ELEMENTARY MECHANICAL VIBRATIONS

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PREFACE

All branches of engineering are confronted with vibration problems. A large percentage of these may be avoided in design or cured by a rational application of the elementary principles. Others, however, call for the use of advanced mathematics combined with an extensive experience in the field; in other words, they require the services of specialists.

This text has been prepared to train students, or enable engineers, to handle the usual problems which arise. It attempts to present in a concise and orderly manner the basic material required for their solution. It is felt that this may be done more successfully by sacrificing mathematical rigor and relying on a physical explanation of the phenomena in most cases.

It must be realized that vibration problems are exceedingly complex and that the material given here serves only as an introduction to a more extensive study. Such subjects as self-excited vibrations, nonlinear systems, harmonic analysis, and damping in systems having more than one degree of freedom are felt to be beyond the scope of this text. Information on these and other advanced topics may be found in the current technical literature, or in more comprehensive books such as *Mechanical Vibrations* by J. P. Den Hartog or *Practical Solution of Torsional Vibration Problems* by W. K. Wilson. No attempt has been made to compile a complete bibliography of the literature, since it is very extensive, but where it has been necessary to draw rather heavily on some of the literature in the preparation of the manuscript, credit has been given. Since balancing is intimately related to vibration, it appears wise to include a chapter covering the basic principles of that subject.

The book is based upon courses given at New York University to various groups, and acknowledgment is made here of the many suggestions received from students to improve the content. It is assumed that the reader has had the usual engineering courses in mechanics and calculus. With this background, little difficulty should be experienced in understanding the subject matter. Extensive use of examples has been made to clarify the text, and at the end of each chapter problems (with answers) are given by which the reader may test his grasp of the material treated.

AUSTIN H. CHURCH

CONTENTS

ELEMENTARY MECHANICAL VIBRATIONS

PARTIAL LIST OF SYMBOLS

Symbol	*Explanation*	*Units*
A, B, C, p	Constants	
a	Linear acceleration	in. per $\sec^2$
a	Axial distance	in.
b	Width	in.
c	Distance from neutral axis to outer fiber	in.
c	Lateral damping factor	lb-sec per in.
c_t	Torsional damping factor	in.-lb-sec per rad
d, D	Diameter	in.
e	Base of natural logarithms = 2.7183	dimensionless
e	Eccentricity	in.
E	Modulus of elasticity (Young's)	psi
f	Cyclic frequency	cps or cpm
F	Force	lb
g	Gravitational acceleration = 386	in. per $\sec^2$
G	Shearing modulus of elasticity	psi
h	Depth or thickness	in.
I	Rectangular moment of inertia	in.^{4}
I_p	Polar moment of inertia	in.^{4}
j	$\sqrt{-1}$	dimensionless
J	Mass moment of inertia	in.-lb-$\sec^2$
k	Lateral spring scale or rate	lb per in.
k_t	Torsional spring scale or rate	in.-lb per rad
L	Length	in.
m	Mass = W/g	lb-$\sec^2$ per in.
M	Bending moment	in.-lb
n	Revolutions per unit time	rpm or rps
n	Ratio of connecting rod length to crank length	dimensionless
r, R	Radius	in.
$\bar{r}$	Radius of gyration	in.
r	Ratio of f/f_n or of ω/ω_n	dimensionless

Symbol	*Explanation*	*Units*
s	Stress	psi
t	Time	sec
t	Transmissibility	dimensionless
T	Torque	in.-lb
v	Velocity	in. per sec
V	Shear	lb
W	Weight	lb
x, y	Lateral displacement	in.
x	Axial distance	in.
x_{st}	Static deflection due to impressed force	in.
α	Angular acceleration	rad per sec²
β, θ	Angular displacement	rad
γ	Specific weight	lb per in.³
δ	Logarithmic decrement	dimensionless
δ_{st}	Static deflection due to weight	in.
θ	Slope of beam or shaft	rad
θ	Angular displacement	rad
μ	Coefficient of friction	dimensionless
μ	Mass ratio $= m_a/m$, or spring scale ratio $= k_a/k$	dimensionless
μ	Weight per unit length	lb per in.
τ	Period	sec
ϕ	Phase angle	rad
ψ	Fixed angle between cranks	rad or deg
ω	Circular frequency or angular velocity	rad per sec

SUBSCRIPTS

a	Actual
c	Critical
d	Damped
e	Equivalent
n	Natural
o	Maximum
t	Torsional

Chapter 1

INTRODUCTION

1.1 *IMPORTANCE AND SCOPE*

The occurrence of vibrations is widespread. Consideration of the topics and problems in this text reveals some of the conditions under which vibrations are present. In some cases their presence is helpful, but generally the reverse is true and they are to be avoided.

Vibrations have been used to relieve the internal cooling stresses set up in castings* and thus eliminate the long periods of time required for natural aging. They have been used in geologic seismic investigations;† to determine endurance limits of materials and machine members; to facilitate the handling of powdered materials apt to pack, such as flour and sand.

On the other hand, the effect of most vibrations is bad in that resonance or near resonance creates high stresses and hastens the time when eventual failure may occur. Moreover, vibration has a bad psychological effect on people in the vicinity, is tiring, slows production, and creates a generally undesirable condition. Any vibration requires energy or power to produce it; hence, the efficiency of the machine is reduced. The air-raid sirens installed in some of the large cities during the Second World War are fine examples of the power required to produce vibrations. The largest type of these sirens, which had a penetration of eight miles, required a 125-hp automobile engine for driving.

* R. T. McGoldrick and H. E. Saunders, "Some Experiments in Stress Relieving Castings," *J. Am. Soc. Naval Engrs.*, November, 1943, p. 589.

† R. K. Bernhard, "Dynamic Tests by Means of Induced Vibrations," *Proc. A.S.T.M.*, 1937, pp. 634–644. J. B. Macelwane, "The Interior of the Earth," *Am. Scientist*, April, 1946, pp. 177–197.

1.2 DEFINITIONS AND TERMINOLOGY

Vibrations occur in *elastic systems* that consist of one or more masses connected by springs. A *vibration* is the motion of a body or system which is repeated after a given interval of time known as the *period*. The number of cycles of motion per unit time is called the *frequency*. The maximum displacement of the body or some part of the system from the equilibrium position is the *amplitude* of the vibration. It should be remembered that the total travel is twice the amplitude.

There are two general types of vibration, namely, *lateral* and *torsional*. In the former, the motion is rectilinear or one of translation, and the amplitude is measured in inches; whereas in the latter, the motion is one of rotation or twisting, and the amplitude is measured in radians or degrees. The equations for the two types and their solutions are similar and hence, will be considered together throughout this text.

If the body or system is given an initial displacement from the equilibrium position and released, it will vibrate with a definite frequency known as the *natural frequency*. The vibration is said to be *free*, since no external forces act upon it after the first displacement. Generally the body vibrates with decreasing amplitude until it comes to rest due to *damping* in the form of friction or air resistance. For some applications, additional damping in the form of a dashpot or rubbing friction may be added to increase this effect. If the amount of damping is very large, the body may not vibrate but merely creep back to the equilibrium position, and the motion is said to be *aperiodic*.

Under some conditions, as in the case of an unbalanced machine, the body or system may be subjected to a periodic external force. In such cases a *forced vibration* occurs. If the frequency of this external force is the same as, or close to, the natural frequency, *resonance* takes place. The body or system then vibrates with large amplitudes, which result in high stresses and possible interference of parts and should be avoided. When resonance occurs in rotating shafts (because of unbalance), the speed of rotation is known as the *critical speed*.

Vibrations may be classified as transient or steady state. A *transient vibration* is a temporary condition which disappears with time, such as a free vibration. A *steady-state vibration* is one in which the motion is repeated exactly in each cycle, as in a forced vibration. A transient vibration may be superimposed upon a steady-state vibration, as when an external load is suddenly applied to a system or body having a steady-state forced vibration. The resultant motion is the vector sum of the two motions considered independently.

It may be observed that for lateral and torsional vibrations the equations are similar. Corresponding terms with their symbols and units are as follows:

Term	Lateral		Torsional	
	Symbol	Units	Symbol	Units
Mass or inertia	m	lb-sec² per in.	J	in.-lb-sec²
Spring scale	k	lb per in.	k_t	in.-lb per rad
Force or torque	F	lb	T	in.-lb
Damping	c	lb-sec per in.	c	in.-lb-sec per rad
Displacement	x, y	in.	β, θ	rad
Velocity	v	in. per sec	ω	rad per sec
Acceleration	a	in. per sec²	α	rad per sec²

Many bodies or systems are able to vibrate in more than one manner and have more than one natural frequency. Systems having many masses or a continuous mass, such as a string or a beam when weight is considered, will have many natural frequencies. Each of them is accompanied by its own *mode* or form of vibration curve. For a lateral vibration, the lowest natural frequency will have no *nodes*, or points, of zero amplitude; the next frequency, one node; and so on. For a torsional vibration, the lowest natural frequency will have one node; the next frequency, two nodes; and so on.

If the motion of the body is constrained so that it can vibrate in only one manner, or mode, it is said to have a *single degree of freedom;* if in two modes, *two degrees of freedom;* and so on. The number of degrees of freedom is equal to the number of coordinates required to specify completely the position of the body or system at any time. A single rigid body, such as a block supported on springs, may have six degrees of freedom, as illustrated in Fig. 1.1. There are three directions of translation, or lateral vibration, and three of rotation, or torsion, about the principal axes as shown. The natural frequency of each mode of vibration is independent of the frequency of the others.

Thus, if a cantilever beam having a rectangular cross section is displaced downward and released, it will vibrate with a certain natural frequency. If the beam is displaced outward and released, it will vibrate with a different natural frequency. If, however, the beam is displaced both downward and outward before being

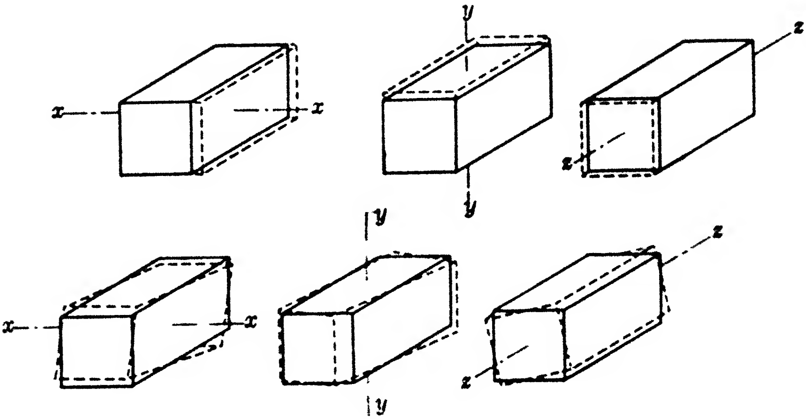


Fig. 1.1

released, both vibrations will occur simultaneously without affecting each other in either frequency or amplitude. The absolute motion of the body will be the resultant of these independent motions (see first example of Sec. 2.2).

The amplitudes of vibrations are usually very small and are commonly linked with the properties of materials; hence, it is convenient to use the inch-pound-second system of units. For this system the acceleration due to gravity is taken as 386 in. per sec^2, rather than the more common equivalent of 32.2 ft per sec^2.

Material	Tensile modulus of elasticity, E, psi	Shear modulus of elasticity, G, psi	Specific weight, γ, lb per in.^{3}
Steel.	$30(10^6)$	$12(10^6)$	0.283
Cast iron.	$17(10^6)$	$7(10^6)$	0.260
Brass and bronze.	$15(10^6)$	$6(10^6)$	0.315
Aluminum.	$10(10^6)$	$4(10^6)$	0.100

The properties of materials vary widely with their chemical composition, method of manufacture, and other factors. The table on page 8 gives average values which may be used in solving problems.

