^z \frac{dt}{1+t^n} = zF\left(\frac{1}{n}, 1, 1+\frac{1}{n}; -z^n\right) \tag{90.8}$$

$$= \cfrac{z}{1+\cfrac{1^2z^n}{n+1+\cfrac{n^2z^n}{2n+1+\cfrac{(n+1)^2z^n}{3n+1+\cfrac{(2n)^2z^n}{4n+1+\cfrac{(2n+1)^2z^n}{5n+1+\cfrac{(3n)^2z^n}{6n+1+\cdots}}}}}}}, \quad [42],$$

valid exterior to the cut along the real axis from -1 to $-\infty$ *in the plane of* z^n $(n = 1, 2, 3, \cdots)$.

From (89.9) we obtain the expansions

$$\frac{\arcsin z}{\sqrt{1-z^2}} = \frac{zF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2)}{F(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}; z^2)}$$

$$= \cfrac{z}{1-\cfrac{1\cdot 2z^2}{3-\cfrac{1\cdot 2z^2}{5-\cfrac{3\cdot 4z^2}{7-\cfrac{3\cdot 4z^2}{9-\cfrac{5\cdot 6z^2}{11-\cfrac{5\cdot 6z^2}{13-\cdots}}}}}}}; \tag{90.9}$$

and

$$\frac{(1+z)^k-(1-z)^k}{(1+z)^k+(1-z)^k} = kz\cdot\frac{F\left(\frac{1-k}{2},\frac{2-k}{2},\frac{3}{2};z^2\right)}{F\left(\frac{1-k}{2},-\frac{k}{2},\frac{1}{2};z^2\right)} \tag{90.10}$$

$$= \cfrac{kz}{1+\cfrac{(k^2-1)z^2}{3+\cfrac{(k^2-4)z^2}{5+\cfrac{(k^2-9)z^2}{7+\cdots}}}} \quad [12.]$$

These are both valid exterior to the cuts along the real axis from -1 to $-\infty$ and from $+1$ to $+\infty$.

From (90.10) one may derive the expansions

$$\left(\frac{z+1}{z-1}\right)^k - 1 = \cfrac{2k}{z-k-\cfrac{1-k^2}{3z-\cfrac{4-k^2}{5z-\cfrac{9-k^2}{7z-\cdots}}}}, \quad [44], \tag{90.11}$$

$$e^{2k\cdot\arctan\frac{1}{z}} = 1+\cfrac{2k}{z-k+\cfrac{1+k^2}{3z+\cfrac{4+k^2}{5z+\cfrac{9+k^2}{7z+\cdots}}}}, \quad [45], \tag{90.12}$$

and

$$\tan k\phi = \cfrac{k\tan\phi}{1-\cfrac{(k^2-1)\tan^2\phi}{3-\cfrac{(k^2-4)\tan^2\phi}{5-\cfrac{(k^2-9)\tan^2\phi}{7-\cdots}}}} \quad [12.] \tag{90.13}$$

The expansion (90.11) is valid for z exterior to the real interval $(-1, +1)$; (90.12) is valid outside the interval on the imaginary axis from $-i$ to $+i$; the range of (90.13) is

$$-\tfrac{1}{2}\pi < \Re(\phi) < +\tfrac{1}{2}\pi.$$

Formula (90.13) shows that for integral values of k, $\dfrac{\tan k\phi}{\tan \phi}$ is a rational function of $\tan^2 \phi$.

91. Certain Meromorphic Functions. On replacing z by z/a in (89.9) and then letting a tend to ∞, we obtain the following expansion for the quotient of two of the series (89.3):

$$\frac{\Phi(b+1, c+1; z)}{\Phi(b, c; z)} = \cfrac{1}{1 - \cfrac{\dfrac{(c-b)}{c(c+1)}z}{1 + \cfrac{\dfrac{(b+1)}{(c+1)(c+2)}z}{1 - \cfrac{\dfrac{(c-b+1)}{(c+2)(c+3)}z}{1 + \cfrac{\dfrac{(b+2)}{(c+3)(c+4)}z}{1 - \cdots}}}}} \tag{91.1}$$

One may readily justify this limiting process directly from the definition of the power series expansion of an S-fraction.

The function $\Phi(b, c; z)$ is an entire function of z, so that the function (91.1) is a meromorphic function of z. Since the sequence of coefficients of z in the continued fraction has the limit zero, it follows from Theorems 54.1 and 54.2 that the continued fraction converges uniformly to this meromorphic function over every closed bounded region containing none of its poles.

On putting $b = 0$, (91.1) becomes

$$\Phi(1, c; z) = \cfrac{1}{1 - \cfrac{z}{c + \cfrac{1\cdot z}{c + 1 - \cfrac{cz}{c + 2 + \cfrac{2\cdot z}{c + 3 - \cfrac{(c+1)z}{c + 4 + \cfrac{3z}{c + 5 - \cdots}}}}}}}. \tag{91.2}$$

In particular,

$$e^z = \cfrac{1}{1 - \cfrac{z}{1 + \cfrac{z}{2 - \cfrac{z}{3 + \cfrac{z}{2 - \cfrac{z}{5 + \cfrac{z}{2 - \cdots}}}}}}}, \tag{91.3}$$

valid for all values of z.

For the quotient of two of the entire functions (89.4) we have:

$$\frac{\Psi(c+1; z)}{\Psi(c; z)} = \cfrac{1}{1 + \cfrac{\frac{z}{c(c+1)}}{1 + \cfrac{\frac{z}{(c+1)(c+2)}}{1 + \cfrac{\frac{z}{(c+2)(c+3)}}{1 + \cdots}}}}, \tag{91.4}$$

valid for all values of z. This is obtained by replacing z by z/b in (91.1) and then letting b tend to ∞. From this one may obtain immediately the expansion

$$\frac{J_{n-1}(z)}{J_n(z)} = \frac{2n}{z} - \cfrac{\frac{z}{2(n+1)}}{1 - \cfrac{\frac{(z/2)^2}{(n+1)(n+2)}}{1 - \cfrac{\frac{(z/2)^2}{(n+2)(n+3)}}{1 - \cdots}}}, \qquad (91.5)$$

where $J_p(z)$ is **Bessel's function.** This expansion is valid for all values of z.

Another special case of (91.4) is **Lambert's continued fraction,**

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{z\Psi\left(\frac{3}{2}; \frac{z}{4}\right)^2}{\Psi\left(\frac{1}{2}; \frac{z}{4}\right)^2} = \cfrac{z}{1 + \cfrac{z^2}{3 + \cfrac{z^2}{5 + \cfrac{z^2}{7 + \cdots}}}}, \quad [46], \qquad (91.6)$$

which is valid for all values of z. On replacing z by iz, this gives

$$\tan z = \cfrac{z}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{7 - \cdots}}}}, \quad [46], \qquad (91.7)$$

valid for all z.

92. A Class of Divergent Series. On replacing z by $-cz$ in the continued fraction of Gauss, and then letting c tend to ∞, we obtain for the quotient of two of the divergent series (89.5), the expansion

$$\frac{\Omega(a, b; -z)}{\Omega(a, b-1; -z)} = \cfrac{1}{1 + \cfrac{az}{1 + \cfrac{bz}{1 + \cfrac{(a+1)z}{1 + \cfrac{(b+1)z}{1 + \cfrac{(a+2)z}{1 + \cfrac{(b+2)z}{1 + \cdots}}}}}}} \tag{92.1}$$

Here, the equality sign is purely formal inasmuch as the series are divergent excepting for $z = 0$. In the special case $b = 1$, we have

$$\Omega(a, 1; -z) = \cfrac{1}{1 + \cfrac{az}{1 + \cfrac{1 \cdot z}{1 + \cfrac{(a+1)z}{1 + \cfrac{2 \cdot z}{1 + \cfrac{(a+2)z}{1 + \cfrac{3 \cdot z}{1 + \cdots}}}}}}} \tag{92.2}$$

Although the power series involved in these formulas are totally divergent, nevertheless the continued fractions converge to analytic functions of z. We shall first prove:

THEOREM 92.1. *Let A and B denote two arbitrary bounded regions of the complex plane. Then, there exists a number $\delta > 0$ such that the continued fraction* (92.1) *converges uniformly for a in A, b in B, and z in the real interval* $(0, \delta)$.

Proof. If $\delta > 0$ is sufficiently small, then $(a + p)z$ and $(b + p)z$ will be in the parabolic region $|w| - \Re(w) \leq \frac{1}{2}$, for all a in A, b in B, and z in $(0, \delta)$, $(p = 0, 1, 2, \cdots)$. The theorem now follows immediately from Theorem 18.1.

THEOREM 92.2. *Let a and b be arbitrary complex constants. Let G be any closed bounded region in the z-plane exterior to the negative half of the real axis. Then the continued fraction* (92.1) *converges over G excepting possibly at certain isolated points, and uniformly over the region obtained from G by removing the interiors of small circles with centers at these points. The value of the continued fraction is an analytic function having these points as poles* [135].

Proof. Let A and B of Theorem 92.1 be the single points a and b, and choose $\delta > 0$ accordingly. We may suppose that G contains the interval $(\delta/2, \delta)$ on the interior, and that G is a connected region. The region G will be contained within the cardioid domain (34.5) provided the parameter $h > 0$ determining that domain is sufficiently small. Then if N is sufficiently large, the numbers $a + n$ and $b + n$ will be in the parabolic region (cf. (34.1)) $|w| - \Re(w) \leq h^2/2$ for $n > N$. Hence, it follows by Theorem 34.1 that for $n > N$, the continued fraction

$$\cfrac{1}{1 + \cfrac{(a+n)z}{1 + \cfrac{(b+n)z}{1 + \cfrac{(a+n+1)z}{1 + \cfrac{(b+n+1)z}{1 + \cdots}}}}}$$

converges uniformly over G to an analytic function $f_n(z)$. Therefore the continued fraction (92.1) converges over G to the value

$$\frac{A_{2n}(z) + (b + n - 1)zf_n(z)A_{2n-1}(z)}{B_{2n}(z) + (b + n - 1)zf_n(z)B_{2n-1}(z)},$$

provided the denominator does not vanish identically. Since the continued fraction converges when z is in the interval $(0, \delta)$, by virtue of Theorem 92.1, we then conclude that it converges except possibly for isolated values of z. The convergence is evidently uniform over the region obtained from G by removing the interiors of small circles with centers at these isolated points.

In order to express the analytic function represented by the continued fraction in terms of integrals, we write

$$\begin{aligned}
\Omega(a, b; -z)& \\
&= 1 - ab\frac{z}{1!} + a(a+1)b(b+1)\frac{z^2}{2!} - \cdots \\
&= \frac{\Gamma(a)}{\Gamma(a)} + \frac{\Gamma(a+1)}{\Gamma(a)}\binom{-b}{1}z + \frac{\Gamma(a+2)}{\Gamma(a)}\binom{-b}{2}z^2 + \cdots \\
&= \frac{1}{\Gamma(a)}\int_0^\infty e^{-u}u^{a-1}du + \frac{1}{\Gamma(a)}\binom{-b}{1}\int_0^\infty e^{-u}u^a du\cdot z \\
&\quad + \frac{1}{\Gamma(a)}\binom{-b}{2}\int_0^\infty e^{-u}u^{a+1}du\cdot z^2 + \cdots \\
&= \frac{1}{\Gamma(a)}\int_0^\infty \left(1 + \binom{-b}{1}zu + \binom{-b}{2}z^2u^2 + \cdots\right)e^{-u}u^{a-1}du \\
&= \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-u}u^{a-1}du}{(1+zu)^b}.
\end{aligned}$$

This *formal* procedure suggests the possibility of the formula

$$\frac{\displaystyle\int_0^\infty \frac{e^{-u}u^{a-1}du}{(1+zu)^b}}{\displaystyle\int_0^\infty \frac{e^{-u}u^{a-1}du}{(1+zu)^{b-1}}} = \cfrac{1}{1 + \cfrac{az}{1 + \cfrac{bz}{1 + \cfrac{(a+1)z}{1 + \cfrac{(b+1)z}{1 + \cfrac{(a+2)z}{1 + \cdots}}}}}}, \quad [63], \tag{92.3}$$

for all values of a, b, and z for which the integrals are defined, namely, for $\Re(a) > 0$, b arbitrary, and for z not on the negative half of the real axis.

We consider first the case where $b = 1$:

$$\frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-u}u^{a-1}du}{1+zu} = \cfrac{1}{1+\cfrac{az}{1+\cfrac{1\cdot z}{1+\cfrac{(a+1)z}{1+\cfrac{2z}{1+\cfrac{(a+2)z}{1+\cfrac{3z}{1+\cdots}}}}}}} \tag{92.4}$$

If a is real and positive, this formula is valid for all z not on the negative half of the real axis. In fact, it is only necessary to apply the theory of § 87. Since the continued fraction is convergent, the Stieltjes moment problem

$$c_p = \int_0^\infty u^p d\phi(u), \quad p = 0, 1, 2, \cdots,$$

is determinate, so that the solution

$$\phi(u) = \frac{1}{\Gamma(a)}\int_0^u e^{-t}t^{a-1}dt$$

is the *only* normalized solution. Therefore (92.4) holds if a is real and positive. To show that (92.4) holds for $\Re(a) > 0$ and for all z not on the negative half of the real axis, it suffices to show that it holds for $\Re(a) > 0$ and for all z on some interval $(0, \delta)$, $(\delta > 0)$. For a fixed $z = x > 0$, the left-hand member of (92.4) is easily seen to be an analytic function of a for $\Re(a) > 0$. By Theorem 92.1, the same is true of the right-hand member for $0 \leq x \leq \delta$, provided δ is a sufficiently small positive number. Since (92.4) holds for a real and positive, it must therefore hold for $\Re(a) > 0$.

We shall now show that (92.3) holds for a and b real and positive, and for z not on the negative half of the real axis. The extension of the result to complex values of a and b for which the integrals exist can be made by means of the argument used above. Suppose that $z = x > 0$, $a > 0$, $b > 0$. Put

$$f(a, b) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u}u^{a-1}du}{(1 + xu)^b}.$$

By integration by parts we readily find the formulas

$$f(a, b) = f(a, b + 1) + axf(a + 1, b + 1),$$

$$f(a, b + 1) = f(a + 1, b + 1) + bxf(a + 1, b + 2).$$

Therefore,

$$\frac{f(a, b + 1)}{f(a, b)} = \cfrac{1}{1 + ax\dfrac{f(a + 1, b + 1)}{f(a, b + 1)}},$$

$$\frac{f(a + 1, b + 1)}{f(a, b + 1)} = \cfrac{1}{1 + (b + 1)x\dfrac{f(a + 1, b + 2)}{f(a + 1, b + 1)}}.$$

These hold for $a > 0$ and for all values of b. We now see immediately from these formulas that

$$\frac{f(a, b)}{f(a, b - 1)} = \frac{A_n(x) + k_nA_{n-1}(x)}{B_n(x) + k_nB_{n-1}(x)},$$

where $A_p(x)$ and $B_p(x)$ are the pth numerator and denominator of the continued fraction in (92.3), with $z = x$, and are *positive*, and where k_n is a positive number. Therefore, the quotient $f(a, b)/f(a, b - 1)$ lies between $A_n(x)/B_n(x)$ and $A_{n-1}(x)/B_{n-1}(x)$. Since the continued fraction converges, its value must then be $f(a, b)/f(a, b - 1)$. Thus, (92.3) holds for a, b and z real and positive. The extension to complex z not on the negative half of the real axis is at once possible, since both members are analytic functions of z. As indicated before, the extension to complex values of a and b for which the integrals exist can be easily made **[135]**.

We shall now derive some important special cases of the formula (92.4).

We first replace the integral in the left-hand member of (92.4) by an integral which converges for all values of a. For this purpose it is only necessary to put $b = 1$ in the identity

$$\frac{1}{\Gamma(b)}\int_0^\infty \frac{e^{-u}u^{b-1}du}{(1+zu)^a} = \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-u}u^{a-1}du}{(1+zu)^b}.$$

We then have:

$$\int_0^\infty \frac{e^{-u}du}{(1+zu)^a} = \cfrac{1}{1+\cfrac{az}{1+\cfrac{1\cdot z}{1+\cfrac{(a+1)z}{1+\cfrac{2\cdot z}{1+\cfrac{(a+2)z}{1+\cfrac{3\cdot z}{1+\cdots}}}}}}}, \tag{92.5}$$

valid for all values of a, and for all z not on the negative half of the real axis.

On replacing z by $1/z$ and then dividing by z in (92.4) we get:

$$\frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-u}u^{a-1}du}{z+u} = \cfrac{1}{z+\cfrac{a}{1+\cfrac{1}{z+\cfrac{(a+1)}{1+\cfrac{2}{z+\cfrac{(a+2)}{1+\cdots}}}}}}, \tag{92.6}$$

valid for $\Re(a) > 0$ and z not on the negative half of the real axis. On taking the even part of the continued fraction in the right-hand member, we obtain

$$\frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-u}u^{a-1}du}{z+u} \tag{92.7}$$

$$= \cfrac{1}{z+a-\cfrac{1\cdot a}{z+a+2-\cfrac{2(1+a)}{z+a+4-\cfrac{3(2+a)}{z+a+6-\cdots}}}},$$

valid in the range of (92.6).