1.3 *SIMPLE HARMONIC MOTION*

All bodies vibrate with either simple harmonic motion or what may be considered to be a combination of simple harmonic motions of different frequencies and amplitudes. It is therefore desirable to review this topic in detail.

If a point P' travels along the circumference of a circle with constant velocity, its projection P on a diameter of the circle will move with simple harmonic motion. This is illustrated in Fig. 1.2, where the radius of the circle, x_o, is the amplitude of the motion, or length of the vector which is rotating about the center O with a con-

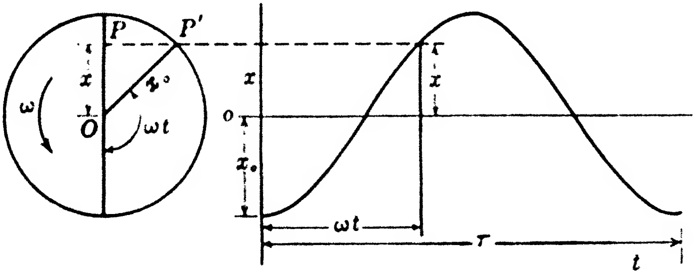


FIG. 1.2

stant angular velocity ω. This angular velocity ω is known as the *circular frequency* and is measured in radians per second. The angle turned through by the vector in time t is ωt, and the corresponding displacement OP is $x = x_o \cos \omega t$. The *cyclic frequency* of the motion is $\frac{\omega}{2\pi}$ in cycles per second, since there are 2π rad in a complete circle.

The velocity of point P can be found by differentiating the displacement with respect to time; thus, $v = dx/dt = -x_o\omega \sin \omega t$. This relation can be shown vectorially, as in Fig. 1.3, where the constant velocity of the point P' equals $x_o\omega$ and is represented by the vector $P'Q$. The component $P'A$ of this velocity parallel to the reference diameter represents the velocity of point P along its path and is equal to $x_o\omega \sin \omega t$, as can be seen from the figure.

The acceleration of the point P can be found by differentiating the equation for the velocity with respect to time; thus,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -x_o\omega^2 \cos \omega t.$$

This relation can also be shown vectorially. Since point P' travels

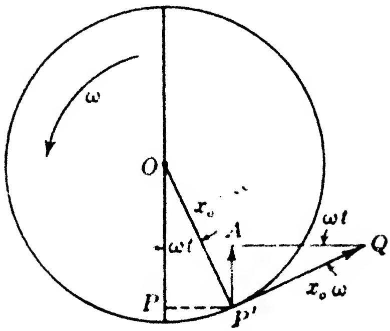


Fig. 1.3

with constant angular velocity ω, the only acceleration is normal, that is, acts toward the center of rotation O and equals the constant value of $x_o\omega^2$. In Fig. 1.4, this acceleration of point P' is represented

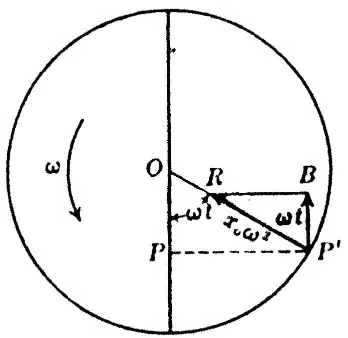


Fig. 1.4

by the vector $P'R$, equal to $x_o\omega^2$, and the component $P'B$ parallel to the reference diameter is the acceleration of point P along its path. This acceleration is $x_o\omega^2 \cos \omega t$, and it should be noted that it *always* acts toward the center point O or opposite to the direction of the displacement for any position of the vector.

To summarize, the equations for simple harmonic motion are

$$x = x_o \cos \omega t, \tag{1.1}$$

$$v = \frac{dx}{dt} = -x_o\omega \sin \omega t, \tag{1.2}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -x_o\omega^2 \cos \omega t. \tag{1.3}$$

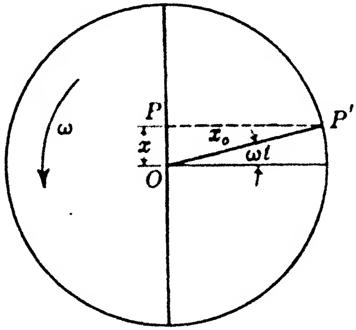


Fig. 1.5

In some cases, time is measured from a horizontal reference line rather than a vertical (Fig. 1.5), and the equations then become

$$x = x_o \sin \omega t,$$

$$v = x_o\omega \cos \omega t,$$

$$a = -x_o\omega^2 \sin \omega t.$$

For either group of equations, the motion is harmonic, the interchange of the sine and cosine terms merely indicating a displacement of 90 deg.

Generally the term simple harmonic is applied to motion, but it may also be applied to a force. Thus, a centrifugal force acts radially, and it may have a component in one particular direction, which acts harmonically. This condition occurs in unbalanced machinery having one degree of freedom. Then the motion vector *OP'* of Fig. 1.2 may be replaced by the centrifugal force; the component *OP* of this force parallel to the diameter is harmonic and is equivalent to the displacement x.

Frequently, two motions, or forces, may be acting about the same point either with the same or with different circular frequencies. The vectors will not coincide but will be separated by a *phase angle* ϕ. If both vectors rotate at the same speed, the phase angle is constant; otherwise, it will vary. The resultant effect may be found by adding the vectors. An illustration of this is the beat phenomenon, where two vectors of about equal magnitude but of slightly different frequencies act together. At one instant of time these vectors will coincide and give large resultant amplitudes of motion; whereas at another instant, they are opposed, and the resultant motion is zero. The beat frequency is the difference in the frequencies of the component vectors.

Chapter 2

UNDAMPED FREE VIBRATIONS—SINGLE DEGREE OF FREEDOM

2.1 *INTRODUCTION*

All actual free vibrations die out in time and thus have some damping. In many systems, however, the amount of damping is so small that it may be neglected. Hence, the natural frequency based upon an undamped free vibration may be very close to the actual for many applications; and the principles discussed in this chapter may be used to determine natural frequencies of many systems having a single degree of freedom with sufficient accuracy for engineering purposes. This problem of the prediction and correction of a resonant condition in a machine or machine part is one of the most common in vibration work.

The effect of larger amounts of damping on free vibrations will be considered in Chap. 4.

2.2 *EQUATION OF MOTION*

For lateral vibrations, the elastic system may be represented by Fig. 2.1, where k is the scale of the spring, or the force in pounds required to deflect the spring 1 in., and W is the weight of the block in pounds.

Due to the weight of the block, the spring will be deflected a distance W/k in., and this point constitutes the equilibrium position where the pull of the weight and the spring forces are balanced. Since this dead weight is constant in magnitude and direction, it may be neglected when considering the forces induced by the vibration.

Assume that the block is displaced downward a distance x

from this equilibrium position, and consider only the vibratory forces that act upon it. The extension of the spring produces an upward force kx, and since the inertia force, or acceleration force, always acts opposite to the direction of the displacement, as shown in Fig. 1.4, or by Eqs. (1.1) and (1.3), it will act upward also and have a magnitude equal to $m\frac{d^2x}{dt^2}$. Considering the block as a free body, the equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0, \tag{2.1}$$

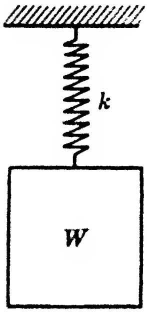


FIG. 2.1

which may be written in the form

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{2.2}$$

The solution of this equation requires a function of x in terms of t, which, when differentiated twice, will give the same function multiplied by the negative constant $-k/m$. The most general solution is

$$x = A\sin\sqrt{\frac{k}{m}}\,t + B\cos\sqrt{\frac{k}{m}}\,t. \tag{2.3}$$

To show that Eq. (2.3) is a solution of Eq. (2.2) it may be differentiated twice; thus,

$$\frac{dx}{dt} = A\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}\,t - B\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}\,t, \tag{2.4}$$

$$\frac{d^2x}{dt^2} = -A\frac{k}{m}\sin\sqrt{\frac{k}{m}}\,t - B\frac{k}{m}\cos\sqrt{\frac{k}{m}}\,t. \tag{2.5}$$

It may be noted that the right side of Eq. (2.5) equals the right side of Eq. (2.3) multiplied by the constant $-k/m$; hence, Eq. (2.2) is satisfied.

Generally, a free vibration is the result of displacing the block a distance x_o and releasing it. At the instant of release the velocity, or dx/dt, of the block is zero. If these conditions ($x = x_o$ and $dx/dt = 0$ when $t = 0$) are inserted in Eqs. (2.3) and (2.4), the constants A and B may be evaluated. Remembering that cos 0 deg = 1 and sin 0 deg = 0, Eq. (2.3) becomes

$$x_o = A \times 0 + B \times 1,$$

and $B = x_o$. Similarly, Eq. (2.4) becomes

$$0 = A\sqrt{\frac{k}{m}} \times 1 - B\sqrt{\frac{k}{m}} \times 0,$$

and $A = 0$. Hence, the solution of Eq. (2.3) for this usual case is

$$x = x_o \cos \sqrt{\frac{k}{m}}\, t. \tag{2.6}$$

Much time and repetition may be saved by generalizing the above discussion for similar cases of free vibrations. If the differential equation of motion is

$$\frac{d^2z}{dt^2} = -Cz, \tag{2.7}$$

the corresponding solution will be

$$z = z_o \cos \sqrt{C}\, t, \tag{2.8}$$

where z is the function of the motion, and C is a constant.

By comparing Eqs. (2.6) and (2.8) with Eq. (1.1), it is obvious that the motion is simple harmonic. A vector of length x_o or z_o rotates in a circle, and its projection on a diameter of the circle gives the displacement x or z at any time t. The angular frequency of the rotation [corresponding to ω in Eq. (1.1)] is $\sqrt{k/m}$ or $\sqrt{C}$. Since this frequency is the natural one by definition, it is designated as ω_n and is measured in radians per second. The cyclic frequency is found by dividing by 2π; thus,

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{C} \text{ cps}, \tag{2.9}$$

and the time required for one cycle, or the period, is

$$\tau = \frac{1}{f_n} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1}{C}} \text{ sec}. \tag{2.10}$$

It should be noted that the frequency depends only on the mass of the block and the spring scale, and is therefore independent of x_o, or the amplitude of the vibration.

The equation for the natural frequency may be put in another form, based upon the spring deflection due to the static weight of the block. This static deflection $\delta_{st} = W/k = mg/k$. Hence, $k/m = g/\delta_{st}$, and Eq. (2.9) may be written

$$f_n = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{1}{\delta_{st}}} = \frac{3.13}{\sqrt{\delta_{st}}}\ \text{cps} = \frac{187.5}{\sqrt{\delta_{st}}}\ \text{cpm.} \tag{2.11}$$

Example 1

A weight of 25 lb is placed at the end of a steel cantilever that is 5 in. long and has a rectangular cross section $\frac{3}{4}$ in. wide and $\frac{1}{4}$ in. deep. Neglecting the mass of the beam, find the natural lateral frequencies in a horizontal and a vertical direction. Plot the path of the resulting motion of the weight, assuming that it is initially displaced downward 0.45 in. and to the right 0.05 in. before being released.