On replacing z by $1/z$ in (92.5) we obtain:

$$\int_0^\infty \frac{e^{-u}du}{(z+u)^a} = \cfrac{z^{1-a}}{z+\cfrac{a}{1+\cfrac{1}{z+\cfrac{(a+1)}{1+\cfrac{2}{z+\cfrac{(a+2)}{1+\cfrac{3}{z+\cdots}}}}}}}, \tag{92.8}$$

valid for all a and for z not on the negative half of the real axis. Let $z = x$ be real and positive, and replace $u + x$ by u in the above integral. On replacing a by $1 - a$, we then have:

$$\int_x^\infty e^{-u}u^{a-1}du = \cfrac{e^{-x}x^a}{x+\cfrac{1-a}{1+\cfrac{1}{x+\cfrac{2-a}{1+\cfrac{2}{x+\cfrac{3-a}{1+\cfrac{3}{x+\cdots}}}}}}}, \quad [50], \tag{92.9}$$

valid for all a if $x > 0$. For $a = 0$, this gives

$$\int_0^{e^{-x}} \frac{du}{\log u} = \cfrac{-e^{-x}}{x + \cfrac{1}{1 + \cfrac{1}{x + \cfrac{2}{1 + \cfrac{2}{x + \cfrac{3}{1 + \cdots}}}}}}, \tag{92.10}$$

valid for $x > 0$; and for $a = \frac{1}{2}$, (92.9) gives

$$\int_{\sqrt{x}}^{\infty} e^{-u^2}\, du = \cfrac{\frac{1}{2}\sqrt{x}e^{-x}}{x + \cfrac{1}{2 + \cfrac{2}{x + \cfrac{3}{2 + \cfrac{4}{x + \cdots}}}}}, \quad [49], \tag{92.11}$$

valid for $x > 0$. With the aid of the formula

$$\int_0^{\infty} e^{-u^2} du = \frac{1}{2}\sqrt{\pi},$$

we then find immediately that

$$\int_0^{x} e^{-u^2} du = \frac{1}{2}\sqrt{\pi} - \cfrac{\frac{1}{2}e^{-x^2}}{x + \cfrac{1}{2x + \cfrac{2}{x + \cfrac{3}{2x + \cfrac{4}{x + \cdots}}}}}, \tag{92.12}$$

valid for $x > 0$.

One may readily obtain the following four formulas, all valid for $x > 0$.

$$\int_0^x e^{-u^2/2}du = \sqrt{\frac{\pi}{2}} - \cfrac{\frac{1}{\sqrt{2}}e^{-x^2/2}}{x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{x + \cdots}}}}}}, \tag{92.13}$$

$$\int_x^\infty e^{-u^2}du = \cfrac{e^{-x^2}}{2x + \cfrac{1}{x + \cfrac{2}{2x + \cfrac{3}{x + \cfrac{4}{2x + \cdots}}}}}, \tag{92.14}$$

$$\int_x^\infty e^{-u^2/2}du = \cfrac{e^{-x^2/2}}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cdots}}}}, \tag{92.15}$$

$$\int_x^\infty \frac{e^{-u}}{u}du = \cfrac{e^{-x}}{x + 1 - \cfrac{1}{x + 3 - \cfrac{4}{x + 5 - \cfrac{9}{x + 7 - \cfrac{16}{x + 9 - \cdots}}}}}. \tag{92.16}$$

Stieltjes gave the following generalization of (92.6):

$$S(a, b, c; z) = \int_0^\infty \left(\frac{1 - c}{e^{u(1-c)} - c^b}\right)^a e^{-zu} du \tag{92.17}$$

$$= \cfrac{m^a}{z + \cfrac{am}{1 + \cfrac{mc^b}{z + \cfrac{(a+1)m}{1 + \cfrac{2mc^b}{z + \cfrac{(a+2)m}{1 + \cfrac{3mc^b}{z + \cdots}}}}}}}, \quad [95],$$

where $m = (1 - c)/(1 - c^b)$. This is valid for $a > 0$, $b > 0$, $c > 0$, and z not on the negative half of the real axis. The integral in (92.6) is the special case $S(a, 1, 1; z)$. If $0 < c < 1$, then

$$S(a, b, c; z) = \sum_{p=0}^{\infty} \frac{a(a+1)(a+2)\cdots(a+p-1)}{p!} \cdot \frac{(1-c)^a c^{bp}}{z + (p+a)(1-c)},$$

and if $c > 1$,

$$S(a, b, c; z) = \sum_{p=0}^{\infty} \frac{a(a+1)(a+2)\cdots(a+p-1)}{p!} \cdot \frac{(c-1)^a c^{-b(p+a)}}{z + p(c-1)}.$$

If

$$\phi(r, z) = \sum_{p=0}^{\infty} \frac{r^p}{1 + pz},$$

then

$$\phi(r, z) = 1 + \frac{r}{z} S\left(1, 1, r; \frac{1-r}{z}\right) = \frac{1}{rz} S\left(1, 1, \frac{1}{r}; \frac{r-1}{z}\right).$$

We therefore have the expansions

$$\phi(r, z) = 1 + \cfrac{r}{1 - r + \cfrac{z}{1 + \cfrac{rz}{1 - r + \cfrac{2z}{1 + \cfrac{2rz}{1 - r + \cfrac{3z}{1 + \cdots}}}}}}, \tag{92.18}$$

and

$$\phi(r, z) = \cfrac{1}{1 - r + \cfrac{rz}{1 + \cfrac{z}{1 - r + \cfrac{2rz}{1 + \cfrac{2z}{1 - r + \cfrac{3rz}{1 + \cdots}}}}}} \tag{92.19}$$

valid for $0 < r < 1$ and for all $z \neq -1/n$, $n = 1, 2, 3, \cdots$.

Exercise 18

18.1. Let $P(z) = \Sigma c_p z^p$ satisfy the differential equation

$$(a + bz)\frac{dP(z)}{dz} + cP(z) + \sum_{p=0}^{n} t_p z^p = 0, \quad a \neq 0, \quad b, c, t_p \text{ constants.}$$

Then

$$P(z) = c_0 + c_1 z + \cdots + c_n z^n + \cfrac{c_{n+1}z^{n+1}}{1 + \cfrac{[c + (n+1)b]z}{(n+2)a + \cfrac{(b - c)z}{n + 3 + \cfrac{(n+2)[c + (n+2)b]z}{(n+4)a + \cfrac{2(2b - c)z}{n + 5 + \cdots}}}}}$$

If $b = 0$, this holds for all values of z; if $b \neq 0$, it holds for all z exterior to the cut along the real axis from -1 to $-\infty$.

18.2. Let $P(z)$ satisfy the differential equation

$$(az + bz^2)\frac{dP(z)}{dz} + (r + sz)P(z) + \sum_{p=0}^{n} t_p z^p = 0, \quad pa + r \neq 0, \quad p = 0, 1, 2, \cdots.$$

Then,

$$P(z) = c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + c_n z^n \cdot \begin{cases} F\left(n + \frac{a}{b}, 1, n + 1 + \frac{r}{a}; -\frac{b}{a}z\right), & a \neq 0, \quad b \neq 0, \\ \Phi\left(1, n + 1 + \frac{r}{a}; -\frac{s}{a}z\right), & a \neq 0, \quad b = 0, \\ \Omega\left(n + \frac{s}{b}, 1; -\frac{b}{r}z\right), & a = 0, \quad b \neq 0, \end{cases}$$

so that continued fraction expansions for $P(z)$ can be written down at once.

Chapter XIX

STIELTJES SUMMABILITY

In this chapter we have applied some of the theory of J-fractions to a number of examples. The pattern of these examples may be described as follows. A definite integral is expanded into a power series which proves to be totally divergent. The power series is then expanded into a J-fraction which turns out to be convergent and to have as its value the definite integral with which we started. The J-fraction then serves as a means for computing the value of the definite integral.

93. Definition and Illustrative Examples. By **Stieltjes summability** we shall understand the process of summing a divergent power series by means of a J-fraction. The examples given in § 92 show that the J-fraction may converge even when the power series expansion of the J-fraction is totally divergent. The J-fraction thus furnishes a generalized sum of the divergent power series.

If there is given a power series $P(1/z) = \Sigma(c_p/z^{p+1})$, then there arise two main problems.

(A) To determine whether or not the power series has a J-fraction expansion.

(B) If the J-fraction exists, to establish convergence, and to determine properties of the function represented.

As to (A), there are several criteria for establishing the existence of the J-fraction, of which the following may be suggested.

(a) There is a bounded nondecreasing function $\phi(u)$, taking on infinitely many different values, such that the coefficients in the power series are given by

$$c_p = \int_{-\infty}^{+\infty} u^p d\phi(u), \quad p = 0, 1, 2, \cdots. \tag{93.1}$$

In case $\phi(u)$ is constant for $u < 0$, this may be written

$$c_p = \int_0^{+\infty} u^p d\phi(u), \quad p = 0, 1, 2, \cdots, \tag{93.2}$$

and then $P(1/z)$ has an S-fraction expansion.

(b) The determinants Δ_p of (86.2) are all different from 0. If the c_p are real and these determinants are all positive, then, and only then, does $P(1/z)$ have a real J-fraction expansion.

(c) There exists a nonrational function $f(z)$ which is analytic and has negative imaginary part for $\mathfrak{I}(z) > 0$, which is asymptotically equal to the *real* power series in the domain $\mathfrak{I}(z) \geq \delta$ for every positive δ. It is actually sufficient if this holds in every domain $c \leq \arg z \leq \pi - c$, $0 < c < \pi/2$. In case $f(z)$ can be expressed in the form

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u}, \tag{93.3}$$

where $\phi(u)$ is bounded and nondecreasing, then it is sufficient that $f(z)$ is asymptotically equal to $P(1/z)$ as $z \to \infty$ along the positive imaginary axis. It may be possible to establish (93.3) by means of Theorem 66.2.

(d) There exists a power series identity of the type required in Theorem 53.1.

(e) The power series has some special property which enables one to obtain its J-fraction expansion, e.g., as is the case for the series of Chapter XVIII.

(*f*) The coefficients in the J-fraction may be computed step by step by means of the algorithm of § 51. If the law of formation of the coefficients is not too complicated, it may be possible to discover and establish it.

As to (B), if the law of the coefficients in the J-fraction is known, it may be possible to use one of the convergence criteria developed in the early chapters of this book. If the law of the coefficients is not known, then it may be possible to establish convergence of the J-fraction by showing that the related moment problem is determinate, e.g., by Theorem 88.1. The given power series may have arisen as the formal solution of a differential equation, or as the formal expansion of a definite integral. One is then naturally interested in knowing whether or not the value of the J-fraction is a solution of the differential equation, or is equal to the integral. Since the value of the J-fraction can always be expressed as a definite integral, it may be possible to find the answers to these questions without much difficulty.

The technique employed depends of course upon the particular problem under consideration. Stieltjes worked out a large number of examples. These will be found in his *Oeuvres*, vol. 2, pp. 184–200, 378–391, and 546–559, and also scattered through the celebrated memoir on pp. 402–566. We shall consider some of these examples here.

Example 1. Let us recall that

$$\log \Gamma(z) = -z + (z - \tfrac{1}{2}) \log z + \log \sqrt{2\pi} + J(z),$$

where, if $\Re(z) > 0$,

$$J(z) = \frac{1}{\pi}\int_0^\infty \log \frac{1}{1 - e^{-2\pi u}} \cdot \frac{z}{z^2 + u^2}\, du. \tag{93.4}$$

The function $J(z)$ has the formal power series expansion

$$P\left(\frac{1}{z}\right) = \sum_{p=0}^{\infty} (-1)^p \frac{c_p}{z^{2p+1}},$$

where

$$c_p = \frac{1}{\pi}\int_0^\infty u^{2p} \log (1 - e^{-2\pi u}) du, \quad p = 0, 1, 2, \cdots. \tag{93.5}$$

Thus,

$$c_p = \frac{(2p)!}{2^{2p+1}\pi^{2p+2}} \sum_{r=1}^{\infty} \frac{1}{r^{2p+2}}, \quad p = 0, 1, 2, \cdots. \tag{93.6}$$

Consequently,

$$c_p = \frac{B_{2p+2}}{(2p + 1)(2p + 2)}, \quad p = 0, 1, 2, \cdots,$$

where $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, $\cdots$ are the Bernoulli numbers. It readily follows from (93.6) that the formal power series $P(1/z)$ is totally divergent.

The equations (93.5) can be written in the form

$$c_p = \int_0^\infty u^p d\phi(u), \quad p = 0, 1, 2, \cdots, \tag{93.7}$$

where

$$\phi(u) = \frac{1}{\pi}\int_0^u \frac{\log(1 - e^{-2\pi\sqrt{t}})}{2\sqrt{t}}\, dt, \tag{93.8}$$

i.e., in the form (93.2), where $\phi(u)$ is bounded and nondecreasing, and takes on infinitely many different values for $u > 0$. Therefore the series $P(1/z)$ has an S-fraction expansion of the form

$$\cfrac{a_0}{z + \cfrac{a_1}{z + \cfrac{a_2}{z + \cdots}}},$$

in which the a_p are positive. On applying Theorem 88.1 to the c_p as given by (93.6) we find that the Stieltjes moment problem (93.7) is determinate. *It follows that the S-fraction must converge for* $\Re(z) > 0$ *to the function* $J(z)$.

By means of the algorithm (51.6) we find that

$$J(z) = \cfrac{\frac{1}{12}}{z + \cfrac{\frac{1}{30}}{z + \cfrac{\frac{53}{210}}{z + \cfrac{\frac{195}{371}}{z + \cfrac{\frac{22999}{22737}}{z + \cfrac{\frac{29944523}{19733142}}{z + \cfrac{\frac{109535241009}{48264275462}}{z + \cdots}}}}}}}. \tag{93.9}$$

The law of formation of the coefficients is extremely complicated. However, even the first approximant $1/(12z)$ gives an excellent approximation to the function $J(z)$ for large positive values of z.

Example 2. By (53.11) we have, formally,

$$-i\int_0^\infty \operatorname{sech}^k u\, e^{izu}du = \cfrac{1}{z - \cfrac{1\cdot k}{z - \cfrac{2(k+1)}{z - \cfrac{3(k+2)}{z - \cdots}}}} \tag{93.10}$$

If k is real and positive, this is a real J-fraction. By Theorem 25.1, the determinate case holds, so that the J-fraction converges for $\mathfrak{J}(z) > 0$. The integral on the left is also convergent for $\mathfrak{J}(z) > 0$. To show that the integral is equal to the J-fraction, we proceed as follows. Denote the value of the integral by $f(z)$. Then

$$\mathfrak{J}(f(z)) = -\int_0^\infty \operatorname{sech}^k u\cdot\cos xu\cdot e^{-yu}du,$$

where we have put $z = x + iy$. It is not difficult to see that $\mathfrak{J}(f(z))$ is negative for $\mathfrak{J}(z) > 0$. Then, by integration by parts,

$$f(z) = \frac{1}{z} + \frac{g(z)}{yz},$$

where

$$g(z) = -ky\int_0^\infty \operatorname{sech}^k u\cdot\tanh u\cdot e^{izu}du.$$

Hence, if $y > 0$,

$$|\, g(z)\, | \le C\int_0^\infty ye^{-yu}du = C,$$

where C is a sufficiently large positive constant. It now follows by § 66 that $f(z)$ can be expressed in the form (93.3), where $\phi(u)$ is bounded and nondecreasing.

Let $P(1/z) = \Sigma(c_p/z^{p+1})$ be the power series expansion of the J-fraction. One may then readily verify that $f(z)$ is asymptotically equal to $P(1/z)$ in the sense of (85.3), as z approaches ∞ along the positive imaginary axis. Since the determinate case holds for the J-fraction, we then conclude by Theorem 85.3 that $f(z)$ is *the* equivalent function of the J-fraction, and consequently (93.10) is a true equality.

Example 3. We now consider the function

$$F(t; z) = \int_0^\infty \frac{1-t}{e^{(1-t)u} - t} e^{-zu} du, \quad t > 0, \tag{93.11}$$

mentioned near the end of § 92. For $t = 1$ this reduces to

$$\int_0^\infty \frac{e^{-u}du}{z+u} = \cfrac{1}{z + \cfrac{1}{1 + \cfrac{1}{z + \cfrac{2}{1 + \cfrac{2}{z + \cfrac{3}{1 + \cdots}}}}}} \tag{93.12}$$

(cf. (92.6) with $a = 1$), which is a formula of Laguerre, valid for all z not on the negative half of the real axis. The power series expansion of this S-fraction is the totally divergent power series $\Sigma(-1)^p p!/z^{p+1}$.