From Fig. 2.6, the static deflection of the weight W is

$$\delta_{st} = \frac{1}{3}\frac{WL^3}{EI}$$

The moment of inertia about a vertical axis is

$$I_h = \frac{b^3h}{12} = \frac{1}{4} \times \left(\frac{3}{4}\right)^3 \times \frac{1}{12} = 0.00879\ \text{in.}^4$$

The horizontal static deflection of the weight is

$$\delta_{st_h} = \frac{25 \times 5^3}{3 \times 30(10^6) \times 0.00879} = 0.00395\ \text{in.}$$

The horizontal natural frequency, from Eq. (2.11), is

$$f_{n_h} = \frac{187.5}{\sqrt{\delta_{st_h}}} = \frac{187.5}{\sqrt{0.00395}} = 2{,}980\ \text{cpm} = 49.7\ \text{cps.}$$

The moment of inertia about a horizontal axis is

$$I_h = \frac{bh^3}{12} = \frac{3}{4} \times \left(\frac{1}{4}\right)^3 \times \frac{1}{12} = 0.000976\ \text{in.}^4$$

The vertical static deflection of the weight is

$$\delta_{st_v} = \frac{25 \times 5^3}{3 \times 30(10^6) \times 0.000976} = 0.0356\ \text{in.}$$

From Eq. (2.11) the vertical natural frequency is

$$f_{n_v} = \frac{187.5}{\sqrt{\delta_{st_v}}} = \frac{187.5}{\sqrt{0.0356}} = 995 \text{ cpm} = 16.6 \text{ cps.}$$

The resultant motion of the weight will have a vertical amplitude of 0.45 in. and a horizontal amplitude of 0.05 in., each of which

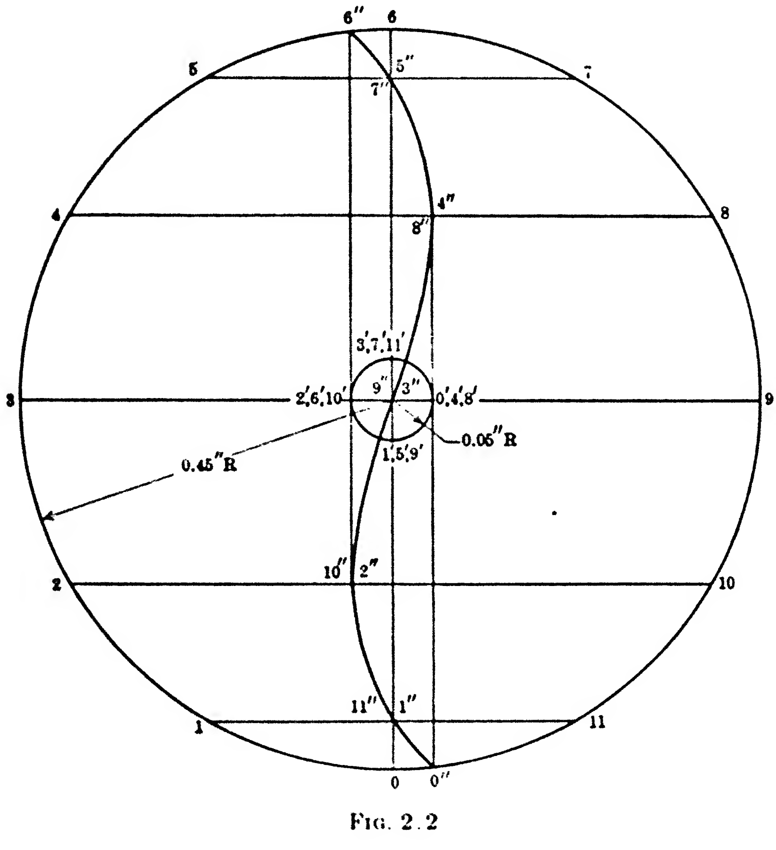


FIG. 2.2

is independent of the other. The frequency of the horizontal vibrations is three times as great as that of the vertical, and hence, the angle turned through by its vector for a given time interval is three times as large. With these facts in mind, the resultant motion of the weight may be plotted as shown in Fig. 2.2. As noted in Sec. 1.2, the resultant position of the weight at any instant is the vector sum of the horizontal and vertical displacements. The numbers on the diagram indicate simultaneous positions of the weight in the two directions. The resultant curve, labeled with double prime, is a function of the amplitudes and the frequencies. In this case the curve retraces itself each half cycle. If the frequencies had been the same in both directions, an ellipse or circle would have been obtained. The curve of the resultant motion is known as a *Lissajous figure*.

A more unusual but interesting case of free vibrations occurs when the body has an initial velocity as well as displacement from the equilibrium position at the instant when the vibration starts. The equation of motion is again given by Eq. (2.3), but the arbitrary constants A and B will have different values. The conditions are now

$$x = x_o \qquad \text{and} \qquad \frac{dx}{dt} = v \qquad \text{when} \qquad t = 0.$$

Equation (2.3) then becomes

$$x_o = A \times 0 + B \times 1,$$

or $B = x_o$ as before. But Eq. (2.4) now becomes

$$v = A\sqrt{\frac{k}{m}} \times 1 - B\sqrt{\frac{k}{m}} \times 0,$$

or $A = v/\sqrt{k/m}$. The solution of Eq. (2.3) then is

$$x = \frac{v}{\sqrt{k/m}} \sin\sqrt{\frac{k}{m}}\,t + x_o \cos\sqrt{\frac{k}{m}}\,t.$$

Since the natural circular frequency ω_n is $\sqrt{k/m}$, the above equation may be written

$$x = \frac{v}{\omega_n} \sin\sqrt{\frac{k}{m}}\,t + x_o \cos\sqrt{\frac{k}{m}}\,t, \tag{2.12}$$

and it may be observed that the resultant motion is made up of two vectors v/ω_n and x_o, which rotate with a circular frequency of ω_n or $\sqrt{k/m}$. Since the trigonometric functions are sine and cosine, respectively, the vectors must be at right angles (see Fig. 2.3).

The resultant motion of the body is $\sqrt{x_o^2 + \left(\frac{v}{\omega_n}\right)^2}$, and this resultant vector lags the x_o vector by the phase angle ϕ, where $\tan \phi = \frac{v/\omega_n}{x_o}$.

EXAMPLE 2

A weight of 200 lb is suspended on a steel wire 0.135 in. in diameter and 3 ft long. It is moving upward with a constant velocity of 1 fps when the upper end is instantaneously stopped. Determine the frequency, the maximum amplitude of the weight, and the maximum stress in the wire.

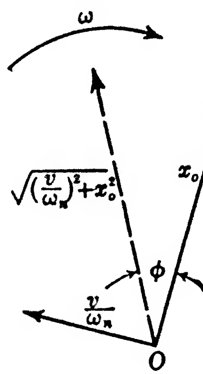


FIG. 2.3

Since the velocity is constant, the initial stretch in the wire is the same as the static deflection δ_{st}.

Area of wire cross section is

$$A = \frac{1}{4}\pi d^2 = \frac{0.135^2\pi}{4} = 0.01432 \text{ sq in.}$$

Static deflection is

$$\delta_{st} = \frac{WL}{AE} = \frac{200 \times 36}{0.01432 \times 30(10^6)} = 0.01675 \text{ in.}$$

The spring scale of the wire is

$$k = \frac{W}{\delta_{st}} = \frac{200}{0.01675} = 11{,}930 \text{ lb per in.}$$

The natural circular frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{11{,}930 \times \tfrac{386}{200}} = 151.6 \text{ rad per sec.}$$

The natural cyclic frequency is

$$f_n = \frac{\omega_n}{2\pi} = \frac{151.6}{2\pi} = 24.15 \text{ cps} = 1{,}450 \text{ cpm.}$$

Since the weight was initially moving with constant velocity, it is in the equilibrium position, and x_o equals zero. Therefore, the amplitude of the weight equals the v/ω_n term only.

The amplitude of the vibration is $\dfrac{v}{\omega_n} = \dfrac{12}{151.6} = 0.0792$ in.

The maximum deflection of the wire is the amplitude due to the vibration (0.0792 in.) plus the dead-load deflection δ_{st}(0.01675 in.), which equals 0.09595 in.

Maximum stress in the wire is

$$s = y\frac{E}{L} = \frac{0.09595 \times 30(10^6)}{36} = 80{,}000 \text{ psi.}$$

Since the amplitude of the vibration is greater than the static deflection, the wire will become slack on the upward displacement. Therefore, the motion of the weight is not simple harmonic. However, the stress in the wire will not exceed 80,000 psi, since the total energy acting in the system is constant, and the stress is directly proportional to the energy in the system.

EXAMPLE 3

Repeat the previous example with the change that the block is moving with an upward acceleration of 40 fps² rather than constant velocity at the instant when the upper end is stopped.

The natural frequency will be 1,450 cpm, as before.

The acceleration force on the wire is

$$F = ma = \frac{200}{386}40 \times 12 = 248 \text{ lb.}$$

The initial deflection of the block from the equilibrium position then is

$$x_o = \frac{FL}{AE} = \frac{248 \times 36}{0.01432 \times 30(10^6)} = 0.0208 \text{ in.}$$

The amplitude due to the initial velocity is $\dfrac{v}{\omega_n} = 0.0792$ in. as before.

The resultant vibratory amplitude then is

$$\sqrt{x_o^2 + \left(\frac{v}{\omega_n}\right)^2} = \sqrt{0.0208^2 + 0.0792^2} = 0.0819 \text{ in.}$$

The maximum stretch in the wire is the resultant amplitude due to the vibration (0.0819 in.) plus the dead-load deflection δ_{st}(0.01675 in.), which is 0.09865 in.

The maximum stress in the wire is

$$s = \frac{yE}{L} = \frac{0.09865 \times 30(10^6)}{36} = 82{,}200 \text{ psi}.$$

Again the amplitude of vibration is greater than the static or dead-load deflection, so the wire will be slack on the upward displacement, and the motion is not simple harmonic.

2.3 *ENERGY SOLUTION*

If it is assumed that the block of Fig. 2.1 vibrates with simple harmonic motion, the equation for the frequency found in the previous section may be determined by energy considerations.

When the block has its maximum displacement x_o, it is at rest and all the energy contained in the system is potential. As the block moves toward the equilibrium position, this potential energy is transformed into kinetic. As the block passes through the equilibrium position, the potential energy is zero, and all the energy is then kinetic. Thus, an interchange of energy continually occurs as the system vibrates.

Since the total energy remains constant (as there is no damping), the maximum potential and kinetic energies may be equated to determine the frequency. Then, the maximum kinetic energy is

$$E_{k_{max}} = \tfrac{1}{2}mv_{max}^2;$$

or, since $v_{max} = x_o\omega$,

$$E_{k_{max}} = \tfrac{1}{2}mx_o^2\omega^2.$$

The maximum potential energy is

$$E_{p_{max}} = \tfrac{1}{2}F_{max}x_{max} = \tfrac{1}{2}(kx_o)x_o = \tfrac{1}{2}kx_o^2.$$

Equating these energies,

$$\tfrac{1}{2}mx_o^2\omega^2 = \tfrac{1}{2}kx_o^2.$$

Then

$$\omega^2 = \frac{k}{m} = \omega_n^2,$$

and the cyclic frequency is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\text{ cps} = \frac{60}{2\pi}\sqrt{\frac{k}{m}}\text{ cpm.}$$

2.4 *TORSIONAL VIBRATIONS*

For torsional vibrations, the system may be represented by a disk shrunk onto a steel shaft, the other end of the shaft being held fixed, as shown in Fig. 2.4. The spring scale of [illegible] required to twist the shaft 1 rad, and the units are inch-pounds per radian. The mass moment of inertia of the disk is J with the units in.-lb-sec², and it equals $\left(\frac{W}{g}\right)\bar{r}^2$, where $\bar{r}$ is the radius of gyration in inches, and equals $\frac{W}{g}\left(\frac{D^2}{8}\right)$ for a solid circular disk having an outside diameter of D in.