The function $F(t; z)$ can be expressed as a Stieltjes integral of the form

$$\int_0^\infty \frac{d\phi(u)}{z+u},$$

also when $t > 0$ and different from 1. In fact, we have

$$F(t; z) = \begin{cases} \displaystyle\sum_{p=1}^\infty \frac{(1-t)t^{p-1}}{z + p(1-t)} & \text{if} \quad 0 < t < 1, \\ \displaystyle\sum_{p=1}^\infty \frac{(t-1)t^{-p}}{z + (p-1)(t-1)} & \text{if} \quad t > 1. \end{cases} \tag{93.13}$$

If $t \neq 1$, $\phi(u)$ is a step-function; and $\phi(u)$ is continuous if $t = 1$.

Moreover, the moments

$$c_p = \int_0^\infty u^p d\phi(u) = \begin{cases} \sum_{k=1}^{\infty} k^p(1-t)^{p+1}t^{k-1} & \text{if} \quad 0 < t < 1, \\ \sum_{k=1}^{\infty} (k-1)^p(t-1)^{p+1}t^{-k} & \text{if} \quad t > 1, \end{cases} \tag{93.14}$$

$$p = 0, 1, 2, \cdots,$$

exist. From (93.14) we readily verify that

$$c_p = t(1-t)\frac{dc_{p-1}}{dt} + [1 + (p-1)t]c_{p-1}, \quad c_0 = 1, \tag{93.15}$$
$$p = 1, 2, 3, \cdots.$$

Thus, $c_p = c_p(t)$ is a polynomial in t of degree $p - 1$, with positive coefficients. One may readily verify that

$$c_p(t) = t^{p-1}c_p\left(\frac{1}{t}\right). \tag{93.16}$$

Consider now the series

$$\Sigma\left(\frac{1}{c_p}\right)^{1/2p}. \tag{93.17}$$

If $t = 1$, then $c_p = p!$, so that this series diverges. If $0 < t < 1$, then obviously $c_p \leq p!$, so that again the series diverges. If $t > 1$, then, by (93.16), $c_p \leq t^{p-1}p!$, and therefore the series (93.17) diverges in this case also. By Theorem 88.1 we now conclude that the Stieltjes moment problem (93.14) is determinate.

From these remarks we conclude that $F(t; z)$ is the value of a convergent S-fraction of the form

$$\cfrac{a_0}{z + \cfrac{a_1}{1 + \cfrac{a_2}{z + \cdots}}},$$

in which the coefficients a_p are positive. By means of the algorithm of § 51 we find that

$$F(t; z) = \cfrac{1}{z + \cfrac{1}{1 + \cfrac{t}{z + \cfrac{2}{1 + \cfrac{2t}{z + \cfrac{3}{1 + \cfrac{3t}{z + \cdots}}}}}}} \tag{93.18}$$

94. List of Expansion Formulas. In this section we have listed, without proof, a number of expansion formulas given by Stieltjes and others.

(*A*) *Integrals involving hyperbolic functions.*

$$\int_0^\infty \frac{e^{-zu}du}{(\cosh u + a \sinh u)^m} \tag{94.1}$$

$$= \cfrac{1}{z+ma+\cfrac{m(1-a^2)}{z+(m+2)a+\cfrac{2(m+1)(1-a^2)}{z+(m+4)a+\cfrac{3(m+2)(1-a^2)}{z+(m+6)a+\cdots}}}} \quad [94.]$$

$$\int_0^\infty \tanh u e^{-zu}du = \cfrac{1}{z^2 + \cfrac{1\cdot 2}{1 + \cfrac{2\cdot 3}{z^2 + \cfrac{3\cdot 4}{1 + \cfrac{4\cdot 5}{z^2 + \cdots}}}}} \quad [94.] \tag{94.2}$$

$$\int_0^\infty c\,\frac{\sinh(au)\sinh(bu)}{\sinh(cu)}e^{-zu}du \tag{94.3}$$

$$= \cfrac{ab}{z^2+b_1-\cfrac{a_1}{z^2+b_2-\cfrac{a_2}{z^2+b_3-\cdots}}}, \quad [94],$$

where

$$b_{p+1} = (2p^2+2p+1)c^2 - a^2 - b^2,$$

$$a_p = (p^2c^2-a^2)(p^2c^2-b^2)\frac{4p^2}{4p^2-1}.$$

$$\int_0^\infty b\,\frac{\sinh(au)}{\sinh(bu)}e^{-zu}du = \cfrac{a}{z+\cfrac{a_1}{z+\cfrac{a_2}{z+\cdots}}}, \quad [94], \tag{94.4}$$

where

$$a_p = (p^2b^2-a^2)\frac{p^2}{4p^2-1}.$$

$$\int_0^\infty F(a, b, \tfrac{1}{2}(a+b+1); -\sinh^2 u)e^{-zu}du \tag{94.5}$$

$$= \cfrac{1}{z+\cfrac{a_1}{z+\cfrac{a_2}{z+\cdots}}}, \quad [94],$$

where $F(a, b, c; x)$ is the hypergeometric series, and

$$a_{p+1} = \frac{4(a+p)(b+p)(p+1)(p+a+b-1)}{(2p+a+b-1)(2p+a+b+1)}.$$

$$\int_0^\infty \frac{ue^{-zu}du}{\sinh u} = \cfrac{1}{z + \cfrac{1^4}{3z + \cfrac{2^4}{5z + \cfrac{3^4}{7z + \cdots}}}} \quad [80.] \tag{94.6}$$

$$\int_0^\infty \frac{ue^{-zu}du}{\cosh u} = \tag{94.7}$$

$$\cfrac{1}{z^2 - 1 + \cfrac{2^2}{1 + \cfrac{2^2}{z^2 - 1 + \cfrac{4^2}{1 + \cfrac{4^2}{z^2 - 1 + \cfrac{6^2}{1 + \cfrac{6^2}{z^2 - 1 + \cdots}}}}}}} \quad [80.]$$

$$\operatorname{Exp}\int_0^\infty \frac{1}{u}\left(1 - \frac{\cosh 2au}{\cosh 2u}\right) e^{-zu}du \tag{94.8}$$

$$= 1 + \cfrac{2(1 - a^2)}{z^2 + \cfrac{(3^2 - a^2)}{1 + \cfrac{(5^2 - a^2)}{z^2 + \cfrac{(7^2 - a^2)}{1 + \cdots}}}} \quad [80.]$$

$$\tanh\left(\frac{1}{2}\int_0^\infty \frac{\sinh 2au}{u \cosh u} e^{-zu}du\right) \tag{94.9}$$

$$= \cfrac{a}{z + \cfrac{(1^2 - a^2)}{z + \cfrac{(2^2 - a^2)}{z + \cfrac{(3^2 - a^2)}{z + \cdots}}}} \quad [80.]$$

$$\tanh\int_0^\infty \frac{\sinh au}{u\cosh u}e^{-zu}du \tag{94.10}$$

$$= \cfrac{a}{z + \cfrac{1^2}{z + \cfrac{(2^2 - a^2)}{z + \cfrac{3^2}{z + \cfrac{(4^2 - a^2)}{z + \cfrac{5^2}{z + \cfrac{(6^2 - a^2)}{z + \cdots}}}}}}}$$

[80.]

(*B*) *Formulas involving* $\Psi(z) = d[\log \Gamma(z)]/dz$. [94.]

$$\frac{1}{2}\Psi(z + b) - \frac{1}{2}\Psi(z + 1 - b) = \cfrac{b - (\frac{1}{2})}{z + \cfrac{a_1}{z + \cfrac{a_2}{z + \cdots}}}, \tag{94.11}$$

where

$$a_p = \frac{p^2(p + 1 - 2b)(p - 1 + 2b)}{4(2p - 1)(2p + 1)}.$$

$$\tfrac{1}{2}[\Psi(z + b) + \Psi(z + 1 - b) - \Psi(z) - \Psi(z + 1)] \tag{94.12}$$

$$= \cfrac{\frac{1}{2}b(1 - b)}{z^2 + \cfrac{p_1}{1 + \cfrac{q_1}{z^2 + \cfrac{p_2}{1 + \cfrac{q_2}{z^2 + \cdots}}}}},$$

where

$$p_n = \frac{n(n - b)(n - 1 + b)}{2(2n - 1)}, \quad q_n = \frac{n(n + b)(n + 1 - b)}{2(2n + 1)}.$$

$$\Psi'(z) = \sum_{p=0}^{\infty} \frac{1}{(z+p)^2} = \cfrac{1}{z - \frac{1}{2} + \cfrac{a_1}{z - \frac{1}{2} + \cfrac{a_2}{z - \frac{1}{2} + \cdot_{\cdot_{\cdot}}}}}, \tag{94.13}$$

where

$$a_p = \frac{p^4}{4(2p-1)(2p+1)}.$$

$$4z^2 \sum_{p=0}^{\infty} \frac{1}{(z+p)^3} - 2z - 2 = \cfrac{1}{z + \cfrac{p_1}{z + \cfrac{q_1}{z + \cfrac{p_2}{z + \cfrac{q_2}{z + \cdot_{\cdot_{\cdot}}}}}}}, \tag{94.14}$$

where

$$p_n = \frac{n^2(n+1)}{4n+2}, \quad q_n = \frac{n(n+1)^2}{4n+2}.$$

$$\frac{1}{2}\left[\Psi\left(\frac{z+1+b}{2}\right) + \Psi\left(\frac{z+2-b}{2}\right) - \Psi\left(\frac{z+b}{2}\right) - \Psi\left(\frac{z+1-b}{2}\right)\right] = \cfrac{1}{z + \cfrac{a_1}{z + \cfrac{a_2}{z + \cdot_{\cdot_{\cdot}}}}}, \tag{94.15}$$

where $a_{2p} = p^2$, $a_{2p+1} = (p+b)(p+1-b)$.

$$\frac{1}{2}\left[\Psi\left(\frac{z+b}{2}\right) + \Psi\left(\frac{z+2-b}{2}\right) - \Psi\left(\frac{z+1-b}{2}\right) - \Psi\left(\frac{z+1+b}{2}\right)\right] = \cfrac{b - (\frac{1}{2})}{z^2 + b_1 - \cfrac{a_1}{z^2 + b_2 - \cfrac{a_2}{z^2 + b_3 - \cdot_{\cdot_{\cdot}}}}}, \tag{94.16}$$

where

$$a_p = p^2\left(\frac{2p-1}{2} + b\right)\left(\frac{2p+1}{2} - b\right),$$

$$b_p = b - b^2 + \tfrac{1}{2}(2p-1)^2.$$

(*C*) *Integrals involving the Jacobi elliptic functions.*

$$\int_0^\infty sn(u, k)e^{-zu}du \tag{94.17}$$

$$= \cfrac{1}{z^2 + a - \cfrac{1\cdot 2^2\cdot 3k^2}{z^2 + 3^2a - \cfrac{3\cdot 4^2\cdot 5k^2}{z^2 + 5^2a - \cfrac{5\cdot 6^2\cdot 7k^2}{z^2 + 7^2a - \cdots}}}}, \quad [93],$$

where

$$a = 1 + k^2.$$

$$\int_0^\infty cn(u, k)e^{-zu}du = \cfrac{1}{z + \cfrac{1^2}{z + \cfrac{2^2k^2}{z + \cfrac{3^2}{z + \cfrac{4^2k^2}{z + \cfrac{5^2}{z + \cfrac{6^2k^2}{z + \cdots}}}}}}}. \quad [93.] \tag{94.18}$$

$$\int_0^\infty dn(u, k)e^{-zu}du = \cfrac{1}{z + \cfrac{k^2}{z + \cfrac{2^2}{z + \cfrac{3^2k^2}{z + \cfrac{4^2}{z + \cfrac{5^2k^2}{z + \cfrac{6^2}{z + \cfrac{7^2k^2}{z + \cdots}}}}}}}}. \quad [93.] \tag{94.19}$$

$$\int_0^\infty sn^2(u, k)e^{-zu}du = \cfrac{2}{z^2 + b_1 - \cfrac{a_1}{z^2 + b_2 - \cfrac{a_2}{z^2 + b_3 - \cdots}}}, \quad [80], \tag{94.20}$$

where

$$a_p = 2p(2p + 1)^2(2p + 2)k^2, \quad b_p = (2p)^2(1 + k^2).$$

$$\int_0^\infty \frac{sn(u, k)cn(u, k)}{dn(u, k)} e^{-zu}du \tag{94.21}$$

$$= \cfrac{1}{z^2 + b_1 - \cfrac{a_1}{z^2 + b_2 - \cfrac{a_2}{z^2 + b_3 - \cdots}}}, \quad [80],$$

where

$$a_p = (2p - 1)(2p)^2(2p + 1)k^4, \quad b_p = 2(2p - 1)^2(2 - k^2).$$

(*D*) *Miscellaneous expansions.*

$$\frac{2}{\pi}\int_0^1 \frac{1 + t}{(1 + t)^2 - 4tu}\sqrt{\frac{1}{u} - 1}\,\frac{du}{1 + 4zu} \tag{94.22}$$

$$= \cfrac{1}{1 + \cfrac{(1 + t)z}{1 + \cfrac{z}{1 + \cfrac{z}{1 + \cfrac{z}{1 + \cdots}}}}}. \quad [88.]$$

Let $a = (\sqrt{p} - \sqrt{q})^2$, $b = (\sqrt{p} + \sqrt{q})^2$, $(p > 0,\ q > 0)$. Let

$$F(z) = \begin{cases} \dfrac{\sqrt{ab}}{z} + \dfrac{1}{2\pi}\displaystyle\int_a^b \frac{\sqrt{(u - a)(b - u)}}{u}\,\frac{du}{z + u} & \text{if } p > q, \\[2ex] \dfrac{1}{2\pi}\displaystyle\int_a^b \frac{\sqrt{(u - a)(b - u)}}{u}\,\frac{du}{z + u} & \text{if } q \geq p. \end{cases}$$

Then

$$F(z) = \cfrac{p}{z + \cfrac{q}{1 + \cfrac{p}{z + \cfrac{q}{1 + \cfrac{p}{z + \cdots}}}}} \tag{94.23}$$

[95.]

Let $c = \prod_{p=0}^{\infty}(1 - a^p b)$ where $0 < a < 1$, $0 < b < 1$, and put

$$M_1 = bc, \quad M_p = \frac{b^p c}{(1-a)(1-a^2)(1-a^3)\cdots(1-a^{p-1})},$$

$$p = 2, 3, 4, \cdots.$$

Then

$$\sum_{p=1}^{\infty} \frac{M_p}{1 + a^p z} = \cfrac{b}{1 + \cfrac{(1-b)az}{1 + \cfrac{(1-a)abz}{1 + \cfrac{(1-ab)a^2 z}{1 + \cfrac{(1-a^2)a^2 bz}{1 + \cfrac{(1-a^2 b)a^3 z}{1 + \cfrac{(1-a^3)a^3 bz}{1 + \cdots}}}}}}} \tag{94.24}$$

[131.]

Chapter XX

THE PADÉ TABLE

With each ordered pair (p, q) of nonnegative integers, there is associated a uniquely determined rational fraction $f_{p,q}(z)$, whose numerator and denominator are of degrees not exceeding q and p, respectively, and whose expansion in ascending powers of z agrees term by term with a given power series $\Sigma c_p z^p$ for more terms than that of any other such rational fraction. These rational fractions are arranged in a table of double entry, by putting $f_{p,q}(z)$ in the $(p + 1)$th row and $(q + 1)$th column of the table $(p, q = 0, 1, 2, \cdots)$. The present chapter is concerned primarily with the investigation of formal properties of this table, and of continued fractions whose approximants are among the fractions of the table.

95. Definitions. Let

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \quad (c_0 \neq 0), \tag{95.1}$$

be a formal power series with constant term different from zero, and let $B(z) = t_0 + t_1 z + \cdots + t_p z^p$ be a polynomial of degree not exceeding p. We form the product

$$P(z)B(z) = c_0 t_0 + (c_0 t_1 + c_1 t_0)z + (c_0 t_2 + c_1 t_1 + c_2 t_0)z^2 + \cdots. \tag{95.2}$$

Since there are $p + 1$ parameters t_k, we may choose them, not all zero, such that the coefficients of z^{q+r}, $r = 1, 2, \cdots, p$, in (95.2) are equal to zero. If we denote by $B_{p,q}(z)$ the corresponding polynomial $B(z)$, and denote by $A_{p,q}(z)$ the sum of the terms of degree less than $q + 1$ in (95.2), we then have

$$P(z)B_{p,q}(z) - A_{p,q}(z) = (z^{p+q+1}), \tag{95.3}$$

where (z^k) denotes a power series beginning with the term in z^k or a higher power of z. If $B_{p,q}'(z)$ and $A_{p,q}'(z)$ are two other polynomials so determined that

$$P(z)B_{p,q}'(z) - A_{p,q}'(z) = (z^{p+q+1}),$$

the polynomials being of degrees not exceeding p and q, respectively, $B_{p,q}'(z) \not\equiv 0$, then we find by composition that

$$A_{p,q}(z)B_{p,q}'(z) - A_{p,q}'(z)B_{p,q}(z) = (z^{p+q+1}).$$

This is a polynomial of degree not exceeding $p + q$, and yet it contains no power of z of degree less than $p + q + 1$. The polynomial must therefore be identically equal to 0, so that

$$\frac{A_{p,q}'(z)}{B_{p,q}'(z)} \equiv \frac{A_{p,q}(z)}{B_{p,q}(z)}.$$

This uniquely determined rational fraction,[29]

$$f_{p,q}(z) = \frac{A_{p,q}(z)}{B_{p,q}(z)} \tag{95.4}$$

is called a **Padé approximant of $P(z)$.**

We associate with the Padé approximants the following geometrical configuration, known as the **Padé table.** To the point

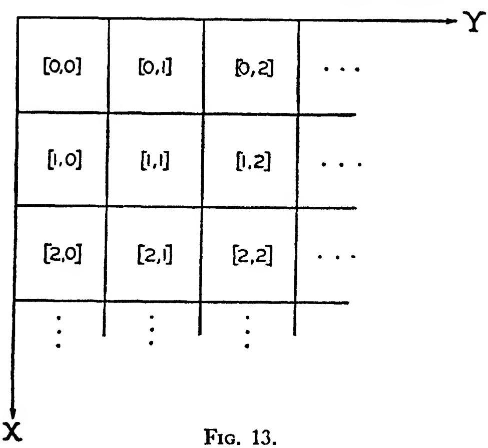

FIG. 13.