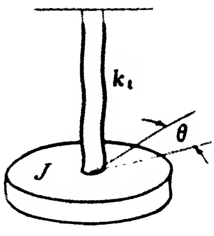


FIG. 2.4

If the disk is displaced through an angle θ and released, the differential equation of the motion, which is similar to the development given in Sec. 2.2 or 2.3, is

$$J\frac{d^2\theta}{dt^2} + k_t\theta = 0 \tag{2.13}$$

which may be written

$$\frac{d^2\theta}{dt^2} = -\left(\frac{k_t}{J}\right)\theta. \tag{2.14}$$

From Eqs. (2.7) and (2.8), the equation of motion of the disk is

$$\theta = \theta_o \cos\sqrt{\frac{k_t}{J}}\,t, \tag{2.15}$$

and from Eq. (2.9), the natural frequency is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_t}{J}}\text{ cps} = \frac{60}{2\pi}\sqrt{\frac{k_t}{J}}\text{ cpm.} \tag{2.16}$$

The angle of twist of a solid circular shaft is given by the equation $\theta = \frac{TL}{GI_p}$. Since $k_t = \frac{T}{\theta}$, it also equals $\frac{GI_p}{L}$ or $\frac{G\pi d^4}{32L}$ for a solid circular shaft. Hence, Eq. (2.16) may be written

$$f_n = \frac{60}{2\pi}\sqrt{\frac{GI_p}{LJ}} = \frac{60}{2\pi}\sqrt{\frac{G\pi d^4}{32LJ}} \text{ cpm.} \tag{2.17}$$

EXAMPLE 1

Determine the natural torsional frequency of a solid steel disk 10 in. in diameter and 3 in. thick, which is shrunk on a 1 in. diameter steel shaft 3 ft long. The other end of the shaft is fixed (see Fig. 2.4). Neglect the mass of the shaft.

Since steel weighs 0.283 lb per cu in., the weight of the disk is

$$W = \frac{\pi D^2}{4} h \times 0.283 = \frac{\pi 10^2}{4} \times 3 \times 0.283 = 66.7 \text{ lb.}$$

The radius of gyration squared is

$$r^2 = \frac{D^2}{8} = \frac{10^2}{8} = 12.5 \text{ in.}^2$$

The mass moment of inertia of the disk is

$$J = \frac{W}{g} r^2 = \frac{12.5 \times 66.7}{386} = 2.16 \text{ in.-lb-sec}^2.$$

The spring scale of the shaft is

$$k_t = G\frac{\pi d^4}{32L} = \frac{12(10^6)\pi \times 1^4}{32 \times 3 \times 12} = 32{,}750 \text{ in.-lb per rad.}$$

The natural frequency is

$$f_n = \frac{60}{2\pi}\sqrt{\frac{k_t}{J}} = 9.55\sqrt{\frac{32{,}750}{2.16}} = 1{,}174 \text{ cpm.}$$

A common situation involving torsional vibrations is two masses connected by a shaft, as shown in Fig. 2.5a. Examples of this application are a motor-generator set, a motor-driven pump, radial aircraft engine and propeller. The two masses will vibrate in opposite directions, and at some point between them there will be a node in the shaft. For purposes of analysis the shaft may be considered to be broken at the node (Fig. 2.5c) and the lengths L_1

and L_2 obtained. For the left portion, the natural frequency from Eq. (2.17) is

$$f_1 = \frac{60}{2\pi}\sqrt{\frac{GI_p}{J_1L_1}};$$

and for the right portion the frequency is

$$f_2 = \frac{60}{2\pi}\sqrt{\frac{GI_p}{J_2L_2}}.$$

Since the two frequencies must be identical and the terms G and I_p

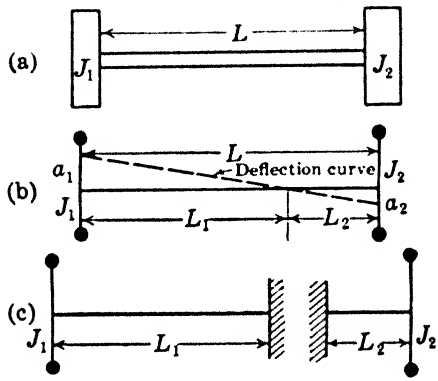


FIG. 2.5

are the same for both portions, the expressions may be equated, with the result that

$$\frac{L_1}{L_2} = \frac{J_2}{J_1};$$

that is, the node divides the total length L inversely with the inertias of the masses. As $L = L_1 + L_2 = L_1\left(1 + \frac{J_1}{J_2}\right)$, the length $L_1 = \frac{J_2L}{J_1 + J_2}$, which locates the position of the node from mass 1. The node location may be important in planning a nodal drive (see Sec. 7.3). The natural frequency is

$$f_n = f_1 = \frac{60}{2\pi}\sqrt{\frac{GI_p}{J_1L_1}} = \frac{60}{2\pi}\sqrt{\frac{GI_p}{L}\frac{(J_1 + J_2)}{J_1J_2}}$$

$$= \frac{60}{2\pi}\sqrt{\frac{k_t(J_1 + J_2)}{J_1J_2}} \text{ cpm.} \tag{2.18}$$

It should be observed that the deflection curve of the shaft between the masses is a straight line, as shown in Fig. 2.5b, since the torque is constant along the shaft.

EXAMPLE 2

A motor-generator set consists essentially of two masses connected by a steel shaft. The motor mass has a J of 2,000 lb-in.-sec^2, while the J of the generator is 1,600 lb-in.-sec^2. The 4 in. diameter shaft connecting the masses is 32 in. long. Determine the natural frequency of the system and the location of the node from the motor end.

The torsional spring scale of the shaft is

$$k_t = \frac{G\pi d^4}{32L} = \frac{12(10^6)\pi 4^4}{32 \times 32} = 9.43\ (10^6)\ \text{in.-lb per rad.}$$

The natural frequency from Eq. (2.18) is

$$f_n = \frac{60}{2\pi}\sqrt{\frac{k_t(J_1 + J_2)}{J_1 J_2}} = 9.55\sqrt{\frac{9.43(10^6)(2{,}000 + 1{,}600)}{2{,}000 \times 1{,}600}}$$

$$= 984 \text{ cpm.}$$

The node position from the motor end is

$$L_1 = \frac{J_2 L}{J_1 + J_2} = \frac{1{,}600 \times 32}{2{,}000 + 1{,}600} = 14.22 \text{ in.}$$

2.5 *EQUIVALENT SYSTEMS*

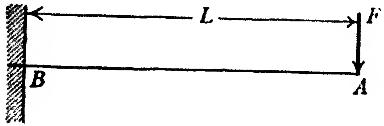
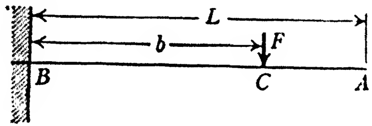
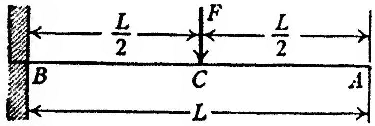
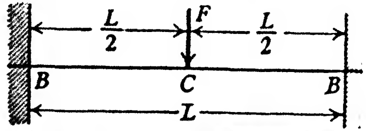
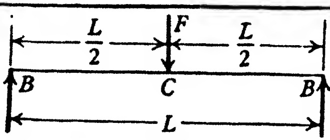
Equivalent torsional systems are treated in detail in Chap. 8, but it is convenient to consider them here briefly. The effect of a distributed mass, types of lateral springs, and the like, are also taken up in this section.

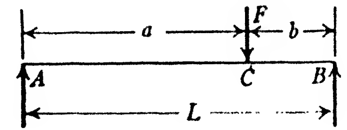
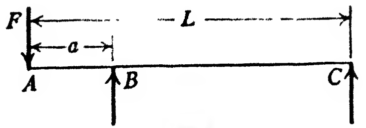
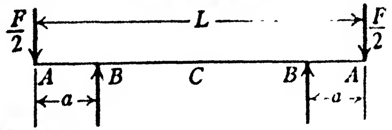
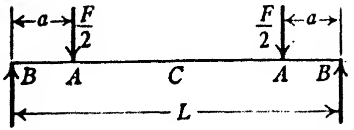
A. *Spring Scale*

The lateral spring scale is the force required to deflect the spring 1 in., that is, $k = F/y$. For helical springs this scale is given by

$$k = \frac{Gd^4}{8D^3N}, \tag{2.19}$$

FIG. 2.6. Uniform-Section Beam Formulas

Loading	Reaction, R	Bending moment, M	Deflection, y	Spring scale, k
	$R_B = F$	$M_B = FL$	$y_A = \frac{1}{3}\frac{FL^3}{EI}$	$k_A = \frac{3EI}{L^3}$
	$R_B = F$	$M_B = Fb$ $M_C = 0$	$y_A = \frac{Fa^2(3L - a)}{6EI}$ $y_C = \frac{1}{3}\frac{Fb^3}{EI}$	$k_A = \frac{6EI}{a^2(3L - a)}$ $k_C = \frac{3EI}{b^3}$
	$R_A = \frac{5}{16}F$ $R_B = \frac{11}{16}F$	$M_B = \frac{3}{16}FL$ $M_C = \frac{5}{32}FL$	$y_C = 0.00912\frac{FL^3}{EI}$	$k_C = \frac{EI}{0.00912L^3}$
	$R_B = \frac{F}{2}$	$M_B = M_C = \frac{FL}{8}$	$y_C = \frac{1}{192}\frac{FL^3}{EI}$	$k_C = \frac{192EI}{L^3}$
	$R_B = \frac{F}{2}$	$M_C = \frac{FL}{4}$	$y_C = \frac{1}{48}\frac{FL^3}{EI}$	$k_C = \frac{48EI}{L^3}$

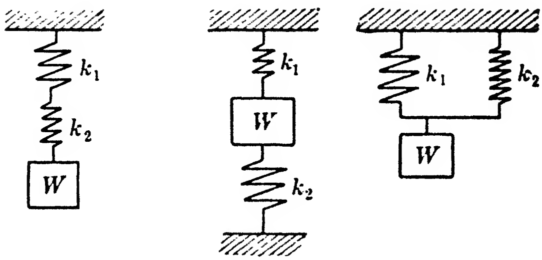
Loading	Reaction, R	Bending moment, M	Deflection, y	Spring scale, k
	$R_A = \frac{Fb}{L}$ $R_B = \frac{Fa}{L}$	$M_C = \frac{Fab}{L}$	$y_C = \frac{Fa^2b^2}{3EIL}$	$k_C = \frac{3EIL}{a^2b^2}$
	$R_B = \frac{FL}{L-a}$ $R_C = \frac{-Fa}{L-a}$	$M_B = Fa$	$y_A = \frac{Fa^2L}{3EI}$	$k_A = \frac{3EI}{a^2L}$
	$R_B = \frac{F}{2}$	$M_B = M_C = \frac{Fa}{2}$	$y_A = \frac{Fa^2(3L - 4a)}{12EI}$	$k_A = \frac{12EI}{a^2(3L - 4a)}$
	$R_B = \frac{F}{2}$	$M_A = M_C = \frac{Fa}{2}$	$y_A = \frac{Fa^2(3L - 4a)}{12EI}$	$k_A = \frac{12EI}{a^2(3L - 4a)}$

and the approximate maximum shear stress, neglecting Wahl's factor, in the wire is given by

$$s_s = \frac{8FD}{\pi d^3}, \tag{2.20}$$

where d = wire diameter in inches;
D = mean coil diameter in inches;
N = active number of coils;
F = maximum load on the spring in pounds;
G = shearing modulus of elasticity in pounds per square inch;
= $12(10^6)$ for steel.

Beams may be used as springs. Formulas for the bending



(a) (b) (c)

Fig. 2.7

moment, deflection, and spring scale of various types of beams are given in Fig. 2.6.