[29] Frobenius [16] made a systematic study of these rational fractions, obtaining recurrence relations connecting numerators and denominators of three contiguous fractions. Padé [66] arranged the fractions in a table of double entry, and investigated the different types of continued fractions whose sequences of approximants are appropriately chosen files in the table.

in the cartesian plane with nonnegative integral co-ordinates (p, q), we shall let correspond the Padé approximant $f_{pq}(z)$. It will be convenient to take the positive y-axis to the right and the positive x-axis downward. We may also regard $f_{pq}(z)$ as occupying the **square** $[p, q]$, with vertices at the points

$$(p, q), \quad (p, q+1),$$
$$(p+1, q), \quad (p+1, q+1).$$

96. The Normal Padé Table. The power series (95.1) and its Padé table will be called **normal** if all the determinants

$$\Delta_p^{(k)} = \begin{vmatrix} c_k, & c_{k+1}, & \cdots, & c_{k+p} \\ c_{k+1}, & c_{k+2}, & \cdots, & c_{k+p+1} \\ & \cdot & \cdot & \cdot \\ c_{k+p}, & c_{k+p+1}, & \cdots, & c_{k+2p} \end{vmatrix}, \quad \begin{aligned} p &= 0, 1, 2, \cdots, \\ k &= 0, 1, 2, \cdots, \end{aligned} \tag{96.1}$$

are different from zero, and if all the like determinants formed with the coefficients of the reciprocal series

$$Q(z) = \frac{1}{P(z)} = d_0 + d_1 z + d_2 z^2 + \cdots \tag{96.2}$$

are different from zero. By Theorem 52.1 it follows that $P(z)$ is normal if, and only if, the series

$$\begin{aligned} P^{(k)}(z) &= c_k + c_{k+1}z + c_{k+2}z^2 + \cdots, \\ Q^{(k)}(z) &= d_k + d_{k+1}z + d_{k+2}z^2 + \cdots, \\ k = 0, 1, 2, \cdots, &\quad (P^{(0)} = P, \quad Q^{(0)} = Q), \end{aligned} \tag{96.3}$$

have S-fraction expansions

$$\cfrac{a_0^{(k)}}{1 - \cfrac{a_1^{(k)}z}{1 - \cfrac{a_2^{(k)}z}{1 - \cdots}}}, \quad (a_p^{(0)} = a_p), \tag{96.4}$$

and

$$\cfrac{\alpha_0^{(k)}}{1 - \cfrac{\alpha_1^{(k)}z}{1 - \cfrac{\alpha_2^{(k)}z}{1 - \cdots}}}, \quad (\alpha_p^{(0)} = \alpha_p), \tag{96.5}$$

respectively, where the $a_p^{(k)}$ and $\alpha_p^{(k)}$ are different from zero.

THEOREM 96.1. *In a normal Padé table, the approximants of the continued fraction*

$$c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-1} z^{k-1} + \cfrac{a_0^{(k)} z^k}{1 - \cfrac{a_1^{(k)} z}{1 - \cfrac{a_2^{(k)} z}{1 - \cdots}}} \tag{96.6}$$

fill the stairlike sequence of squares

$$\begin{matrix} [0, k-1][0, k] & & \\ & [1, k][1, k+1] & \\ & & [2, k+1][2, k+2] \\ & & \ddots, \end{matrix} \tag{96.7}$$

while the approximants of the continued fraction

$$\cfrac{1}{d_0 + d_1 z + d_2 z^2 + \cdots + d_{k-1} z^{k-1} + \cfrac{z^k \alpha_0^{(k)}}{1 - \cfrac{\alpha_1^{(k)} z}{1 - \cfrac{\alpha_2^{(k)} z}{1 - \cdots}}}} \tag{96.8}$$

fill the squares

$$\begin{matrix} [k-1, 0] & & \\ [k, 0] & [k, 1] & \\ & [k+1, 1][k+1, 2] & \\ & & [k+2, 2] \\ & & \ddots. \end{matrix} \tag{96.9}$$

[66.]

Proof. Let $A_p^{(k)}(z)$ and $B_p^{(k)}(z)$ denote the pth numerator and denominator of (96.4). Then the numerators and denominators of (96.6) are given by

$$\begin{aligned} A_{p,p+k-1}(z) &= (c_0 + c_1 z + \cdots + c_{k-1} z^{k-1}) B_{2p}^{(k)}(z) \\ &\quad + z^k A_{2p}^{(k)}(z), \\ A_{p,p+k}(z) &= (c_0 + c_1 z + \cdots + c_{k-1} z^{k-1}) B_{2p+1}^{(k)}(z) \\ &\quad + z^k A_{2p+1}^{(k)}(z), \\ B_{p,p+k-1}(z) &= B_{2p}^{(k)}(z), \\ B_{p,p+k}(z) &= B_{2p+1}^{(k)}(z). \end{aligned} \tag{96.10}$$

The polynomials are of degrees $p + k - 1$, $p + k$, p and p, respectively. Moreover,

$$P(z)B_{p,p+k-1}(z) - A_{p,p+k-1}(z) = z^k[P^{(k)}(z)B_{2p}^{(k)}(z) - A_{2p}^{(k)}(z)]$$
$$= (z^{2p+k}).$$

Consequently, $A_{p,p+k-1}(z)/B_{p,p+k-1}(z)$ is the Padé approximant $f_{p,p+k-1}(z)$ of $P(z)$. Similarly, $A_{p,p+k}(z)/B_{p,p+k}(z) = f_{p,p+k}(z)$. Therefore the approximants of (96.6) are the Padé approximants in the file (96.7). Now one may verify at once that if $F_{p,q}$ is a Padé approximant of $Q(z) = 1/P(z)$, then $1/F_{p,q}$ is the Padé approximant $f_{q,p}$ of $P(z)$. From this remark it follows that the approximants of the continued fraction (96.8) make up the file (96.9).

We note that (96.4) with $k = 0$ is the same as (96.8) with $k = 1$. Hence, we have the relations

$$\frac{1}{d_0} = a_0, \quad -\frac{\alpha_0^{(1)}}{d_0} = a_1, \quad \alpha_p^{(1)} = a_{p+1}, \qquad p = 1, 2, 3, \cdots. \tag{96.11}$$

Inasmuch as two of the stairlike files (96.7) or (96.9), formed for consecutive values of k, overlap, it follows that if all the continued fractions (96.6) and (96.8) converge, then they must have a common value. In order to investigate the convergence of these continued fractions one needs to have formulas expressing their approximants in terms of the approximants of some one of them, for instance, (96.4) with $k = 0$. We shall proceed to obtain such formulas.

The approximants of the even part of (96.6), namely,

$$c_0 + c_1 z + \cdots + c_{k-1}z^{k-1} \tag{96.12}$$
$$+ \cfrac{a_0^{(k)}z^k}{1 - a_1^{(k)}z - \cfrac{a_1^{(k)}a_2^{(k)}z^2}{1 - (a_2^{(k)} + a_3^{(k)})z - \cfrac{a_3^{(k)}a_4^{(k)}z^2}{1 - (a_4^{(k)} + a_5^{(k)})z - \cdots}}},$$

occupy the diagonal file of squares

$$\begin{matrix} [0, k-1] & & & \\ & [1, k] & & \\ & & [2, k+1] & \\ & & & \ddots, \end{matrix}$$

and the approximants of the odd part of (96.6), namely,

$$c_0+c_1z+c_2z^2+\cdots+c_{k-1}z^{k-1}+a_0^{(k)}z^k \\ +\cfrac{a_0^{(k)}a_1^{(k)}z^{k+1}}{1-(a_1^{(k)}+a_2^{(k)})z-\cfrac{a_2^{(k)}a_3^{(k)}z^2}{1-(a_3^{(k)}+a_4^{(k)})z-\cfrac{a_4^{(k)}a_5^{(k)}z^2}{1-(a_5^{(k)}+a_6^{(k)})z-\cdots}}}, \tag{96.13}$$

occupy the file

$$\begin{matrix} [0, k] & & & \\ & [1, k+1] & & \\ & & [2, k+2] & \\ & & & \ddots. \end{matrix}$$

It follows that (96.13) with k replaced by $k-1$ must be identical with (96.12). Because of the uniqueness of the J-fraction expansion, we are justified in equating corresponding coefficients. Therefore

$$\begin{aligned} a_0^{(k-1)} &= c_{k-1}, & a_1^{(k-1)}+a_2^{(k-1)} &= a_1^{(k)}, \\ a_0^{(k-1)}a_1^{(k-1)} &= a_0^{(k)}, & a_3^{(k-1)}+a_4^{(k-1)} &= a_2^{(k)}+a_3^{(k)}, \\ a_2^{(k-1)}a_3^{(k-1)} &= a_1^{(k)}a_2^{(k)}, & a_5^{(k-1)}+a_6^{(k-1)} &= a_4^{(k)}+a_5^{(k)}, \\ & \cdots & & \cdots\cdots \end{aligned} \tag{96.14}$$

These can be most easily handled if we make the substitution

$$a_0^{(r)} = \frac{1}{b_1^{(r)}}, \quad a_p^{(r)} = \frac{1}{b_p^{(r)}b_{p+1}^{(r)}}, \quad p = 1, 2, 3, \cdots. \tag{96.15}$$

One may then readily establish the relations

$$b_{2p-1}^{(k)} = b_{2p}^{(k-1)} \left(\sum_{r=1}^{p} b_{2r-1}^{(k-1)} \right)^2,$$

$$b_{2p}^{(k)} = \frac{b_{2p+1}^{(k-1)}}{\sum_{r=1}^{p} b_{2r-1}^{(k-1)} \cdot \sum_{r=1}^{p+1} b_{2r-1}^{(k-1)}}, \quad p = 1, 2, 3, \cdots, \tag{96.16}$$

and also,

$$b_1^{(k-1)} = \frac{1}{c_{k-1}}, \quad b_3^{(k-1)} = \frac{b_2^{(k)}}{c_{k-1}(c_{k-1} - b_2^{(k)})},$$

$$b_2^{(k-1)} = b_1^{(k)} c_{k-1}^2,$$

$$b_{2p+1}^{(k-1)} = \frac{b_{2p}^{(k)}}{\left(c_{k-1} - \sum_{r=1}^{p-1} b_{2r}^{(k)} \right) \left(c_{k-1} - \sum_{r=1}^{p} b_{2r}^{(k)} \right)}, \tag{96.17}$$

$$b_{2p}^{(k-1)} = b_{2p-1}^{(k)} \left(c_{k-1} - \sum_{r=1}^{p-1} b_{2r}^{(k)} \right)^2. \quad [95.]$$

Again by comparison of (96.12) and (96.13) we obtain the relations

$$\begin{aligned} B_{2p}^{(k)}(z) &= B_{2p+1}^{(k-1)}(z), \\ zA_{2p}^{(k)}(z) &= A_{2p+1}^{(k-1)}(z) - c_{k-1} B_{2p+1}^{(k-1)}(z). \end{aligned} \tag{96.18}$$

By means of the fundamental recurrence formulas we then get:

$$\begin{aligned} B_{2p+1}^{(k)}(z) &= B_{2p+3}^{(k-1)}(z) + z a_{2p+1}^{(k)} B_{2p+1}^{(k-1)}(z), \\ zA_{2p+1}^{(k)}(z) &= A_{2p+3}^{(k-1)}(z) + z a_{2p+1}^{(k)} A_{2p+1}^{(k-1)}(z) \\ &\quad - c_{k-1}[B_{2p+3}^{(k-1)}(z) + z a_{2p+1}^{(k)} B_{2p+1}^{(k-1)}(z)]. \end{aligned} \tag{96.19}$$

Corresponding to the substitution (96.15) we now put

$$\begin{aligned} G_p^{(r)}(z) &= b_1^{(r)} b_2^{(r)} \cdots b_p^{(r)} A_p^{(r)}(z), \\ H_p^{(r)}(z) &= b_1^{(r)} b_2^{(r)} \cdots b_p^{(r)} B_p^{(r)}(z). \end{aligned} \tag{96.20}$$

The relations (96.18) and (96.19) then become, if we use (96.16),

$$H_{2p}^{(k)}(z) = \frac{H_{2p+1}^{(k-1)}(z)}{h_p^{(k-1)}},$$

$$G_{2p}^{(k)}(z) = \frac{G_{2p+1}^{(k-1)}(z) - c_{k-1}H_{2p+1}^{(k-1)}(z)}{zh_p^{(k-1)}},$$

$$h_p^{(q)} = \sum_{r=0}^{p} b_{2r+1}^{(q)}. \tag{96.21}$$

$$H_{2p+1}^{(k)}(z) = h_{p+1}^{(k-1)}H_{2p+2}^{(k-1)}(z) - H_{2p+3}^{(k-1)}(z),$$

$$zG_{2p+1}^{(k)}(z) = h_{p+1}^{(k-1)}G_{2p+2}^{(k-1)}(z) - G_{2p+3}^{(k-1)}(z) - c_{k-1}[h_{p+1}^{(k-1)}H_{2p+2}^{(k-1)}(z) - H_{2p+3}^{(k-1)}(z)].$$

THEOREM 96.2. *For $k = 0, 1, 2, \cdots$, the Padé approximant $f_{p,p+k}(z)$ is given by*

$$f_{p,p+k}(z) = \frac{M_k(z)G_{2p+k+1}(z) - N_k(z)G_{2p+k+2}(z)}{M_k(z)H_{2p+k+1}(z) - N_k(z)H_{2p+k+2}(z)}, \tag{96.22}$$

where $M_k(z)$ and $N_k(z)$ are polynomials in z whose coefficients are rational functions of the quantities $h_p^{(q)}$. [119.]

Proof. Since

$$f_{p,p}(z) = \frac{G_{2p+1}(z)}{H_{2p+1}(z)}, \quad p = 0, 1, 2, \cdots,$$

the formula (96.22) holds for $k = 0$ with $M_0(z) = 1$, $N_0(z) = 0$. Using induction, we shall assume that (96.22) holds for k and prove it for $k + 1$. By hypothesis, (96.22) holds for the series $c_1 + c_2z + c_3z^2 + \cdots$, and for k. That is, the approximant in the square $[p, p + k]$ of the Padé table for this series is given by

$$\frac{M_k^{(1)}(z)G_{2p+k+1}^{(1)}(z) - N_k^{(1)}(z)G_{2p+k+2}^{(1)}(z)}{M_k^{(1)}(z)H_{2p+k+1}^{(1)}(z) - N_k^{(1)}(z)H_{2p+k+2}^{(1)}(z)},$$

where the superscript indicates that all the b_p which are involved in the coefficients are replaced by $b_p^{(1)}$. If we multiply this expression by z and add c_0, we obtain a rational fraction whose numerator and denominator are of degrees not exceeding $p + k + 1$ and p, respectively, which is obviously the Padé approximant $f_{p,p+k+1}(z)$

for $P(z)$. By means of (96.21), this expression may be reduced to (96.22) with k replaced by $k + 1$, and, according as $k = 2r$ or $k = 2r + 1$:

$$\begin{aligned} M_{2r+1}(z) &= h_{p+r+1} M_{2r}^{(1)}(z), \\ N_{2r+1}(z) &= M_{2r}^{(1)}(z) + N_{2r}^{(1)}(z)/h_{p+r+1}, \\ M_{2r+2}(z) &= M_{2r+1}^{(1)}(z)/h_{p+r+1} - zN_{2r+1}^{(1)}(z), \\ N_{2r+2}(z) &= h_{p+r+1} N_{2r+1}^{(1)}(z). \end{aligned} \tag{96.23}$$

Therefore (96.22) holds with k replaced by $k + 1$, where $M_{k+1}(z)$ and $N_{k+1}(z)$ are polynomials in z whose coefficients are rational functions of the quantities $h_p^{(q)}$, and the theorem is proved.

From the recurrence formulas (96.23) we find that

$$\frac{M_{2r+1}(z)}{zN_{2r+1}(z)} = \cfrac{h_{p+r+1}}{\cfrac{h_{p+r}^{(1)}}{z - \cfrac{\cfrac{h_{p+r+1}}{h_{p+r}^{(2)}}}{\cfrac{h_{p+r}^{(1)}}{1 - \cfrac{\cfrac{h_{p+r-1}^{(3)}}{h_{p+r}^{(2)}}}{z - \cfrac{\cfrac{h_{p+r-1}^{(4)}}{h_{p+r-1}^{(3)}}}{1 - \cfrac{\cfrac{h_{p+r-2}^{(5)}}{h_{p+r-1}^{(4)}}}{z - \cfrac{}{1 - \cdot \, \cdot \, \cdot - \cfrac{\cfrac{h_{p+1}^{(2r-1)}}{h_{p+2}^{(2r-2)}}}{\cfrac{h_{p+1}^{(2r)}}{1 - \cfrac{h_{p+1}^{(2r-1)}}{z}}}}}}}}}}}}, \tag{96.24}$$

and

$$-\frac{N_{2r+2}(z)}{M_{2r+2}(z)} = \cfrac{+h_{p+r+1}}{\cfrac{h_{p+r+1}^{(1)}}{h_{p+r+1}} z - \cfrac{\cfrac{h_{p+r}^{(2)}}{h_{p+r+1}^{(1)}}}{1 - \cfrac{\cfrac{h_{p+r}^{(3)}}{h_{p+r}^{(2)}}}{z - \cfrac{\cfrac{h_{p+r-1}^{(4)}}{h_{p+r}^{(3)}}}{1 - \cfrac{}{z - \ddots - \cfrac{\cfrac{h_{p+1}^{(2r)}}{h_{p+2}^{(2r-1)}}}{\cfrac{h_{p+1}^{(2r+1)}}{h_{p+1}^{(2r)}} z - \cfrac{}{1}}}}}}} \qquad (96.25)$$

Inasmuch as the approximants $f_{p+k,p}(z)$ for $P(z)$ are the reciprocals of the approximants $F_{p,p+k}(z)$ for $Q(z) = 1/P(z)$, analogous results hold for the approximants $f_{p+k,p}(z)$. By (96.22) we have

$$f_{p+k,p}(z) = \frac{1}{F_{p,p+k}(z)} = \frac{M_k'(z)H_{2p+k+1}'(z) - N_k'(z)H_{2p+k+2}'(z)}{M_k'(z)G_{2p+k+1}'(z) - N_k'(z)G_{2p+k+2}'(z)}, \qquad (96.26)$$

where the primes denote that the polynomials of (96.22) have been formed for the series $Q(z) = 1/P(z)$. In particular, $G_p'(z)/H_p'(z)$ is the pth approximant of the S-fraction

$$\cfrac{1}{\beta_1 - \cfrac{z}{\beta_2 - \cfrac{z}{\beta_3 - \ddots}}} \qquad (96.27)$$

where, analogous to (96.15),

$$\alpha_0^{(r)} = \frac{1}{\beta_1^{(r)}}, \quad \alpha_p^{(r)} = \frac{1}{\beta_p^{(r)}\beta_{p+1}^{(r)}}.$$

On comparing the reciprocal of (96.27) with (96.6) with $k = 1$, we find that

$$G_p'(z) = (-1)^{p-1}H_{p-1}^{(1)}(z),$$

$$H_p'(z) = (-1)^{p-1}[c_0H_{p-1}^{(1)}(z) + zG_{p-1}^{(1)}(z)].$$

By (96.21), these may be written

$$\begin{aligned} G_{2p}'(z) &= H_{2p+1}(z) - h_pH_{2p}(z), \\ G_{2p+1}'(z) &= H_{2p+1}(z)/h_p, \\ H_{2p}'(z) &= G_{2p+1}(z) - h_pG_{2p}(z), \\ H_{2p+1}'(z) &= G_{2p+1}(z)/h_p. \end{aligned} \tag{96.28}$$

On substituting from (96.28) into (96.26) we obtain

$$f_{p+k,p}(z) = \frac{K_k(z)G_{2p+k+1}(z) - L_k(z)G_{2p+k+2}(z)}{K_k(z)H_{2p+k+1}(z) - L_k(z)H_{2p+k+2}(z)}, \tag{96.29}$$

where $K_k(z)$ and $L_k(z)$ are polynomials given by

$$\begin{aligned} K_{2k}(z) &= M_{2k}'(z)/h_{p+k} + zN_{2k}'(z), \\ L_{2k}(z) &= -N_{2k}'(z)h_{p+k}, \\ K_{2k+1}(z) &= M_{2k+1}'(z)h_{p+k+1}, \\ L_{2k+1}(z) &= -N_{2k+1}'(z)/h_{p+k+1} + M_{2k+1}'(z). \end{aligned} \tag{96.30}$$

Let

$$g_p^{(k)} = \sum_{r=0}^{p} \beta_{2r+1}^{(k)}, \quad (g_p^{(0)} = g_p).$$

Then we find that $g_p = 1/h_p$. Also, $\beta_p^{(1)} = -b_{p+1}$, $p = 1, 2, 3,$

$\cdots$. The $g_p^{(k)}$ may be computed by means of (96.16). By (96.28), (96.24) and (96.25), we now readily obtain

$$\frac{L_{2r+1}(z)}{K_{2r+1}(z)} = \cfrac{g_{p+r}^{(1)}}{z - \cfrac{\dfrac{g_{p+r}^{(2)}}{g_{p+r}^{(1)}}}{1 - \cfrac{\dfrac{g_{p+r-1}^{(3)}}{g_{p+r}^{(2)}}}{z - \cfrac{\dfrac{g_{p+r-1}^{(4)}}{g_{p+r-1}^{(3)}}}{1 - \cdot \cdot \cdot - \cfrac{\dfrac{g_{p+1}^{(2r-1)}}{g_{p+2}^{(2r-2)}}}{z - \cfrac{\dfrac{g_{p+1}^{(2r)}}{g_{p+1}^{(2r-1)}}}{1}}}}}}, \tag{96.31}$$

and

$$\frac{K_{2r+2}(z)}{zL_{2r+2}(z)} = \cfrac{-g_{p+r+1}^{(1)}}{z - \cfrac{\dfrac{g_{p+r}^{(2)}}{g_{p+r+1}^{(1)}}}{1 - \cfrac{\dfrac{g_{p+r}^{(3)}}{g_{p+r}^{(2)}}}{z - \cfrac{\dfrac{g_{p+r-1}^{(4)}}{g_{p+r}^{(3)}}}{1 - \cdot \cdot \cdot - \cfrac{\dfrac{g_{p+1}^{(2r)}}{g_{p+2}^{(2r-1)}}}{1 - \cfrac{\dfrac{g_{p+1}^{(2r+1)}}{g_{p+1}^{(2r)}}}{z}}}}}}. \tag{96.32}$$

We conclude the present section with the following theorem.

THEOREM 96.3. *In a normal Padé table, the approximant occupying the square* $[p, q]$ *has, in its simplest terms, numerator and denominator whose degrees are exactly* q *and* p, *respectively* [66].

Proof. We shall prove this for $f_{p,p+k}(z)$, using (96.10). Since

$$A_{p,p+k-1}(z)B_{p,p+k}(z) - A_{p,p+k}(z)B_{p,p+k-1}(z)$$
$$= z^k[A_{2p}{}^{(k)}(z)B_{2p+1}{}^{(k)}(z) - A_{2p+1}{}^{(k)}(z)B_{2p}{}^{(k)}(z)] = cz^{k+2p+1},$$

it follows that the only possible common factor of $A_{p,p+k}(z)$ and $B_{p,p+k}(z)$ is a power of z. Since $B_{p,p+k}(0) \neq 0$, there can be no common factor, so that $A_{p,p+k}(z)/B_{p,p+k}(z)$ is irreducible. Since the degrees of numerator and denominator are $p + k$ and p, respectively, the theorem is proved for this case. The proof for $f_{p+k,p}(z)$ can be made in the same way.

97. The Padé Table for the Series of Stieltjes. A power series (95.1) with real coefficients, such that the determinants $\Delta_p = \Delta_p{}^{(0)}$ and $\Omega_p = \Delta_p{}^{(1)}$, $p = 0, 1, 2, \cdots$ (cf. (96.1)) are all positive, is called a **series of Stieltjes.** We saw in § 87 that the conditions $\Delta_p > 0$, $\Omega_p > 0$ are necessary and sufficient for the existence of a bounded nondecreasing function $\phi(u)$, taking on infinitely many different values for $0 \leq u < +\infty$, such that

$$c_p = \int_0^{+\infty} u^p d\phi(u), \quad p = 0, 1, 2, \cdots.$$

This implies that the quadratic forms

$$\sum_{p,q=0}^{n} c_{p+q+k}X_pX_q = \int_0^{+\infty} u^k(X_0 + X_1u + \cdots + X_nu^n)^2 d\phi(u),$$
$$n, k = 0, 1, 2, \cdots,$$

are all positive definite. *Hence, the determinants* (96.1) *are all positive when* $P(z)$ *is a series of Stieltjes.* By Theorem 52.1 and the remark following that theorem, $P^{(k)}(z) = c_k + c_{k+1}z + c_{k+2}z^2 + \cdots$ has an S-fraction expansion

$$\cfrac{1}{b_1{}^{(k)} - \cfrac{z}{b_2{}^{(k)} - \cfrac{z}{b_3{}^{(k)} - \cdots}}}, \tag{97.1}$$

in which the $b_p^{(k)}$ are all positive. The reciprocal series $Q(z) = 1/P(z) = d_0 + d_1 z + d_2 z^2 + \cdots$ has the S-fraction expansion

$$\cfrac{1}{c_0 + \cfrac{z}{b_1^{(k)} - \cfrac{z}{b_2^{(k)} - \ddots}}}, \tag{97.2}$$

and therefore the determinants Δ_p and Ω_p formed for $Q(z)$ are different from zero. Also, $-Q^{(1)}(z)$ has the S-fraction expansion

$$\cfrac{1}{b_2 - \cfrac{z}{b_3 - \cfrac{z}{b_4 - \ddots}}}, \tag{97.3}$$

so that $-Q^{(1)}(z)$ is a series of Stieltjes. It follows that all the determinants (96.1) formed for $Q(z)$ are different from zero. Hence, we have proved

THEOREM 97.1. *The series of Stieltjes is normal* [109].

In order to connect with the work of § 87 we now replace z by $1/z$ and divide by z in the power series $P(z)$ and in all its Padé approximants. Put (cf. (96.20))

$$\begin{aligned} R_{2n}(z) &= (-z)^{n-1} G_{2n}\left(\frac{1}{z}\right), \\ R_{2n+1}(z) &= (-z)^{n} G_{2n+1}\left(\frac{1}{z}\right), \\ S_{2n}(z) &= (-z)^{n} H_{2n}\left(\frac{1}{z}\right), \\ S_{2n+1}(z) &= (-z)^{n+1} H_{2n+1}\left(\frac{1}{z}\right). \end{aligned} \tag{97.4}$$

Then $R_n(z)$ and $S_n(z)$ are the nth numerator and denominator of the S-fraction

$$\cfrac{1}{b_1 z - \cfrac{1}{b_2 - \cfrac{1}{b_3 z - \cfrac{1}{b_4 - \cdots}}}} \tag{97.5}$$

The Padé approximants (96.22) become, under the indicated change of variable,

$$\begin{aligned}
&\frac{1}{z} f_{p,p+2k}\left(\frac{1}{z}\right) \\
&= \frac{M_{2k}\left(\frac{1}{z}\right) R_{2p+2k+1}(z) - N_{2k}\left(\frac{1}{z}\right) R_{2p+2k+2}(z)}{M_{2k}\left(\frac{1}{z}\right) S_{2p+2k+1}(z) - N_{2k}\left(\frac{1}{z}\right) S_{2p+2k+2}(z)}, \\
&\frac{1}{z} f_{p,p+2k+1}\left(\frac{1}{z}\right) \\
&= \frac{N_{2k+1}\left(\frac{1}{z}\right) R_{2p+2k+3}(z) + zM_{2k+1}\left(\frac{1}{z}\right) R_{2p+2k+2}(z)}{N_{2k+1}\left(\frac{1}{z}\right) S_{2p+2k+3}(z) + zM_{2k+1}\left(\frac{1}{z}\right) S_{2p+2k+2}(z)}.
\end{aligned} \tag{97.6}$$

Since $P(z)$ is a series of Stieltjes, the numbers b_p are positive, and, by (96.16), the numbers $b_p^{(k)}$ are all positive. Moreover, if the series Σb_p converges, we see that the series $\sum_{(p)} b_p^{(k)}$ are convergent. The corresponding series associated with $Q(z)$ are also convergent, since, as we saw in § 96, $\beta_p^{(1)} = -b_{p+1}, p = 1, 2, 3, \cdots$. For the numbers $h_p^{(k)}$ and $g_p^{(k)}$ introduced in § 96 we then have

$$\begin{gathered}
\lim_{p=\infty} h_p^{(k)} = h^{(k)} > 0, \quad \lim_{p=\infty} g_p^{(k)} = -g^{(k)} < 0, \\
k = 1, 2, 3, \cdots, \quad \lim_{p=\infty} h_p = h > 0.
\end{gathered} \tag{97.7}$$

By Theorem 6.1, there exist four entire functions $r(z)$, $r_1(z)$, $s(z)$, $s_1(z)$, such that $r_1(z)s(z) - r(z)s_1(z) = 1$, and such that

$$\lim_{n=\infty} R_{2n}(z) = r(z), \quad \lim_{n=\infty} R_{2n+1}(z) = r_1(z),$$
$$\lim_{n=\infty} S_{2n}(z) = s(z), \quad \lim_{n=\infty} S_{2n+1}(z) = s_1(z). \tag{97.8}$$

Let $\theta_k(z)$ and $\phi_k(z)$ denote the kth numerator and denominator of the S-fraction

$$\cfrac{zh}{1 - \cfrac{zh^{(1)}/h}{1 - \cfrac{zh^{(2)}/h^{(1)}}{1 - \cfrac{zh^{(3)}/h^{(2)}}{1 - \cdots}}}} \tag{97.9}$$

By (96.24), (96.25) and (97.7), we then have

$$\lim_{p=\infty} \frac{zM_{2r+1}\left(\frac{1}{z}\right)}{N_{2r+1}\left(\frac{1}{z}\right)} = \frac{\theta_{2r+1}(z)}{\phi_{2r+1}(z)}$$

$$\lim_{p=\infty} \frac{N_{2r+2}\left(\frac{1}{z}\right)}{M_{2r+2}\left(\frac{1}{z}\right)} = -\frac{\theta_{2r+2}(z)}{\phi_{2r+2}(z)}.$$

Therefore, by (97.6) and (97.8),

$$\lim_{p=\infty} \frac{1}{z} f_{p,p+k}\left(\frac{1}{z}\right) = \frac{\phi_k(z)r_1(z) - \theta_k(z)r(z)}{\phi_k(z)s_1(z) - \theta_k(z)s(z)}, \tag{97.10}$$

$$k = 0, 1, 2, \cdots.$$

This is a meromorphic function, $f_k(z)$. Since $\theta_r(z)/\phi_r(z) \not\equiv \theta_s(z)/\phi_s(z)$ if $r \neq s$, it follows that $f_r(z) \not\equiv f_s(z)$ if $r \neq s$.

By means of (96.29) we find in like manner that

$$\lim_{p=\infty} \frac{1}{z} f_{p+k,p}\left(\frac{1}{z}\right) = \frac{\phi_k'(z)r_1(z) - \theta_k'(z)r(z)}{\phi_k'(z)s_1(z) - \theta_k'(z)s(z)}, \qquad (97.11)$$

where the $\phi_k'(z)$ and $\theta_k'(z)$ are polynomials. This is a meromorphic function $F_k(z)$; and $F_r(z) \not\equiv F_s(z)$ if $r \neq s$. It can be shown that the functions $f_k(z)$ all have $z = 0$ as a pole, while the functions $F_k(z)$ are all regular at $z = 0$. *We therefore conclude that when the series Σb_p is convergent, then every diagonal file of the Padé table converges, and no two different diagonal files have a common limit* [**117**].