Frequently springs may be used in combination. The two combination types are series and parallel. The former is illustrated in Fig. 2.7a, where the two springs act in series. The total deflection of the weight equals the sum of the deflections of each spring as if it alone were present; thus, $y_{tot} = y_1 + y_2$. The equivalent spring scale may be found from $\frac{W}{k_e} = \frac{W}{k_1} + \frac{W}{k_2}$. Solving this for k_e gives

$$k_e = \frac{k_1 k_2}{k_1 + k_2}. \tag{2.21}$$

The spring arrangements shown in Figs. 2.7b and 2.7c represent parallel operation. In Fig. 2.7c the weight must be so placed that each spring will be deflected the same amount. The deflection of the weight is $y_{tot} = \dfrac{W}{k_1 + k_2}$. The equivalent spring scale is

$$\frac{W}{k_e} = \frac{W}{k_1 + k_2},$$

and

$$k_e = k_1 + k_2. \tag{2.22}$$

B. *Equivalent Shaft Length*

Generally shafts of machines are stepped between the masses, hence do not have a constant diameter. Therefore, it is necessary to reduce such shafts to that of an equivalent shaft of one diameter before using the equations of Sec. 2.4 or determining the torsional spring scale.

The torsional spring scale is $k_t = \dfrac{G\pi d^4}{32L}$. It is necessary to find the equivalent length of a constant-diameter shaft that will have the same spring scale as the stepped shaft. To keep k_t the same,

$$k_t = \frac{G\pi d_a^4}{32L_a} = \frac{G\pi d_e^4}{32L_e},$$

and

$$\frac{d_a^4}{L_a} = \frac{d_e^4}{L_e}.$$

Hence,

$$L_e = \frac{d_e^4}{d_a^4} L_a, \tag{2.23}$$

where the subscripts a and e represent the actual and equivalent values respectively. Frequently the diameter of the equivalent shaft is taken as 1 in. to simplify the calculations, since d_e^4 is then unity.

EXAMPLE

Determine the equivalent length of shafting 1 in. in diameter which will have the same torsional stiffness as the stepped shaft shown in Fig. 2.8.

Applying Eq. (2.23), the equivalent length is

$$L_e = \frac{d_e^4}{d_a^4} L_a = \frac{10 \times 1^4}{2^4} + \frac{10 \times 1^4}{\frac{1}{2}^4} = \frac{5}{8} + 160 = 160\tfrac{5}{8} \text{ in.}$$

Thus, the length of a shaft 1 in. in diameter which is equivalent to the left portion of the stepped shaft is only $\frac{5}{8}$ in., and the length equivalent to the right portion is 160 in. long. A given torque on either the shaft shown in Fig. 2.8 or the 1 in. diameter shaft that is $160\frac{5}{8}$ in. long will produce the same angular deflection.

C. *Effect of Distributed Mass*

It may be observed in Fig. 2.1 that the part of the spring near the stationary support remains at rest during a vibration, while the portion of the spring near the vibrating block has the same amplitude as the block. Hence, for a more accurate determination of the natural frequency, a certain proportion of the spring mass should be added to that of the block. It will be shown below that this portion should be one third.

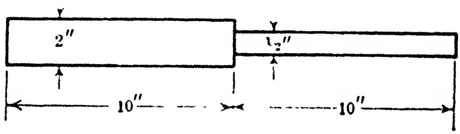


FIG. 2.8

The kinetic energy of a vibrating mass is $\frac{1}{2}mv^2$. The value of v of the spring is a maximum at the block and zero at the support. At a distance y from the support, the velocity is $\frac{y}{L} v_{max}$, if the total length of the spring is L. If the mass per unit length of the spring is m_s/L, the kinetic energy of a differential length dy is $\frac{1}{2}\frac{m_s}{L} dy \left(\frac{y}{L} v_{max}\right)^2$, and the total kinetic energy of the entire spring is

$$E_k = \int_0^L \frac{1}{2}\frac{m_s}{L} dy \left(\frac{y}{L} v_{max}\right)^2 = \frac{1}{2}\frac{m_s}{L^3} v_{max}^2 \int_0^L y^2 dy = \frac{1}{2}\frac{m_s}{L^3}\frac{L^3}{3} v^2_{max}$$

$$= \frac{1}{2}\frac{m_s}{3} v_{max}^2.$$

One third of the mass of the spring should therefore be added to the mass of the block before applying the frequency formula Eq. (2.9), and so on. In a similar manner it can be shown that one third of the mass moment of inertia of a shaft between the node and the disk should be added to the mass moment of inertia of the disk.

This rule does not apply to systems in which the deflection curve is not a straight line, such as beams of various types. For a cantilever beam with a weight at the end, approximately one quarter of the beam weight may be assumed to act with the weight.* If the mass of the spring is large compared with mass attached to it, the natural frequency may be appreciably changed.†

2.6 *GENERAL PROCEDURE FOR DETERMINING NATURAL FREQUENCIES*

It is occasionally necessary to determine the natural frequency of elastic systems where the solution is not obvious. There are two general methods of attacking such problems: namely, the force or torque method and the energy method.

A. *Force or Torque Method*

The general procedure in the force or torque method is to displace the body, or system, slightly and consider the forces, or torques that act upon it as a free body. There will be an inertia force equal to its mass, or torque equal to its mass moment of inertia, times the second derivative of the displacement with respect to time. This inertia force always acts toward the equilibrium position of the body, or system. The second force is the restoring force, or torque, due to the action of a spring or of gravity which also tends to return the body, or system, to its equilibrium position and equals the spring scale times the displacement. The sum of these two forces or torques, equals zero; hence, an equation similar to Eq. (2.7) may be set up, and the solution will be similar to that of Eq. (2.8) with a circular frequency of $\omega_n = \sqrt{C}$.

Applications of this method to pendulums are used in the following section.

B. *Energy Method*

The second method—the energy method—involves the equating of the maximum potential and kinetic energies in a manner similar

* J. P. Den Hartog, *Mechanical Vibrations*, p. 192, McGraw-Hill, New York, 1947.

† S. Timoshenko, *Vibration Problems in Engineering*, pp. 317–325, Van Nostrand, 1937.

to that followed in Sec. 2.3. This method is not as precise as the force method, since the type of motion (simple harmonic) must be assumed rather than having it in the solution, but it is frequently much shorter and easier to apply.

The body is displaced an amount x_o or θ_o, and expressions for the maximum potential and kinetic energies are set up. The maximum kinetic energy equals $\frac{1}{2}mv^2$ or $\frac{1}{2}J\omega^2$. The maximum velocity, assuming simple harmonic motion, is $x_o\omega_n$ for lateral vibrations or $\theta_o\omega_n$ for torsionals; hence, the maximum kinetic energy will be $\frac{1}{2}mx_o^2\omega_n^2$ or $\frac{1}{2}J\theta_o^2\omega_n^2$.

The maximum potential energy equals one half the maximum restoring force times the maximum displacement, or one half the spring scale times the maximum displacement squared ($\frac{1}{2}kx_o^2$ for lateral vibrations or $\frac{1}{2}k_t\theta_o^2$ for torsionals).

These two maximum energies may be equated and the natural circular frequency ω_n obtained. From this the natural cyclic frequency f_n is derived.

EXAMPLE

A mass m is attached at the mid-point of a thin wire of length L. The wire has a high initial tension T. Determine the natural frequency of the mass for small oscillations in a vertical direction if the weight of the wire is neglected.

Force Method. Since the tension T is large, it remains practically constant for small displacements x. Hence, the restoring force per side which acts toward the equilibrium position is $\frac{x}{L/2}T$, or $\frac{2Tx}{L}$ (see Fig. 2.9). The total restoring force acting upon the mass is double this, that is, $\frac{4Tx}{L}$.

The inertia force of the mass is $m\frac{d^2x}{dt^2}$. Then, for equilibrium

$$m\frac{d^2x}{dt^2} + \frac{4T}{L}x = 0;$$

and

$$\frac{d^2x}{dt^2} = -\frac{4T}{Lm}x.$$

And from Eqs. (2.7) and (2.9)

$$f_n = \frac{1}{2\pi}\sqrt{\frac{4T}{Lm}} \text{ cps.}$$

Energy Method. Assuming that the mass moves with simple harmonic motion of amplitude x_o and circular frequency ω_n, the maximum velocity is $x_o\omega_n$.

The maximum kinetic energy is $\frac{1}{2}mv_{max}^2 = \frac{1}{2}mx_o^2\omega_n^2$.

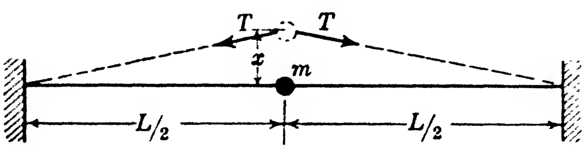


FIG. 2.9

The spring scale of the system is $k = F/x$, which from the previous method is $\frac{4Tx}{Lx} = \frac{4T}{L}$, and the maximum potential energy is

$$\frac{1}{2}kx_o^2 = \frac{1}{2}\frac{4T}{L}x_o^2.$$

Equating the maximum energies,

$$\frac{1}{2}mx_o^2\omega_n^2 = \frac{1}{2}\frac{4T}{L}x_o^2;$$

or

$$\omega_n^2 = \frac{4T}{Lm},$$

and

$$f_n = \frac{1}{2\pi}\sqrt{\frac{4T}{Lm}} \text{ cps.}$$

2.7 *PENDULUMS*

A pendulum is an example of a vibrating system in which the restoring force is generally due to gravity rather than to spring action. There are four general types: namely, the simple, the compound, the torsional, and the oscillating wire, which will be discussed in the order named. An important application of pendulums is in the determination of the mass moment of inertia J of a complicated body, such as a gear rotor, a flywheel, a turbine rotor with blades, and so on.

A. *Simple Pendulum*

In a simple pendulum the mass may be considered to be concentrated at a point on the end of a weightless rod. The solution may be found on a torsional basis by applying the principles of the force or torque method of the preceding section, that is, by equating the sum of the inertia and restoring torques to zero.

In Fig. 2.10, the force F tending to restore the bob, or mass, to the equilibrium position equals $W \sin \theta$, and the corresponding torque is $WR \sin \theta$. If θ is small (less than about 10 deg), the sine and the angle measured in radians are nearly the same; hence, the restoring torque may be taken as $WR\theta$. The inertia torque about the pivot point equals $J_o\alpha$, where J_o is the mass moment of inertia of the weight W about the pivot point O and equals $(W/g)R^2$; and α is the angular acceleration of the weight about O and equals $d^2\theta/dt^2$. Therefore, the inertia torque is

$$\frac{W}{g} R^2 \frac{d^2\theta}{dt^2}.$$

The equation of motion then becomes

$$\frac{W}{g} R^2 \frac{d^2\theta}{dt^2} + WR\theta = 0,$$

and

$$\frac{d^2\theta}{dt^2} = -\frac{g}{R}\theta.$$

And from Eqs. (2.7) to (2.10)

$$\theta = \theta_o \cos \sqrt{\frac{g}{R}}\, t \text{ rad.} \tag{2.24}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{R}} \text{ cps,} \tag{2.25}$$

$$\tau = 2\pi\sqrt{\frac{R}{g}} \text{ sec.} \tag{2.26}$$

It should be noted that the frequency of the pendulum is independent of the weight of the bob and depends only upon the length of the rod R and the acceleration of gravity.

B. *Compound Pendulum*

A compound pendulum has the weight distributed along its length, as illustrated in Fig. 2.11, where the pivot point is O. The weight W may be assumed to be concentrated at the center of gravity, which is at a distance R from the pivot.

The procedure for finding the equation of motion and frequency is the same as in the previous case. The restoring torque is WR

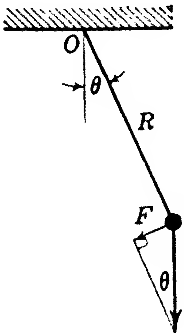


Fig. 2.10

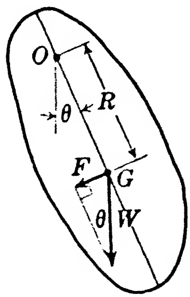


Fig. 2.11

$\sin\theta$; or, for small values of θ, it equals $WR\theta$, approximately. The inertia torque about the pivot is $J_o\,\alpha$ or $J_o \dfrac{d^2\theta}{dt^2}$. Hence, the differential equation is

$$J_o \frac{d^2\theta}{dt^2} + WR\theta = 0,$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{WR}{J_o}\theta.$$

And from Eqs. (2.7) to (2.10)

$$\theta = \theta_o \cos \sqrt{\frac{WR}{J_o}}\, t \text{ rad,} \qquad (2.27)$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{WR}{J_o}} \text{ cps,} \qquad (2.28)$$

$$\tau = 2\pi\sqrt{\frac{J_o}{WR}} \text{ sec.} \qquad (2.29)$$

Equations (2.28) and (2.29) may be used to determine the mass moment of inertia J_o of a complicated body by observing the period, or frequency, with which it oscillates when mounted as a compound pendulum. If it is desired to find the mass moment of inertia about the center of gravity, the following transfer formula of mechanics may be used:

$$J_o = J_G + \frac{W}{g} R^2, \tag{2.30}$$

where J_o = mass moment of inertia about the pivot;
J_G = mass moment of inertia about the center of gravity;
R = distance between the pivot and center of gravity.

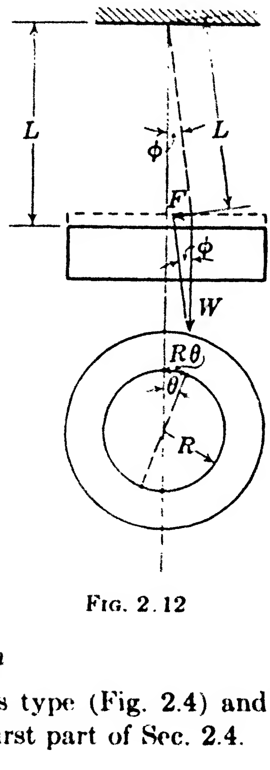


FIG. 2.12

C. *Torsional Pendulum*

A pendulum of this type (Fig. 2.4) and the derived equations were discussed in the first part of Sec. 2.4.

D. *Oscillating Pendulum*

The mass moment of inertia J of complicated bodies about their center of gravity may be found by suspending the body by wires placed equidistant from the center of gravity and allowing it to oscillate freely through small angles θ. From the observed period, or frequency, the value of J may be determined. This principle is illustrated in Fig. 2.12 for a disk that is suspended by two wires of length L, which are attached at a distance R from the center of gravity. As the disk is displaced through an angle θ, the wire will deflect through an angle ϕ from the vertical. From the figure it may be observed that $\sin \phi = R\theta/L$. For small angles of twist the restoring torque about the center of gravity equals RF or $WR \sin \phi$; or for small values of θ it equals $RW(R\theta/L)$ or $WR^2\theta/L$. The inertia torque about the center of gravity is $J\alpha$ or $J\, d^2\theta/dt^2$. Then,

$$J \frac{d^2\theta}{dt^2} + \frac{WR^2\theta}{L} = 0,$$

$$\frac{d^2\theta}{dt^2} = -\frac{WR^2}{JL}\theta.$$

And from Eqs. (2.7) to (2.10)

$$\theta = \theta_o \cos \sqrt{\frac{WR^2}{JL}} \text{ rad,} \tag{2.31}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{WR^2}{JL}} \text{ cps,} \tag{2.32}$$

$$\tau = 2\pi \sqrt{\frac{JL}{WR^2}} \text{ sec.} \tag{2.33}$$

Therefore, the mass moment of inertia about the center of gravity is

$$J = \frac{WR^2\tau^2}{4\pi^2 L} = \frac{WR^2}{4\pi^2 L f_n^2} \text{ in.-lb-sec}^2. \tag{2.34}$$

It may be observed that the above equations are independent of the number of wires used. Generally, three wires will give the greatest stability.

PROBLEMS

2.1. A weight of 30 lb is suspended on a spring having a scale k of 120 lb per in. Determine the natural frequency and the period of vibration of the system. *Ans.* 6.26 cps; 0.16 sec.

2.2. When a weight of 100 lb is placed at the mid-point of a simple beam, the beam has a maximum deflection of 0.360 in. Determine the natural frequency and the period of the vibration. *Ans.* 5.2 cps; 0.192 sec.

2.3. A weight W of 10 lb is supported on five springs, as shown in Fig. 2.13. Each spring has a scale of 50 lb per in. If the weight is displaced downward $\frac{1}{8}$ in. and to the right $\frac{1}{16}$ in. and allowed to vibrate freely, determine the resultant motion of its center of gravity. *Ans.* See Fig. 2.13A.

2.4. A weight of 75 lb is suspended on a steel helical spring having 10 active coils of W. & M. No. 2 wire (diameter 0.2625 in.). The mean diameter of the coils is 3 in. The system moves downward with a constant velocity of $1\frac{1}{2}$ fps when the upper end of the spring is instantaneously stopped. Determine the frequency of the vibration and the maximum stress in the spring, neglecting Wahl's factor. *Ans.* 111 cpm; 49,000 psi.

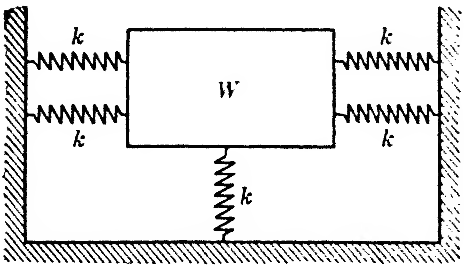


FIG. 2.13

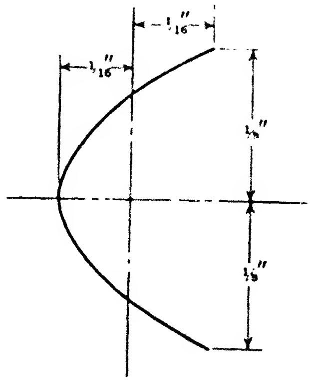


FIG. 2.13A

2.5. Repeat Prob. 2.4 with the change that the weight is being decelerated at the rate of 10 fps^2, but moving with a velocity of $1\frac{1}{2}$ fps as before. *Ans.* 111 cpm; 51,500 psi.

2.6. An elevator weighing 4 tons moves downward with a constant velocity of 300 fpm. At the instant when the rope length is 50 ft, an accident occurs and causes the drum to stop rotating. If the modulus of elasticity of the steel wire rope is $12(10^6)$ and the area of metal is 2 sq in., determine the frequency and the maximum stress in the rope due to the accident if the rope weight is neglected.

Ans. 420 cpm; 31,240 psi.

2.7. If a spring having a scale k of 1,500 lb per in. is placed between the end of the rope and the cage of the elevator in Prob. 2.6, determine the new frequency and maximum rope stress.

Ans. 80 cpm; 9,200 psi.

2.8. Repeat the first example of Sec. 2.4, but include the mass moment of inertia of the shaft. *Ans.* 1,173 cpm.

2.9. Determine the natural torsional frequency of the motor-driven pump set shown in Fig. 2.14. Locate the position of the node from the motor end of the steel shaft. *Ans.* 446 cpm; 57.2 in.

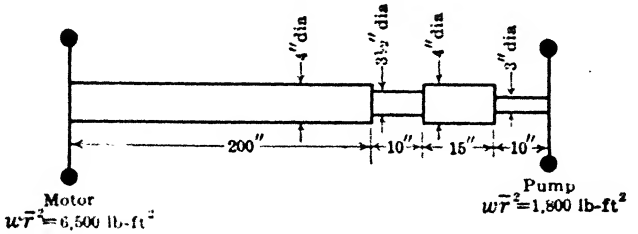


FIG. 2.14

2.10. Determine the natural torsional frequency of a radial airplane engine and propeller system, given the following data: J_{prop} = 132 in.-lb-sec^2; J_{eng} = 7 in.-lb-sec^2; k_t of shaft = 2.5(10^6) in.-lb per rad. *Ans.* 5,860 cpm.

2.11. Determine the natural frequency of the torsional pendulum shown in Fig. 2.15. Neglect the mass of the steel shaft. If the weight has an amplitude of 0.2 rad, find the angular velocity as the disk passes through the equilibrium position.

7.6 rad/sec

Ans. 362 cpm; ~~3.9~~ rad per sec.

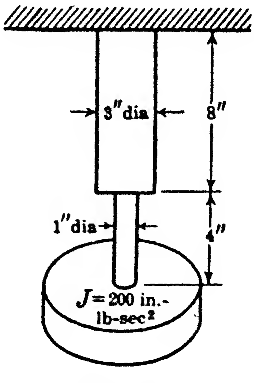


FIG. 2.15

2.12. In Fig. 2.7, let k_1 = 40 lb per in., k_2 = 45 lb per in., and W = 25 lb. Find the natural lateral frequency for each of the three cases. If the amplitude of vibration in each case is 1 in., find the energy involved and the velocity of the weight as it passes through the equilibrium position.

Ans. (a) 173 cpm, 10.6 in.-lb, 18.1 in. per sec; (b) and (c) 346 cpm, 42.5 in.-lb, 36.2 in. per sec.

2.13. Following the method outlined in Sec. 2.5C, show that one-third of the mass moment of inertia of the shaft of the torsional pendulum (Fig. 2.4) should be added to that of the disk in determining the natural frequency.

2.14. A weight of $\frac{1}{4}$ lb is to be suspended by four identical springs, as shown in Fig. 2.16, so that it will have a natural lateral frequency of 6 cps. What should be the scale of each of the four springs? *Ans.* 0.23 lb per in.

2.15. Determine an expression for the natural frequency of

4k

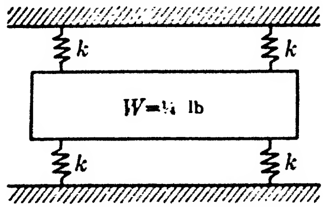


FIG. 2.16

the weight W in each case of Fig. 2.17. Neglect the weights of the steel beams or springs.

Ans. (a) $\frac{1}{2\pi}\sqrt{\frac{3EILg}{Wa^2b^2}}$ cps; (b) $\frac{1}{2\pi}\sqrt{\frac{192EIg}{WL^3}}$ cps;

(c) $\frac{1}{2\pi}\sqrt{\frac{3EIkg}{(3EI + kL^3)W}}$ cps; (d) $\frac{1}{2\pi}\sqrt{\frac{48EIkg}{(48EI + kL^3)W}}$ cps.

2.16. (a) Set up the equation of motion and solve for the natural frequency of a column of water in a manometer U tube. Let the

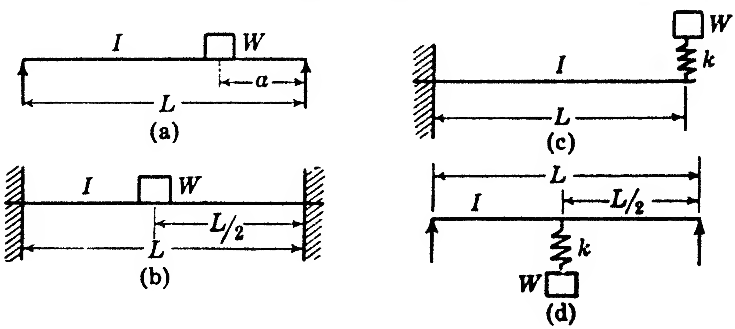


FIG. 2.17

diameter of the tube be d in., and let the total length of the column be L in. (b) If mercury having a specific gravity of 13.6 is substituted for the water, what will be the effect, if any?

Ans. (a) $f = \frac{1}{2\pi}\sqrt{\frac{2g}{L}}$ cps; (b) none.

2.17. A wooden plank has a specific gravity s and the following dimensions: thickness h, length L, width b. The plank is placed in a tank of water and depressed slightly a distance x. (a) Set up the equation of the vibratory motion, and solve for the natural frequency. (b) If the plank is placed in a tank of mercury having a specific gravity of 13.6, what will be the effect, if any, on the above?