We turn now to the case where the series Σb_p is **divergent.** One may show by means of examples that the series $\sum_{(p)} b_p^{(k)}$, $k = 1, 2, 3, \cdots$, may all diverge, or they may diverge for $k = 1, 2, 3, \cdots, r$, $r \geq 1$, and converge for $k > r$. In the first case, all the continued fractions (96.6) converge to a function $f(z)$ which is analytic over the entire plane excepting the whole or a part of the positive half of the real axis. Then, for $k = 0, 1, 2, \cdots$,

$$\lim_{p=\infty} f_{p,p+k}(z) = \lim_{p=\infty} f_{p+1,p}(z) = f(z).$$

If the series $\sum_{(p)} \beta_p^{(k)}$ also diverges, then

$$\lim_{p=\infty} f_{p+k,p}(z) = f(z), \quad k = 2, 3, 4, \cdots.$$

If $\sum_{(p)} b_p^{(k)}$ diverges for $k \leq r$ and converges for $k > r$, then

$$\lim_{p=\infty} f_{p,p+k}(z) = \lim_{p=\infty} f_{p+1,p}(z) = f(z),$$

for $k = 0, 1, 2, \cdots, r$. For $k > r$, the limits

$$\lim_{p=\infty} f_{p,p+k}(z), \quad k = r+1, \quad r+2, \quad r+3, \quad \cdots,$$

exist and are distinct analytic functions. An analogous statement holds for the sequences $f_{p+k,p}(z)$, $p = 0, 1, 2, \cdots$, $(k \geq 1)$.

98. General Theorems on the Padé Table. We shall now investigate properties of the Padé table for an arbitrary power series (95.1).

THEOREM 98.1. *Let* $f_{m,n}(z) = A_{m,n}(z)/B_{m,n}(z)$ *be the Padé approximant for* $P(z)$ *occupying the square* $[m, n]$, *and suppose that* $A_{m,n}(z)$ *and* $B_{m,n}(z)$ *have no zero in common. Let the degrees of* $A_{m,n}(z)$ *and* $B_{m,n}(z)$ *be exactly* q *and* p, *respectively. There exists an integer* r, $r \geq 0$, *such that*

$$P(z)B_{m,n}(z) - A_{m,n}(z) = k_{m,n}z^{p+q+r+1} + k_{m,n}'z^{p+q+r+2} + \cdots, \quad (98.1)$$

where $k_{m,n} \neq 0$, *or else the left-hand member of* (98.1) *is identically equal to zero. In the latter event, we set* $r = \infty$. *Moreover, in the Padé table, the squares*

$$\begin{array}{llll} [p, q], & [p, q+1], & \cdots, & [p, q+r] \\ [p+1, q], & [p+1, q+1], & \cdots, & [p+1, q+r] \\ & \cdot \quad \cdot \quad \cdot & & \\ [p+r, q], & [p+r, q+1], & \cdots, & [p+r, q+r] \end{array} \quad (98.2)$$

are all occupied by the approximant $f_{m,n}(z)$, *while no other square in the table is occupied by* $f_{m,n}(z)$. [66.]

Note. We shall call (98.2) a **block of order** r. If $r = \infty$ we shall understand that the block extends to infinity to the right and downward.

Proof. By hypothesis, there exists an integer h, $h \geq 0$, and a polynomial $g_0 + g_1z + \cdots + g_kz^k$, $g_0 \neq 0$, where $h + k + q \leq n$, $h + k + p \leq m$, such that

$$[P(z)B_{m,n}(z) - A_{m,n}(z)]z^h(g_0 + g_1z + \cdots + g_kz^k) = (z^{m+n+1}).$$

Inasmuch as $g_0 \neq 0$, it follows that

$$P(z)B_{m,n}(z) - A_{m,n}(z) = (z^{m+n+1-h}).$$

Now, $p + h \leq m$, $q + h \leq n$, $m + n + 1 - h \geq p + q + h + 1$, and consequently there exists an integer r, $r \geq 0$, such that $m + n + 1 - h = p + q + 1 + r$. We then have the inequalities:

$$p + h \leq m, \quad q + h \leq n, \quad p + q + r + h \geq m + n, \quad (98.3)$$

so that

$$p + q + r + h \geq m + n \geq \begin{cases} h + q + h, \\ h + p + n, \end{cases}$$

or

$$p + r \geq m, \quad q + r \geq n. \quad (98.4)$$

From (98.3) and (98.4) it follows that m and n can have only the values $m = p, p + 1, p + 2, \cdots, p + r, n = q, q + 1, q + 2, \cdots, q + r$; and one may verify immediately that these values are actually allowable. Therefore, $f_{p+i,q+j}(z) = f_{p,q}(z)$ if, and only if, i and j have independently the values $0, 1, \cdots, r$, where

$$P(z)B_{m,n}(z) - A_{m,n}(z) = (z^{p+q+r+1}). \tag{98.5}$$

This last may hold for $r = 0, 1, 2, \cdots$, or there may be a largest r for which it holds.

If (98.5) holds for $r = 0, 1, 2, \cdots$, then we shall say that $f_{p,q}(z)$ is a Padé approximant of **infinite order**, while if there is a largest $r = r_0$ for which (98.5) holds, then we shall say that $f_{p,q}(z)$ is a Padé approximant of **order** r_0.

THEOREM 98.2. *Let*

$$\Delta_{m,n} = \begin{vmatrix} c_{n-m}, & c_{n-m+1}, & \cdots, & c_n \\ c_{n-m+1}, & c_{n-m+2}, & \cdots, & c_{n+1} \\ & \cdot \quad \cdot \quad \cdot & & \\ c_n, & c_{n+1}, & \cdots, & c_{n+m} \end{vmatrix}, \quad m, n = 0, 1, 2, \cdots, \tag{98.6}$$

where $c_p = 0$ if $p < 0$. A necessary and sufficient condition for the Padé table for $P(z)$ to contain the block (98.2) of order r is that the following five conditions hold:

$$\begin{aligned} &\text{(a)} && \Delta_{p-1,q} \neq 0, \\ &\text{(b)} && \Delta_{p-1,q+1} \neq 0, \\ &\text{(c)} && \Delta_{p,q} \neq 0, \\ &\text{(d)} && \Delta_{p+k,q+k+1} = 0, \quad k = 0, 1, 2, \cdots, r-1, \\ &\text{(e)} && \Delta_{p+r,q+r+1} \neq 0. \quad [14.] \end{aligned} \tag{98.7}$$

Note. If $r = 0$, the condition (d) is not present. If $p = 0$, we shall set

$$\Delta_{-1,q} = 1, \tag{98.8}$$

so that (a) and (b) always hold for $p = 0$.

Proof. The Padé table contains the block (98.2) if, and only if (cf. Theorem 98.1),

$$
\begin{aligned}
&(\text{a}') && f_{p,q}(z) \neq f_{p-1,q-1}(z), \\
&(\text{b}') && f_{p,q}(z) \neq f_{p-1,q}(z), \\
&(\text{c}') && f_{p,q}(z) \neq f_{p,q-1}(z), \\
&(\text{d}') && f_{p+k,q+k}(z) = f_{p,q}(z), \quad k = 1, 2, 3, \cdots, r, \\
&(\text{e}') && f_{p+r+1,q+r+1}(z) \neq f_{p,q}(z).
\end{aligned}
\tag{98.9}
$$

It is convenient to include the condition (a′), although it is a consequence of (b′) and (c′). Let $f_{p,q}(z) = A/B$, where $A = a_0 + a_1 z + \cdots + a_q z^q$, $B = b_0 + b_1 z + \cdots + b_p z^p$, and let

$$
\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_q \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad \mathbf{C}_{m,n} = \begin{bmatrix} c_n, & c_{n-1}, & \cdots, & c_{n-m} \\ c_{n+1}, & c_n, & \cdots, & c_{n-m+1} \\ & \cdot & \cdot & \cdot \\ c_{n+m-1}, & c_{n+m-2}, & \cdots, & c_{n-1} \end{bmatrix}.
$$

The equation $P(z)B(z) - A(z) = (z^{p+q+1})$ can then be expressed in the form

$$
\mathbf{C}_{p,q+1}\mathbf{b} = \mathbf{0}, \quad \mathbf{C}_{q+1,0}\mathbf{b} = \mathbf{a}. \tag{98.10}
$$

If $\Delta_{p-1,q} = 0$, then the first equation (98.10) has a solution $\mathbf{b} \neq \mathbf{0}$ where $b_0 = 0$. Then the value of $\mathbf{a}$ given by the second equation has $a_0 = 0$. This means that $A(z)$ and $B(z)$ have the common factor z, and $P(z)[B(z)/z] - [A(z)/z] = (z^{p+q})$, i.e., $f_{p,q}(z) = f_{p-1,q-1}(z)$. Thus, (a) is necessary for (a′).

If $\Delta_{p-1,q+1} = 0$, then (98.10) has a solution $\mathbf{b} \neq \mathbf{0}$ in which $b_p = 0$. Therefore $f_{p-1,q}(z) = f_{p,q}(z)$. Hence (b) is necessary for (b′).

If $\Delta_{p,q} = 0$, there is a solution with $\mathbf{b} \neq \mathbf{0}$, $a_q = 0$, so that $f_{p,q-1}(z) = f_{p,q}(z)$. Thus (c) is necessary for (c′).

The condition (e) is necessary for (e′) for the same reason that (a) is necessary for (a′).

It remains to be shown that (d) is necessary for (d′). In order for (d′) to hold (cf. the proof of Theorem 98.1), it is necessary that

$$P(z)z^kB(z) - z^kA(z) = (z^{p+q+2k+1}), \quad k = 1, 2, 3, \cdots, r. \qquad (98.11)$$

Let $\mathbf{b}^{(k)}$ denote the matrix obtained from $\mathbf{b}$ by inserting k zeros above b_0. Then the condition (98.11) can be written

$$\mathbf{C}_{p+k,q+k+1}\mathbf{b}^{(k)} = \mathbf{0}, \quad k = 1, 2, 3, \cdots, r,$$

where $b^{(k)} \neq 0$. But this is impossible unless (d) holds.

We now prove that (98.7) is sufficient for (98.9).

From (a) we conclude that the equation $\mathbf{C}_{p,q+1}\mathbf{b} = \mathbf{0}$ has a solution $\mathbf{b} \neq \mathbf{0}$, and that every other solution is a constant multiple of this particular solution. Moreover, b_0 is proportional to $\Delta_{p-1,q}$ and is not zero if $\mathbf{b} \neq \mathbf{0}$. The fraction A/B determined by (98.10) with $\mathbf{b} \neq \mathbf{0}$ is irreducible. For if $A/B \equiv A'/B'$, where A' and B' are of lower degree by $h > 0$ than A and B, then $P(z)z^hB'(z) - z^hA'(z) = (z^{p+q+1})$, and therefore $z^hA'(z)$ and $z^hB'(z) \not\equiv 0$ can be determined by solving (98.10). But, in this solution we would have $b_0 = b_1 = \cdots = b_{h-1} = 0$, which we have seen is impossible. From (b) and (c) it follows that A and B are exactly of degree q and p, respectively. Hence, (a′), (b′) and (c′) all hold.

We now let $\mathbf{b} \neq \mathbf{0}$ be a fixed solution of (98.10). Let

$$f_{p+k,q+k}(z) = \frac{a_0^{(k)} + a_1^{(k)}z + \cdots + a_{q+k}^{(k)}z^{q+k}}{b_0^{(k)} + b_1^{(k)}z + \cdots + b_{p+k}^{(k)}z^{p+k}} = \frac{A^{(k)}(z)}{B^{(k)}(z)}.$$

Let $\mathbf{a}^{(k)}$ and $\mathbf{b}^{(k)}$ be the one-column matrices analogous to $\mathbf{a}$ and $\mathbf{b}$, formed with the coefficients of $A^{(k)}(z)$ and $B^{(k)}(z)$. The condition $P(z)B^{(k)}(z) - A^{(k)}(z) = (z^{p+q+2k+1})$ may now be written

$$\mathbf{C}_{p+k,q+k+1}\mathbf{b}^{(k)} = \mathbf{0}, \quad \mathbf{C}_{q+k+1,0}\mathbf{b}^{(k)} = \mathbf{a}^{(k)}. \qquad (98.12)$$

Since $\Delta_{p,q+1} = 0$, we see that (98.12) with $k = 1$ has the solution $\mathbf{a}^{(1)}$, $\mathbf{b}^{(1)}$ in which $b_0^{(1)} = 0$, $b_s^{(1)} = b_{s-1}$, $s = 1, 2, 3, \cdots, p + 1$, $a_0^{(1)} = 0$, $a_s^{(1)} = a_{s-1}$, $s = 1, 2, 3, \cdots, q + 1$. Hence it follows that $A^{(1)}(z)/B^{(1)}(z) \equiv A(z)/B(z)$. Then, since $\Delta_{p+1,q+2} = 0$, it follows that for $k = 2$, (98.12) has the solution $\mathbf{a}^{(2)}$, $\mathbf{b}^{(2)}$ where $b_0^{(2)} = 0$, $b_s^{(2)} = b_{s-1}^{(1)}$, $s = 1, 2, 3, \cdots, p+2$, $a_0^{(2)} = 0$, $a_s^{(2)} = a_{s-1}^{(1)}$, $s = 1, 2, 3, \cdots, q+2$, where $\mathbf{a}^{(1)}, \mathbf{b}^{(1)} \neq \mathbf{0}$ is *some* solution of (98.12) for $k = 1$. Consequently, $A^{(2)}(z)/B^{(2)}(z) \equiv A^{(1)}(z)/B^{(1)}(z) \equiv A(z)/B(z)$. On continuing this argument, we conclude that (d′) holds.

Finally, (e′) holds. For if not we would have

$$P(z)z^{r+1}B(z) - z^{r+1}A(z) = (z^{p+q+2r+3}),$$

which is impossible by virtue of (e).

This completes the proof of Theorem 98.2.

The preceding argument contains the proof of the following theorem.

THEOREM 98.3. *The relations*

$$\begin{aligned} &(1) \qquad f_{s-1,t-1}(z) \neq f_{s,t}, \\ &(2) \qquad f_{s+k,t+k}(z) = f_{s,t}(z), \quad k = 1, 2, 3, \cdots, r, \\ &(3) \qquad f_{s,t}(z) \neq f_{s+r+1,t+r+1}(z), \end{aligned} \tag{98.13}$$

hold if, and only if,

$$\begin{aligned} &(1') \qquad \Delta_{s-1,t} \neq 0, \\ &(2') \qquad \Delta_{s+k,t+k+1} = 0, \quad k = 0, 1, 2, \cdots, r-1, \\ &(3') \qquad \Delta_{s+r,t+r+1} \neq 0, \end{aligned} \tag{98.14}$$

hold [14].

In fact, we saw that (1′), (2′) and (3′) are necessary for (1), (2), and (3), respectively. Conversely, (1′) is sufficient for $A(z)/B(z)$ to be irreducible, and for the numerator *or* the denominator to be exactly of degree t or s, respectively, so that (1) holds. The proofs of (2) and (3) are the same as before.

It follows from Theorem 98.3 and Theorem 96.3 that a *necessary* condition for the Padé table to be normal is that

$$\Delta_{m,n} \neq 0, \quad m, n = 0, 1, 2, \cdots. \tag{98.15}$$

This condition is also *sufficient*. For if (98.15) holds, then the determinants (96.1), being contained among the determinants $\Delta_{m,n}$, are different from zero. Let $\Delta_{m,n}'$ denote the determinants

$\Delta_{m,n}$ formed with the coefficients of the reciprocal series $Q(z) = 1/P(z)$. Then, as Hadamard [25] showed,

$$\Delta_{p-1,n+p}{}' = \frac{(-1)^{3[n/2]+p}\Delta_{n+p-1,p}}{c_0^{\,2p+n+2}}, \qquad (98.16)$$

so that the determinants (96.1) formed for $Q(z)$ must be different from zero. Hence, we have:

THEOREM 98.4. *The Padé table is normal if, and only if,* (98.15) *holds* [66].

When (98.15) holds, then every approximant $f_{p,q}(z)$, when in simplest terms, has numerator and denominator exactly of degree q and p, respectively. Therefore, by Theorems 96.3 and 98.4 we have:

THEOREM 98.5. *The Padé table is normal if, and only if, every approximant $f_{p,q}(z)$, when in simplest terms, has numerator and denominator exactly of degree q and p, respectively* [66].

99. C-fractions. We have seen that a power series has an S-fraction expansion only when the determinants Δ_p and Ω_p are all different from zero, and has a J-fraction expansion only when the Δ_p are all different from zero. We shall now introduce a type of continued fraction expansion which exists for arbitrary power series. We shall assume for the sake of simplicity that the power series has constant term different from zero:

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \quad (c_0 \neq 0). \qquad (99.1)$$

The continued fractions are of the form

$$c_0 + \cfrac{a_1 z^{\alpha_1}}{1 + \cfrac{a_2 z^{\alpha_2}}{1 + \cfrac{a_3 z^{\alpha_3}}{1 + \cdots}}} \qquad (99.2)$$

in which the a_p are complex constants different from zero, and the α_p are positive integers. This is called a "corresponding type" continued fraction,[30] or simply a **C-fraction.** By a **terminating C-fraction** we shall understand a continued fraction of the form (99.2) with but a finite number of partial quotients.

We shall show that there is a one-to-one correspondence between C-fractions and power series (99.1) which do not represent rational functions of z, and also a one-to-one correspondence between terminating C-fractions and power series (99.1) representing rational functions of z.