Ans. (a) $\dfrac{1}{2\pi}\sqrt{\dfrac{g}{sh}}$ cps; (b) $\dfrac{1}{2\pi}\sqrt{\dfrac{13.6g}{sh}}$ cps.

2.18. Determine the natural lateral frequency of the weight

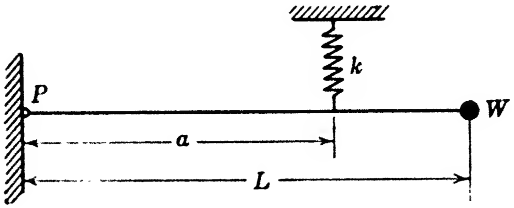


FIG. 2.18

shown in Fig. 2.18 if the weight of the bar pivoted at P may be neglected. *Ans.* $f_n = \dfrac{1}{2\pi}\dfrac{a}{L}\sqrt{\dfrac{kg}{W}}$ cps.

2.19. Determine the natural lateral frequency of the weight shown in Fig. 2.19, assuming that the rod is weightless, perfectly rigid, and pivoted at P. *Ans.* $\dfrac{1}{2\pi}\sqrt{\dfrac{kg}{9W}}$ cps.

2.20. If the mechanism of Fig. 2.19 is rotated clockwise 90

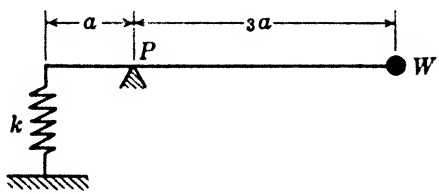


FIG. 2.19

deg, that is, so that the rod is vertical with the weight W at the bottom, what is the natural frequency?

Ans. $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{3a} + \dfrac{kg}{9W}}$ cps.

(*Note* that the first term under radical is the gravity or pendulum effect, while the second is the spring effect.)

2.21. What length of simple pendulum should be used on a clock so that it will have a period of 1 sec? ½ sec? *Ans.* 9.76 in., 2.44 in.

2.22. The pendulum shown in Fig. 2.20 is pivoted at O and consists of two weights W and w supported on a weightless rod. (a) For small displacements, derive an equation for the natural frequency. (b) If $W = 10$ lb, $w = 2$ lb, $R = 5$ in., $r = 2$ in., find the natural frequency. (c) If the small weight w is removed, find the natural frequency for the conditions set forth in part (b).

Ans. (a) $f_n = \dfrac{60}{2\pi}\sqrt{\dfrac{g(WR - wr)}{WR^2 + wr^2}}$ cpm; (b) 79.1 cpm; (c) 84 cpm.

2.23. What will be the frequency of oscillation of a steel disk 30 in. in diameter and 4 in. thick if it is pivoted about a point 10 in. from its center of gravity?

Ans. 40.6 cpm.

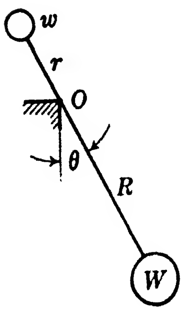


FIG. 2.20

2.24. A second-reduction welded gear rotor weighing 25,000 lb is suspended on a knife-edge located 50 in. from its center of gravity. Small oscillations about the knife-edge occur with a frequency of 21 cpm. What is the mass moment of inertia J of the rotor about its center of gravity?

Ans. 97,000 in.-lb-sec².

2.25. It is desired to obtain the mass moment of inertia J of a flywheel weighing 1,000 lb. It is suspended from the ceiling on three wires, each 4 ft long and attached to the wheel at a radial distance of 12 in. from its center. The time required for 100 complete oscillations is 145 sec. Determine the mass moment of inertia about the center of gravity. *Ans.* 159.5 in.-lb-sec².

Chapter 3

UNDAMPED FORCED VIBRATIONS—SINGLE DEGREE OF FREEDOM

3.1 *IMPORTANCE AND APPLICATIONS*

As was pointed out in Sec. 2.1, the amount of damping that is present in many actual vibrating systems is extremely small. Hence, the principles developed in this chapter may be used to solve and explain many actual cases of systems having forced vibrations.

The chief applications of these principles are the isolation of vibrating machinery forces from their surroundings (for example, isolation of automobile-engine vibrations from the body of the car, refrigerator-compressor vibrations from the house), and the design of isolators to prevent outside vibrations from reaching delicate instruments (for example, radios in airplanes, seismic apparatus). An explanation of the action that takes place in a single disk as the rotation passes through the critical speed is given.

3.2 *EQUATION OF MOTION*

The system for the case of undamped forced vibrations may be visualized by Fig. 3.1, in which a harmonic force $F_o \cos \omega t$ acts upon the mass m suspended on a spring having a scale k.

Considering the mass as a free body that is displaced downward from the equilibrium position a distance x due to the impressed force $F_o \cos \omega t$ and neglecting the static weight of the block, which is constant in magnitude and direction, the following dynamic forces are exerted:

a. The spring force acts upward (always toward the equilibrium position) and equals kx.

b. The inertia force also acts upward (always toward the equilibrium position) and equals $m \frac{d^2x}{dt^2}$.

c. The impressed force acts downward to depress the mass and equals $F_o \cos \omega t$. For equilibrium of the vertical vibratory forces, the equation is

$$m \frac{d^2x}{dt^2} + kx = F_o \cos \omega t. \tag{3.1}$$

A solution of this equation is that the block moves harmonically with the same frequency as the impressed force; thus,

$$x = x_o \cos \omega t. \tag{3.2}$$

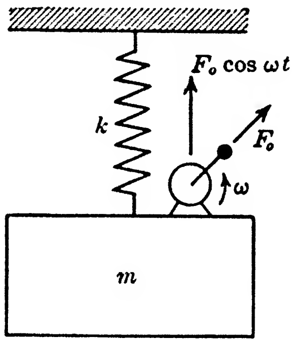


FIG. 3 1

Since the second derivative of Eq. (3.2) with respect to time is

$$\frac{d^2x}{dt^2} = -x_o\omega^2 \cos \omega t,$$

Eq. (3.1) may be written

$$-m\omega^2 x_o \cos \omega t + kx_o \cos \omega t = F_o \cos \omega t,$$

or

$$-m\omega^2 x_o + kx_o = F_o.$$

Dividing through by k and rearranging terms,

$$x_o\left(1 - \frac{m}{k}\omega^2\right) = \frac{F_o}{k}.$$

In Sec. 2.2 it was seen the $\frac{m}{k} = \frac{1}{\omega_n^2}$; hence, this equation becomes

$$x_o\left(1 - \frac{\omega^2}{\omega_n^2}\right) = \frac{F_o}{k}.$$

Two new notations may now be introduced. The first is that $r = \dfrac{\omega}{\omega_n} = \dfrac{f}{f_n}$; and the second that $\dfrac{F_o}{k} = x_{st}$. The term x_{st} is the

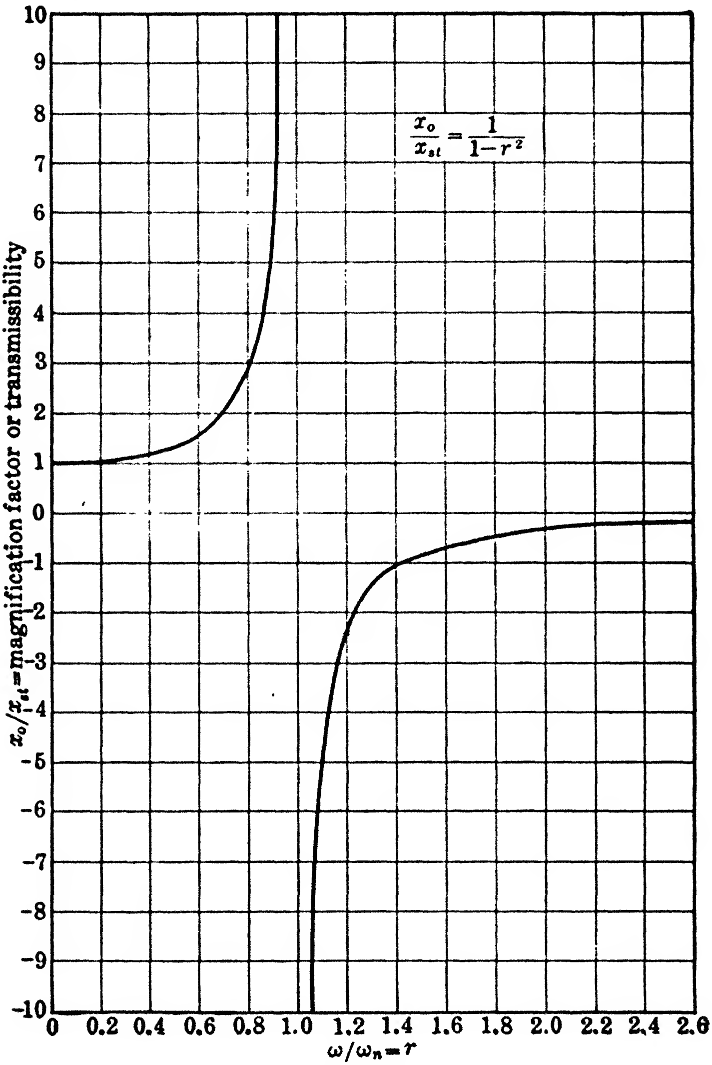


FIG. 3.2

deflection of the spring due to the impressed force F_o acting as a static load. It should not be confused with δ_{st} used previously, which is the spring deflection due to the static weight of the block.

With the above notations, the preceding equation may be written

$$\frac{x_o}{x_{st}} = \frac{1}{1 - r^2}. \tag{3.3}$$

A plot of x_o/x_{st} against r or ω/ω_n is shown in Fig. 3.2. When a particular impressed force F_o acts upon a given mass and spring system, the values of x_{st} and ω_n are fixed and constant. Thus, this curve shows the amplitude of the mass x_o for various frequencies ω of the impressed force.

The ordinates of the curve of Fig. 3.2 are known as the *magnification factor*, since they represent the ratio of the actual deflection of the mass at a given frequency to the deflection that would be obtained if the impressed force were applied as a static load.

If the force acts at a very low frequency (r or ω near zero), the mass will be deflected through an amplitude of x_{st} or F_o/k, that is, x_o/x_{st} is close to unity, and the motion of the mass will be in phase with the force. If the force is applied with a frequency close to the natural one of the system (r about unity), the force and motion of the block act together to produce large amplitudes. This condition results in tremendous stresses and rapid failure of the parts. If the force is applied at a very high frequency, the inertia of the mass prevents it from following the force, with the result that the block remains practically stationary in space, that is, x_o approaches zero. It may also be noted that in this region the value of x_o/x_{st} is negative, indicating that x_o and x_{st} (or F_o/k) oppose each other. Therefore, they are out of phase, and as the force acts downward, the mass is displaced upward, and vice versa.

EXAMPLE

An impressed force of 10 lb acts harmonically on a 20-lb block suspended on a spring having a scale of 40 lb per in. Determine the amplitude of the block if the frequency of the impressed force is (a) 10 cpm, (b) 250 cpm, and (c) 4,000 cpm. Assume that the block is constrained to a single lateral degree of freedom.

The static deflection due to the weight of the block is

$$\delta_{st} = \frac{W}{k} = \frac{20}{40} = 0.5 \text{ in.}$$

The natural frequency of the system from Eq. (2.11) is

$$f_n = \frac{187.5}{\sqrt{\delta_{st}}} = \frac{187.5}{\sqrt{0.5}} = 265 \text{ cpm.}$$

The static deflection of the block due to the impressed force F_o is

$$x_{st} = \frac{F_o}{k} = \frac{10}{40} = 0.25 \text{ in.}$$

a. For an impressed frequency of 10 cpm, the ratio is

$$r = \frac{f}{f_n} = \frac{10}{265} = 0.0377,$$

and $r^2 = 0.001422$.