Let us suppose first that we are given a C-fraction (99.2). We denote its pth numerator and denominator by $A_p(z)$ and $B_p(z)$, respectively, and put

$$h_p = \alpha_1 + \alpha_2 + \cdots + \alpha_{p+1}.$$

It follows from the determinant formula

$$A_p(z)B_{p+1}(z) - A_{p+1}(z)B_p(z) = (-1)^{p+1}a_1a_2 \cdots a_{p+1}z^{h_p}, \quad (99.3)$$

just as in the case of J-fractions or S-fractions, that there is determined uniquely a power series $P(z)$ such that the power series expansion in ascending powers of z of $A_p(z)/B_p(z)$ agrees term by term with the power series $P(z)$ for a number of consecutive terms beginning with the first which increases with p. In fact,

$$P(z)B_p(z) - A_p(z) = (-1)^p a_1a_2 \cdots a_{p+1}z^{h_p} + \cdots. \quad (99.4)$$

This statement holds also for terminating C-fractions, except that in this case there is a k for which $P(z)B_k(z) - A_k(z) \equiv 0$.

We shall call this uniquely determined power series the **power series expansion** of the C-fraction or of the terminating C-fraction.

THEOREM 99.1. *The power series expansion of a C-fraction cannot represent a rational function of z.* **[54.]**

Proof. Let it be supposed that $P(z) = N(z)/D(z)$, a rational function of z. Then by (99.4),

$$N(z)B_p(z) - D(z)A_p(z) = (z^{h_p}).$$

Let s_p and t_p denote the degrees of $A_p(z)$ and $B_p(z)$, respectively. Then the degree of $N(z)$ must be at least $h_p - t_p$, or that of $D(z)$ at least $h_p - s_p$. Now, one may readily show by mathematical

[30] These continued fractions were first investigated by Leighton and Scott [54]. They showed that every power series has a unique C-fraction expansion. Scott and Wall [85] and Frank [14] considered the connection of these C-fractions with the Padé table.

induction, using the fundamental recurrence formulas, that $h_p - s_p \geq [(p+2)/2]$, and that a similar inequality holds for $h_p - t_p$. This shows that the assumption that $P(z)$ is a rational function is untenable.

We shall now start with a power series (99.1), and show that there is uniquely determined a C-fraction or a terminating C-fraction, of which it is the power series expansion. There is no restriction upon the coefficients $c_0, c_1, c_2, \cdots$ of the power series; it may diverge for every value of $z \neq 0$, or may even be a polynomial.

If $c_p = 0$ for $p > 0$, then $P(z) = c_0$, which is a terminating C-fraction. If the c_p are not all zero for $p > 0$, let c_k be the first which is not zero. Put $a_1 = c_k, \alpha_1 = k$, and the fraction starts out

$$c_0 + \frac{a_1 z^{\alpha_1}}{1}. \tag{99.5}$$

If $c_p = 0$ for $p > k$, then $P(z)$ is equal to the terminating C-fraction (99.5). If the c_p are not all zero for $p > k$, let c_{k+r} be the first which is not zero. Put $a_2 = -c_{k+r}/c_k$, $\alpha_2 = r$, and at this step we have

$$c_0 + \cfrac{a_1 z^{\alpha_1}}{1 + \cfrac{a_2 z^{\alpha_2}}{1}}. \tag{99.6}$$

The three denominators of (99.6) are

$$B_0(z) = 1, \quad B_1(z) = 1, \quad B_2(z) = 1 + a_2 z^{\alpha_2}.$$

If we put $B_p(z) = \delta_{p,0} + \delta_{p,1}z + \delta_{p,2}z^2 + \cdots$, then the formulas for determining the a_p and α_p may be written in matrix form as follows [**14**].

$$(c_n, c_{n-1}, c_{n-2}, \cdots) \begin{bmatrix} \delta_{p,0} \\ \delta_{p,1} \\ \delta_{p,2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{cases} 0 \quad \text{if} \quad \alpha_0 + \alpha_1 + \cdots + \alpha_p < n < \alpha_1 \\ \quad + \alpha_2 + \cdots + \alpha_{p+1}, \quad (\alpha_0 = 0); \\ (-1)^p a_1 a_2 \cdots a_{p+1} \quad \text{if} \quad n = \alpha_1 \\ \quad + \alpha_2 + \cdots + \alpha_{p+1}. \end{cases} \tag{99.7}$$

Explanation. Supposing that $a_1, a_2, \cdots, a_p, \alpha_1, \alpha_2, \cdots, \alpha_p$ have been determined, then α_{p+1} is *defined* as the least positive integer s such that the product in the left-hand member of (99.7) is not zero for $n = \alpha_1 + \alpha_2 + \cdots + \alpha_p + s$. If the value of this product is c, then a_{p+1} is determined by the relation $c = (-1)^p a_1 a_2 \cdots a_{p+1}$. If $P(z)$ does not represent a rational function of z, it is evident that a C-fraction is determined by this procedure. We must show that the power series expansion of this C-fraction is the given power series $P(z)$. If $P(z)$ is a rational function of z, we shall find that the process leads to a terminating C-fraction equal to $P(z)$.

We suppose first that $P(z)$ is not a rational function of z. We are to show that the C-fraction determined by (99.7) in the manner indicated satisfies (99.4) for $p = 0, 1, 2, \cdots$. It is immediately evident that this holds for $p = 0$ and $p = 1$. Supposing that (99.4) holds for $p < m$, $m \geq 2$, we shall prove it for $p = m$. We have:

$$
\begin{aligned}
P(z)B_m(z) - A_m(z)& \\
&= P(z)B_{m-1}(z) - A_{m-1}(z) + a_m z^m[P(z)B_{m-2}(z) - A_{m-2}(z)] \\
&= (-1)^{m-1}a_1a_2 \cdots a_m z^{h_{m-1}} + \text{(higher powers)} \\
&\qquad + (-1)^{m-2}a_1a_2 \cdots a_m z^{h_{m-1}} + \text{(higher powers)} \\
&= (z^{1+h_{m-1}}).
\end{aligned}
$$

Thus the formal power series $P(z)B_m(z) - A_m(z)$ contains no power of z of degree less than $1 + h_{m-1} = 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_m$. Now, it is easy to see that the degree of $A_m(z)$ is less than $\alpha_1 + \alpha_2 + \cdots + \alpha_m$, Therefore, the first nonvanishing term in the power series $P(z)B_m(z) - A_m(z)$ is equal to the first nonvanishing term in the product $P(z)B_m(z)$. By (99.7), this term is $(-1)^m a_1 a_2 \cdots a_{m+1} z^{h_m}$. Consequently, (99.4) holds for $p = m$, and therefore the algorithm (99.7) is established for the case where $P(z)$ is not a rational function of z.

If $P(z)$ is a rational function of z, it follows from Theorem 99.1 that there must be a last a_p, say, a_m, which can be determined by means of (99.7). Then it is clear that (99.4) holds for $p = 0$,

$1, 2, \cdots, m-1$, while $P(z)B_m(z) - A_m(z) \equiv 0$. Therefore,

$$P(z) \equiv c_0 + \cfrac{a_1 z^{\alpha_1}}{1 + \cfrac{a_2 z^{\alpha_2}}{1 + \cdots + \cfrac{a_m z^{\alpha_m}}{1}}}. \tag{99.8}$$

If two C-fractions have one and the same power series expansion $P(z)$, they must evidently agree in the constant term c_0. Assuming that they agree up to and including the kth partial quotient, it is easy to prove that they must agree in the $(k+1)$th partial quotient, and hence they agree throughout.

If we put $A_p(z) = \sigma_{p,0} + \sigma_{p,1}z + \sigma_{p,2}z^2 + \cdots$, and equate coefficients of corresponding powers of z from the 0th to the s_pth, inclusive, in (99.4), where s_p is the degree of $A_p(z)$, we obtain the following formula for determining $A_p(z)$:

$$(\sigma_{0,p}, \sigma_{1,p}, \sigma_{2,p}, \cdots) = (\delta_{0,p}, \delta_{1,p}, \delta_{2,p}, \cdots) \cdot \begin{pmatrix} c_0, & c_1, & c_2, & \cdots \\ 0, & c_0, & c_1, & \cdots \\ 0, & 0, & c_0, & \cdots \\ & \cdot & \cdot & \cdot \end{pmatrix}. \tag{99.9}$$

We remark that the relations (99.7) serve also to determine the power series $P(z)$ when the C-fraction is given.

Example 1. Let $P(z) = 1 - z^2 + z^4$, a polynomial. We find that $a_1 = -1$, $\alpha_1 = 2$, $a_2 = 1$, $\alpha_2 = 2$. Then, $B_2 = 1 + z^2$, and, by (99.7) with $p = 2$, $n > \alpha_1 + \alpha_2 = 4$, we have

$$(0, 1, 0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0, \quad (0, 0, 1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 = a_1 a_2 a_3 = -a_3,$$

so that $a_3 = -1$, $\alpha_2 = 2$, $B_3 = 1$, and the process terminates. Hence, we have

$$1 - z^2 + z^4 = 1 - \cfrac{z^2}{1 + \cfrac{z^2}{1 - \cfrac{z^2}{1}}}.$$

Example 2. Consider the following C-fraction:

$$1+\cfrac{z}{1+\cfrac{z^2}{1+\cfrac{z^3}{1+\cdots}}}, \tag{99.10}$$

in which $a_p = 1$, $\alpha_p = p$, $p = 1, 2, 3, \cdots$. We first compute the first few denominators:

$$B_0(z) = 1, \quad B_1(z) = 1, \quad B_2(z) = 1 + z^2,$$
$$B_3(z) = 1 + z^2 + z^3, \cdots.$$

By means of (99.7) we then readily find that the power series expansion $P(z)$ of (99.10) is, to 31 terms:

$$\begin{aligned} P(z) = 1 + z - z^3 + z^5 + z^6 - z^7 - 2z^8 + 2z^{10} + 2z^{11} - z^{12} \\ - 3z^{13} - z^{14} + 3z^{15} + 3z^{16} - 2z^{17} - 5z^{18} - z^{19} + 6z^{20} \\ + 5z^{21} - 3z^{22} - 6z^{23} - 2z^{24} + 8z^{25} + 7z^{26} - 5z^{27} \\ - 12z^{28} - 2z^{29} + 13z^{30} + \cdots. \end{aligned} \tag{99.11}$$

Example 3. Let $P(z) = 1 + zP'(z)$, where $P(z)$ is the series (99.11). We find by means of (99.7) that the C-fraction for $P'(z)$ is, to 8 partial quotients,

$$1-\cfrac{z^2}{1+\cfrac{z^2}{1-\cfrac{z}{1+\cfrac{z}{1-\cfrac{z^3}{1+\cfrac{z^3}{1-\cfrac{z^2}{1+\cfrac{z^2}{1+\cdots}}}}}}}} \tag{99.12}$$

One might conjecture that the next partial numerators are $-z^4, +z^4, -z^3, +z^3, -z^5, \cdots$. It is not difficult to prove that this conjecture is correct. This can be done by multiplying (99.12) by z, adding 1, and taking the even part of the resulting continued fraction. This gives a continued fraction whose sequence of approximants is easily seen to be the same as the sequence of approximants of (99.10) with the 0th approximant omitted. It then follows immediately that the power series expansion of (99.10) is $1 + zP'(z)$, where $P'(z)$ is the power series expansion of (99.12).

It can be shown that the corresponding result holds for any C-fraction whose exponents α_p satisfy the inequalities

$$\alpha_2 + \alpha_4 + \cdots + \alpha_{2p+2} > \alpha_3 + \alpha_5 + \cdots + \alpha_{2p+1} > \alpha_2 + \alpha_4 + \cdots + \alpha_{2p},$$

$$p = 1, 2, 3, \cdots. \quad [\mathbf{14.}]$$

Example 4. Let the coefficients a_p and exponents α_p in the C-fraction (99.2) satisfy the relations

$$\left.\begin{aligned} a_{p\cdot 2^{q+1}-2^q+1} &= (-1)^{p-1} a_{2^q+1}, \\ \alpha_{p\cdot 2^{q+1}-2^q+1} &= \alpha_{2^q+1}, \end{aligned}\right\} \quad p, q = 1, 2, 3, \cdots.$$

Then the power series expansion of the C-fraction is $c_0 + c_1 z^{\lambda_1} + c_2 z^{\lambda_2} + c_3 z^{\lambda_3} + \cdots$, where

$$\begin{aligned} &\lambda_1 = \alpha_1, \quad \lambda_p = 2\lambda_{p-1} - \alpha_1 + \alpha_{2^{p-1}+1}, \quad p = 2, 3, 4, \cdots, \\ &c_1 = a_1, \quad c_2 = \frac{{c_1}^2 a_3}{a_1}, \quad c_p = -\frac{{c_{p-1}}^2 a_{2^{p-1}+1}}{a_1}, \quad p = 3, 4, \cdots. \end{aligned} \tag{99.13}$$

Conversely, if we are given a power series with gaps such that (99.13) holds, then the C-fraction can be written down by means of the above formulas. For the proof, see [**85**, § 3].

100. Regular C-fractions and Power Series. A C-fraction and its power series expansion are called **regular** if every approximant of the C-fraction is a Padé approximant of the power series. This property is dependent upon the degrees s_p and t_p of the numerators and denominators of the C-fraction, and upon the exponents α_p. In fact, if we compare (95.3) with (99.4) we find immediately that the C-fraction (99.2) is regular if, and only if, there exist integers r_p such that

$$\begin{gathered} \alpha_1 + \alpha_2 + \cdots + \alpha_{p+1} = r_p + s_p + t_p + 1, \\ r_p \geq 0, \quad p = 0, 1, 2, \cdots. \end{gathered} \tag{100.1}$$

If this condition is satisfied, it follows from Theorem 98.1 that the pth approximant $A_p(z)/B_p(z)$ of the C-fraction occupies all the squares of a block Q_p of order r_p in the Padé table, and occurs nowhere else in the table. We shall investigate properties of the geometrical configuration in the Padé table which is formed by the sequence of blocks Q_0, Q_1, Q_2, $\cdots$.

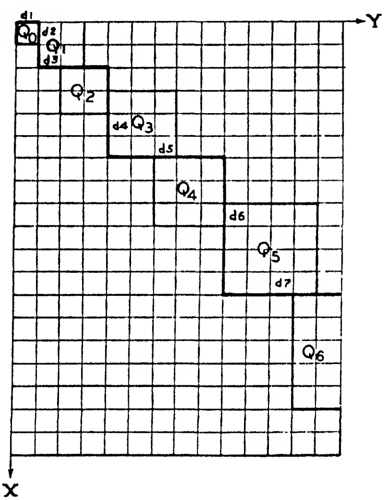

FIG. 14.

If $p = 0$, then $s_0 = t_0 = 0$, so that the order of Q_0 is $r_0 = \alpha_1 - 1 \geq 0$. The block Q_0 has vertices $(0, 0)$, $(0, \alpha_1)$, $(\alpha_1, 0)$, (α_1, α_1). Since $s_1 = \alpha_1$, $t_1 = 0$, then $r_1 = \alpha_2 - 1 \geq 0$. The vertices of Q_1 are $(0, \alpha_1)$, $(0, \alpha_1+\alpha_2)$, (α_2, α_1), $(\alpha_2, \alpha_1 + \alpha_2)$. Since $t_2 = \alpha_2$, and $s_2 < \alpha_1 + \alpha_2$, it follows that Q_2 lies below Q_1. Moreover, it is easily seen that the lower side of Q_1 and the upper side of Q_2 have a line segment in common. We shall now proceed to show that the blocks Q_p fit together as shown in Fig. 14.

The co-ordinates of the vertices of Q_p are

$$(t_p, s_p), \quad (t_p, s_p + r_p + 1), \quad (t_p + r_p + 1, s_p),$$
$$(t_p + r_p + 1, s_p + r_p + 1).$$

For all points on the diagonal connecting the second and third of these vertices, the sum of the co-ordinates is $s_p + t_p + r_p + 1$. By (100.1), this sum increases with increasing p. Hence, as Q_p and Q_{p+1} do not overlap, it follows that for any given p either

$$s_{p+1} > s_p + r_p, \tag{100.2}$$

or

$$t_{p+1} > t_p + r_p. \tag{100.3}$$

By the fundamental recurrence formulas it then follows that

$$s_{p+1} = s_{p-1} + \alpha_{p+1}, \tag{100.4}$$

or

$$t_{p+1} = t_{p-1} + \alpha_{p+1}, \tag{100.5}$$

according as (100.2) or (100.3), respectively, holds. Therefore,

$$s_{p+1} - (s_p + r_p) = t_p - (t_{p-1} + r_{p-1}) > 0, \tag{100.6}$$

if (100.2) holds, while

$$t_{p+1} - (t_p + r_p) = s_p - (s_{p-1} + r_{p-1}) > 0, \tag{100.7}$$

if (100.3) holds. One may now readily show by mathematical induction that

$$\begin{matrix} s_{2p+1} > s_{2p} + r_{2p}, & s_{2p+2} \leq s_{2p+1} + r_{2p+1}, \\ t_{2p+1} \leq t_{2p} + r_{2p}, & t_{2p+2} > t_{2p+1} + r_{2p+1}, \end{matrix} \quad p = 0, 1, 2, \cdots. \tag{100.8}$$

Hence, by (100.4) and (100.5),

$$\begin{aligned} s_{2p-1} &= \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1}, \\ t_{2p} &= \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \end{aligned} \quad p = 1, 2, 3, \cdots. \tag{100.9}$$

Then, by (100.1),

$$\begin{aligned} 1 + s_{2p} + r_{2p} &= \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1}, \\ 1 + t_{2p-1} + r_{2p-1} &= \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \end{aligned} \tag{100.10}$$

or

$$\begin{aligned} s_{2p+1} &= s_{2p} + r_{2p} + 1, \\ t_{2p} &= t_{2p-1} + r_{2p-1} + 1, \end{aligned} \quad p = 1, 2, 3, \cdots. \tag{100.11}$$

Turning now to the Padé table (Fig. 14) we draw the *α-polygon* $0V_0V_1V_2 \cdots$ with vertices

$$0 = (0, 0), \quad V_0 = (0, \alpha_1),$$

$$V_{2p-1} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1}),$$

$$V_{2p} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1}),$$

$$p = 1, 2, 3, \cdots.$$

By (100.9) and (100.10), the vertices of Q_{2p} are

$$W_{2p} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, s_{2p}), \quad V_{2p},$$
$$T_{2p} = (t_{2p} + r_{2p} + 1, s_{2p}),$$
$$U_{2p} = (t_{2p} + r_{2p} + 1, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1}),$$

and the vertices of Q_{2p+1} are

$$U_{2p+1} = (t_{2p+1}, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1}),$$
$$T_{2p+1} = (t_{2p+1}, s_{2p+1} + r_{2p+1} + 1), \quad V_{2p+1},$$
$$W_{2p+1} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, s_{2p+1} + r_{2p+1} + 1).$$

Thus, the α-polygon forms part of the boundary of every block Q_p. By (100.8) we see that Q_p and Q_{p+1} have a line segment d_p of the α-polygon as common boundary.

Let $L(k)$ denote the straight line whose equation is $y = x + k$. Then it is clear that there exist one or more integers k_p such that the lines $L(k_p)$ and $L(1 + k_p)$ have points in common with the line segment d_{2p}, the common boundary of Q_{2p} and Q_{2p+1}. The principal diagonal of Q_{2p} lies on or below $L(k_p)$, and the principal diagonal of Q_{2p+1} lies on or above $L(1 + k_p)$. Hence, we see that W_{2p} must lie on or below $L(k_p)$, U_{2p+1} must lie on or above $L(1 + k_p)$, V_{2p} must lie on or above $L(1 + k_p)$, and V_{2p+1} must lie on or below $L(k_p)$. These statements are equivalent to the following inequalities:

$$\begin{aligned}
&\text{(a)} \qquad s_{2p} \leq t_{2p} + k_p, \\
&\text{(b)} \qquad s_{2p+1} \geq t_{2p+1} + k_p + 1, \quad p = 0, 1, 2, \cdots. \qquad (100.12)\\
&\text{(c)} \qquad s_{2p+1} \geq t_{2p} + k_p + 1, \\
&\text{(d)} \qquad s_{2p+1} \leq t_{2p+2} + k_p,
\end{aligned}$$

We shall now prove that the necessary conditions (100.9) and (100.12) are together sufficient for regularity. We suppose that (100.9) holds, and that integers $k_0, k_1, k_2, \cdots$ exist such that (100.12) holds. Then $s_{2p} + t_{2p} = s_{2p} + \alpha_2 + \alpha_4 + \cdots + \alpha_{2p} \leq t_{2p} + k_p + \alpha_2 + \alpha_4 + \cdots + \alpha_{2p} \leq s_{2p+1} - 1 + \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}$, or

$$s_{2p} + t_{2p} + 1 \leq \alpha_1 + \alpha_2 + \cdots + \alpha_{2p+1}.$$

Similarly, $s_{2p+1} + t_{2p+1} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1} + t_{2p+1} \leq \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1} + s_{2p+1} - k_p - 1 \leq \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1} + t_{2p+2} - 1$, or

$$s_{2p+1} + t_{2p+1} + 1 \leq \alpha_1 + \alpha_2 + \cdots + \alpha_{2p+2}.$$

Consequently, (100.1) holds, so that the C-fraction is regular. We have proved the following theorem.

THEOREM 100.1. *The C-fraction* (99.2) *is regular if, and only if, integers k_p can be found such that* (100.12) *holds and, in addition,* (100.9) *holds* [**14**].

101. α-regular C-fractions. If (100.9) holds, and there exists an integer $k \geq 0$ such that the straight lines $L(k)$ and $L(1 + k)$ intersect *all* the line segments d_p, $p = 0, 1, 2, \cdots$, then the C-fraction and its power series expansion are called **α-regular.** We shall prove the following theorem.[31]

THEOREM 101.1. *The C-fraction* (99.2) *is α-regular if, and only if, there exists an integer $k \geq 0$ such that*

$$\begin{gathered}\alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1} \geq \alpha_0 + \alpha_2 + \alpha_4 + \cdots + \alpha_{2p} + k + 1,\\ \alpha_2 + \alpha_4 + \cdots + \alpha_{2p+2} \geq \alpha_1 + \alpha_3 + \alpha_5 + \cdots + \alpha_{2p+1} - k,\\ p = 0, 1, 2, \cdots,\end{gathered} \tag{101.1}$$

where $\alpha_0 = 0$. [**14**.]

Proof. If (99.2) is α-regular, then (100.12) holds with $k_p = k$, $p = 0, 1, 2, \cdots$. Hence, by (100.9) and (100.12) (c), (d), we conclude that (101.1) holds.

If, conversely, (101.1) holds, then we shall prove by mathematical induction that

$$\begin{gathered}s_{2p} - k \leq t_{2p} = \alpha_0 + \alpha_2 + \alpha_4 + \cdots + \alpha_{2p},\\ p = 0, 1, 2, \cdots, \quad (\alpha_0 = 0),\\ k + 1 + t_{2p-1} \leq s_{2p-1} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1},\\ p = 1, 2, 3, \cdots,\end{gathered} \tag{101.2}$$

from which it will follow immediately that (99.2) is α-regular. The first relation (101.2) holds for $p = 0$, since $k \geq 0$. Since

[31] Scott and Wall [85] investigated C-fractions for which (101.1) holds with $k = 0$.

$s_1 = \alpha_1$, $t_1 = 0$, $s_2 \leq \max(\alpha_1, \alpha_2)$, $t_2 = \alpha_2$, we readily verify (101.2) for $p = 1$. Assuming that the relations hold for $p \leq n$, $n \geq 1$, we shall prove them for $p = n + 1$. From the fundamental recurrence formulas we see that $s_{2n+1} = \max(s_{2n}, s_{2n-1} + \alpha_{2n+1})$ if $s_{2n} \neq s_{2n-1} + \alpha_{2n+1}$. But, by our assumption, $s_{2n} \leq k + \alpha_2 + \alpha_4 + \cdots + \alpha_{2n}$ and, by (101.1), $s_{2n-1} + \alpha_{2n+1} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2n+1} > \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} + k$, so that $s_{2n+1} = \alpha_{2n+1} + s_{2n-1} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2n+1}$. Again, $t_{2n+1} \leq \max(t_{2n}, t_{2n-1} + \alpha_{2n+1})$, so that

$$k + 1 + t_{2n+1} \leq \begin{Bmatrix} k + 1 + t_{2n} \\ k + 1 + t_{2n-1} + \alpha_{2n+1} \end{Bmatrix}$$
$$\leq \alpha_1 + \alpha_3 + \cdots + \alpha_{2n+1} = s_{2n+1}.$$

Next, since $t_{2n+2} = \max(t_{2n+1}, t_{2n} + \alpha_{2n+2})$ provided $t_{2n+1} \neq t_{2n} + \alpha_{2n+2}$, we conclude that $t_{2n+2} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2n+2}$, inasmuch as, by what we have just proved and (101.1), $t_{2n+1} \leq \alpha_1 + \alpha_3 + \cdots + \alpha_{2n+1} - k - 1 < \alpha_2 + \alpha_4 + \cdots + \alpha_{2n+2} = t_{2n} + \alpha_{2n+2}$. Finally, we have

$$s_{2n+2} \leq \begin{Bmatrix} s_{2n+1} \\ s_{2n} + \alpha_{2n+2} \end{Bmatrix}$$
$$\leq \alpha_2 + \alpha_4 + \cdots + \alpha_{2n+2} + k = t_{2n+2} + k.$$

The proof of Theorem 101.1 is now complete.

Regularity depends in general upon the coefficients a_p and the exponents α_p of the C-fraction. The conditions (101.1) involve only the exponents α_p, so that α-regularity depends upon only the exponents α_p.

The condition for α-regularity can be formulated in terms of the coefficients in the power series expansion of the C-fraction, using for that purpose Theorem 98.3. These conditions involve the determinants $\Delta_{m,n}$, and are somewhat complicated [**14**].

102. Concluding Remarks on the Padé Table. It is difficult to appraise the significance of the Padé table in the theory of continued fractions and power series. We feel that any appraisal must await further and deeper investigations. We might be permitted to conjecture that the nature of the "blocks" in the table have a bearing upon the nature of the function represented by the power series. Consider, for example, a series with "Hadamard gaps." Then the sequence of blocks along the upper border of the table have orders equal to the sizes of these gaps. Perhaps any power series whose Padé table contains a sequence of blocks whose orders increase sufficiently rapidly represents a function with a natural boundary. This conjecture is supported by several examples. For instance, the continued fraction of Ramanujan [76]

$$\cfrac{1}{1+\cfrac{z}{1+\cfrac{z^2}{1+\cfrac{z^3}{1+\cdots}}}}$$

represents a function having the unit circle as natural boundary although the power series does not have Hadamard gaps. The gaps are "concealed" further down in the Padé table. The same is true of the continued fraction [85]

$$\cfrac{1}{1+\cfrac{az}{1+\cfrac{az^m}{1+\cfrac{az^{m^2}}{1+\cdots}}}},$$

at least if m is odd and $a \neq 0$, and if m is even and $a < 0$, (a real).

Another example which leads to interesting speculations is as follows. The continued fraction

$$\cfrac{1}{1+\cfrac{z}{1+\cfrac{z}{1+\cdots}}}$$

has a normal Padé table, and represents an algebraic function.

By suitably changing the signs in the sequence of partial numerators, this can be made the continued fraction for the series $1 + z + z^2 + z^4 + z^8 + z^{16} + \cdots$, which has the unit circle as natural boundary [**54**]. This is a special case of Example 4, § 99.

We have not touched upon the horizontal files of the Padé table. If $P(z)$ is a power series representing a function having only polar singularities on its circle of convergence, then the horizontal files of the table are closely connected with the investigations of Hadamard's [**25**] thesis. Using Hadamard's results, de Montessus de Balloire [**8**] proved the following theorem.

Let $P(z)$ be a power series representing a function which is regular for $|z| \leq R$ except for m poles within this circle. Then the $(m + 1)$st *horizontal file of the Padé table for $P(z)$ converges to $P(z)$ uniformly in the domain obtained from $|z| \leq R$ by removing the interiors of small circles with centers at these poles* [**8**].

Perron [**69**] gives an interesting example of a power series $P(z)$ having a finite or even infinite radius of convergence for which the *second* horizontal file in the Padé table diverges over an everywhere dense set in the circle of convergence of the power series.

Considerable attention has been given to the problem of determining explicitly the numerators and denominators of the Padé approximants of a power series $P(z)$ satisfying a differential equation

$$L(z)P'(z) + M(z)P(z) + N(z) = 0,$$

in which $L(z)$, $M(z)$, $N(z)$ are polynomials. Laguerre [**43**] showed that the problem can be reduced to the problem of determining other polynomials which are essentially simpler than the numerators and denominators of the approximants. However, according to Perron [**69**], the actual computation has been successfully carried out in but three particular cases. In two of these, the approximants can be obtained by much easier means. (Cf. Exercise 20.9.) We should like to emphasize the main idea in Laguerre's theory. If $P(z)$ satisfies the differential equation

$$L(z)\frac{dw}{dz} + M(z)w + N(z) = 0, \tag{102.1}$$

then the nth approximant, $A_n(z)/B_n(z)$, of the J-fraction for $P(z)$ satisfies the differential equation

$$L(z)\frac{dw}{dz} + M(z)w + N(z) = \frac{z^{2n-1}G_n(z)}{B_n^2(z)},$$

where $G_n(z)$ is a polynomial whose degree is at most one unit more than the largest of the degrees of $L(z)$, $M(z)$ and $N(z)$. This can be proved very easily by eliminating $P(z)$ and $P'(z)$ from the set of equations

$$LP' + MP + N = 0,$$

$$B_nP - A_n = z^{2n}Q,$$

$$B_nP' + B_n'P - A_n' = z^{2n-1}Q_1,$$

where Q and Q_1 are power series, and the third equation is obtained from the second by differentiation with respect to z.

Exercise 20

20.1. Let $a_1, a_2, \cdots, a_k (k \geq 1)$ be real or complex numbers different from zero, let $n_1, n_2, \cdots, n_k$ be positive integers, and let m be an integer > 1. The continued fraction

$$1 + \cfrac{a_1x^{n_1}}{1 + \cfrac{a_2x^{n_2}}{1 + \cdots + \cfrac{a_kx^{n_k}}{1 + \cfrac{a_1x^{mn_1}}{1 + \cfrac{a_2x^{mn_2}}{1 + \cdots + \cfrac{a_kx^{mn_k}}{1 + \cfrac{a_1x^{m^2n_1}}{1 + \cdots}}}}}}}$$

represents a function $f(x)$ which is analytic for $|x| < 1$, with the possible exception of poles, and satisfies the functional equation

$$f(x) = \frac{f(x^m)A_{k-1}(x) + a_kx^{n_k}A_{k-2}(x)}{f(x^m)B_{k-1}(x) + a_kx^{n_k}B_{k-2}(x)}. \quad \textbf{[85.]}$$

20.2. If there is a point $x_0 = e^{i\theta}$ upon the circle $|x| = 1$ where $f(x)$ is analytic, then $f(x)$ is a meromorphic function of x. **[85.]**

20.3. Let $s = 1$ or -1 if the exponents n_p are all odd, and $s = 1$ if any of them is even. Put

$$Z(s) = [B_k(s) - B_{k-1}(s) + A_{k-1}(s)]^2 - 4(-s)^k a_1 a_2 \cdots a_k.$$

Then $f(x)$ has the unit circle as natural boundary if the a_p are real, $B_{k-1}(s) \neq 0$, and $Z(s) < 0$. [85.]

20.4. If a is real, the function

$$f(x) = 1 + \cfrac{ax}{1 + \cfrac{ax^m}{1 + \cfrac{ax^{m^2}}{1 + \cfrac{ax^{m^3}}{1 + \cdots}}}} \qquad (m > 1),$$

has the unit circle as natural boundary if (a) m is odd and $a \neq 0$, or (b) m is even and $a < 0$. [85.]

20.5. Let u and v be parameters, $u \neq v$, and m an integer > 1. Then the continued fraction

$$1 - \cfrac{(u+v)x}{1 - \cfrac{(u-v)x}{1 - \cfrac{(u+v)x^m}{1 - \cfrac{(u-v)x^m}{1 - \cfrac{(u+v)x^{m^2}}{1 - \cdots}}}}}$$

represents a function having the unit circle as natural boundary if u and v are real and $u > v^2 + \frac{1}{4}$. If m is odd, the same holds if $u < -v^2 - \frac{1}{4}$. [85.]

20.6. Let $P(z) = 1 + 4z^5 + 6z^9$. Show that

$$\frac{1}{P(z)} = \cfrac{1}{1 + \cfrac{4z^5}{1 - \cfrac{(\frac{3}{2})z^4}{1 + \cfrac{(\frac{3}{2})z^4}{1}}}},$$

and

$$\frac{1}{z^9 P\left(\frac{1}{z}\right)} = \cfrac{\frac{1}{6}}{1 + \cfrac{(\frac{2}{3})z^4}{1 - \cfrac{(\frac{1}{4})z^5}{1 + \cfrac{(\frac{1}{4})z^5}{1}}}},$$

and hence show that the zeros of $P(z)$ are in the domain $.63 < |z| < 1.08$. (Cf. Exercise 10.8).

20.7. Investigate the questions of convergence and the character of the limits of the diagonal files of the Padé table for a series $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ which has a real J-fraction expansion [118].

20.8. Let $P(z)$ be a power series having a real J-fraction expansion. Let $P(z)$ be Borel-summable over a domain G. Then, in the Padé table for $P(z)$, all the diagonal files which start on the upper border of the table converge uniformly over every finite region whose distance from the real axis is positive, to a function $F(z)$ which is the Borel sum of the series in G [**120**].

20.9. Apply Theorem 96.1 and the results of Exercise 18 to obtain continued fractions whose approximants fill all the squares $[p, p + k]$, for sufficiently large k, in the Padé table for a power series satisfying the differential equations of Exercise 18.

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