Hence, by Eq. (3.3), the amplitude of the block is

$$x_o = x_{st}\frac{1}{1 - r^2} = 0.25\frac{1}{1 - 0.001422} = 0.25+ \text{ in.},$$

which is practically the same as x_{st}.

b. For an impressed frequency of 250 cpm, the ratio is

$$r = \frac{250}{265} = 0.9435, \qquad r^2 = 0.89,$$

and the block amplitude is

$$x_o = 0.25\frac{1}{1 - 0.89} = 2.27 \text{ in.},$$

which is quite large and accompanied by high spring stresses.

c. For an impressed frequency of 4,000 cpm, the ratio is

$$r = \frac{4{,}000}{265} = 15.1, \qquad r^2 = 228,$$

and the block amplitude is

$$x_o = 0.25\frac{1}{1 - 228} = -0.0011 \text{ in.},$$

which is practically stationary in space compared with x_{st}.

3.3 *TRANSMISSIBILITY*

One of the common problems occurring in vibration work is the isolation of forces that are set up in the machine from being transmitted to the ground or surrounding structure. This isolation may be accomplished by mounting the machine on flexible supports, such as steel springs; or rubber, felt, or cork pads. Since the damping effect of steel springs is very small, only they will be considered in this chapter. The other types of isolators will be discussed in Chap. 5.

The measure of the effectiveness of a support is the *transmissibility,* which is the ratio of the force transmitted to the foundation to the force developed in the machine. Any transmitted force is due to the change in the length of the supporting spring; hence, its maximum value is kx_o, whereas the maximum impressed force is F_o. The ratio of these two, or the transmissibility, is $\frac{kx_o}{F_o}$. This ratio is the same as $\frac{x_o}{x_{st}}$, since $\frac{k}{F_o}$ is $\frac{1}{x_{st}}$; so that Eq. (3.3) and Fig. 3.2 apply to this case, and the transmissibility is

$$t = \frac{1}{1 - r^2}.$$

The ideal condition would be to have no transmitted force, that is, a transmissibility of zero, which means that the scale of the springs would have to be zero. Obviously this is impossible. If the machine is mounted directly on the foundation or structure without springs, the equivalent spring scale is infinite, and all the impressed force is transmitted through the isolator.

In Fig. 3.2, it may be observed that for a given system the transmissibility is unity or 100 per cent if the impressed force acts very slowly (r near zero); that is, the springs deflect an amount x_{st} or F_o/k, and thus all the force is transmitted. On the other hand, for high impressed frequencies (r large), the mass, owing to its inertia, is not able to follow the impressed force and hence, remains almost stationary in space. The accompanying small value of x_o or spring deflection results in low transmitted forces and consequently low transmissibilities.

It should be noted also that for values of r between zero and $\sqrt{2}$, or 1.41, the value of x_o/x_{st}, or transmissibility, is greater than one. In this region the inertia of the mass acts with the impressed force to make the transmitted force greater than that impressed. Therefore, the use of springs under these conditions makes matters worse rather than better and so should be avoided.

Isolators designed to give frequency ratios of r greater than $\sqrt{2}$ under operating conditions will reduce the impressed forces transmitted to the foundation; the higher the ratio, the lower the transmissibility. From a practical standpoint the ratio r should be at least 2.5 or 3, which corresponds to transmissibilities of 18.9 or 12.5 per cent.

Care must be exercised in the use of signs. In the region where the transmissibility is effective, the displacement of the block and the impressed force are opposed, and the transmissibility, or x_o/x_{st}, will have a negative sign.

EXAMPLE

A printing press creates a disturbing force of 800 lb at a frequency of 1,450 cpm. The press weighs 2,500 lb. It is desired to reduce the force transmitted to the building to not more than 20 lb by supporting the press on six springs, each taking an equal share of the total load. Determine the scale of each spring, assuming that the press can move only in a vertical direction; that is, that it cannot "rock" or move horizontally.

$$\text{Transmissibility} = \frac{\text{transmitted force}}{\text{impressed force}} = \frac{x_o}{x_{st}} = \frac{20}{800} = \frac{1}{40},$$

and, since it is less than 1, should be considered as negative.

Using Eq. (3.3),

$$\frac{x_o}{x_{st}} = -\frac{1}{40} = \frac{1}{1 - r^2},$$

and

$$r^2 - 1 = 40 \qquad \text{and} \qquad r^2 \doteq 41 = \frac{\omega^2}{\omega_n{}^2}.$$

The impressed circular frequency is $\omega = \dfrac{1,450}{60} 2\pi = 151.8$ rad per sec, and $\omega^2 = 23,000$.

Since $\omega_n^2 = \frac{k}{m}$ and $\frac{\omega^2}{\omega_n^2} = 41$, the total spring scale is

$$k = \frac{m\omega^2}{41} = \frac{2{,}500}{386} \times \frac{23{,}000}{41} = 3{,}635 \text{ lb per in.},$$

and the spring scale of each spring is $\frac{3{,}635}{6} = 606$ lb per in.

3.4 *RELATIVE MOTION OF BLOCK AND SUPPORT*

Another type of forced-vibration problem occurs when the support of a spring-mass system vibrates with simple harmonic motion.

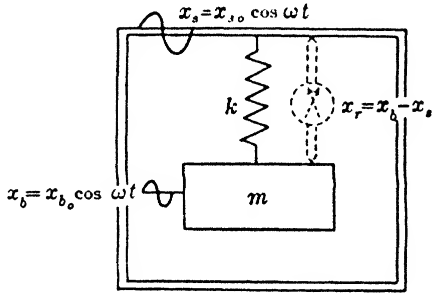


FIG. 3.3

The solution of this problem finds application in the design of vibrometers to measure amplitudes, in seismic apparatus, and similar instruments.

In Fig. 3.3, let it be assumed that the support to which the spring is attached vibrates with simple harmonic motion described by the equation

$$x_s = x_{s_o} \cos \omega t. \tag{3.5}$$

The action of this motion is to cause the block to vibrate harmonically at the same frequency with the motion

$$x_b = x_{b_o} \cos \omega t. \tag{3.6}$$

The change in length of the spring then is $x_b - x_s$, which creates an impressed force on the block through the spring of $k(x_b - x_s)$.

If we take the block as a free body and consider the forces acting upon it, the following equations result:

$$m \frac{d^2 x_b}{dt^2} + k(x_b - x_s) = 0,$$

or

$$m\frac{d^2x_b}{dt^2} + kx_b = kx_s. \tag{3.7}$$

If kx_{s_o} is considered to be the impressed force acting on the block, Eq. (3.7) is similar to Eq. (3.1); that is, x_{s_o} corresponds to the x_{st} term of that equation. By a development similar to that given there, an equation similar to Eq. (3.3) results:

$$\frac{x_{b_o}}{x_{s_o}} = \frac{1}{1 - r^2}. \tag{3.8}$$

To measure amplitudes of vibration, a mechanics' dial gauge is placed between the support and block as shown by the dotted lines in Fig. 3.3. The hand of the gauge will vibrate at the impressed frequency and, since the motion at the end of the swing is relatively slow, can be read quite accurately. The motion of the hand is double the relative amplitude of $x_{b_o} - x_{s_o}$, which is x_{r_o}; hence, the total travel of the hand must be divided by 2 to get this value. Care must be exercised to select an indicator with a small inertia in its moving parts so that the motion will be followed completely.

Since x_{s_o} is the amplitude to be measured and x_{r_o} the value read on the vibrometer gauge, it is desirable to determine the relationship between them. This may be accomplished as follows:

$$x_{b_o} = x_{s_o}\frac{1}{1 - r^2}. \tag{3.8}$$

$$x_{r_o} = x_{b_o} - x_{s_o} = x_{s_o}\frac{1}{1 - r^2} - x_{s_o} = x_{s_o}\left(\frac{1}{1 - r^2} - 1\right)$$

$$= x_{s_o}\left(\frac{r^2}{1 - r^2}\right). \tag{3.9}$$

The values of x_{r_o}/x_{s_o} are plotted against r in Fig. 3.4, and it may be noted that the curve shape is similar to that of Fig. 3.2, the only difference being that in the latter the ordinates are one point lower than in Fig. 3.2. When the frequency of the support is low (r near zero), the whole system moves as a unit, so that x_{r_o}, or the stretch of the spring, is practically zero. For high frequencies of the support (r large), the mass, because of its inertia, tends to remain stationary in space, and the stretch of the spring x_{r_o} is approximately equal to the motion of the support.

It should again be observed that for values of r greater than 1, the sign of x_{r_o}/x_{s_o} is negative and must be so used. It should also be observed that the gauge reading x_r will be closer to the motion of

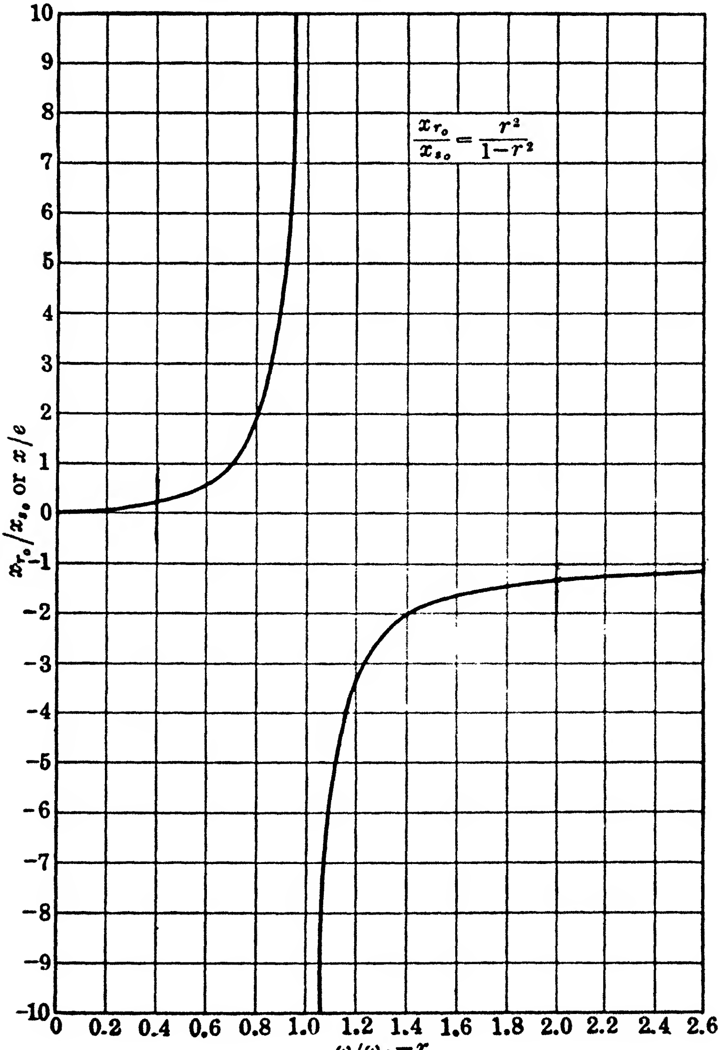


Fig. 3.4

the support x_s as the ratio r is increased, that is, by the use of a large mass and/or weak springs. The mass size is sometimes limited by the size of the body whose amplitude is being measured.

In aircraft work the mass may be as small as 1 lb or less; whereas for heavy work, such as in building vibrations, weights of 25 lb or greater may be used, and for seismographs the weight may be tons.

A similar device may be used to determine accelerations. The spring scale is made large in such cases for stiffness and strength purposes, and the resulting deflections are consequently small.

Figure 3.5 illustrates an instrument that may be used to determine the path of the resultant motion of a vibrating body. A low-power microscope is mounted in a heavy block suspended on weak

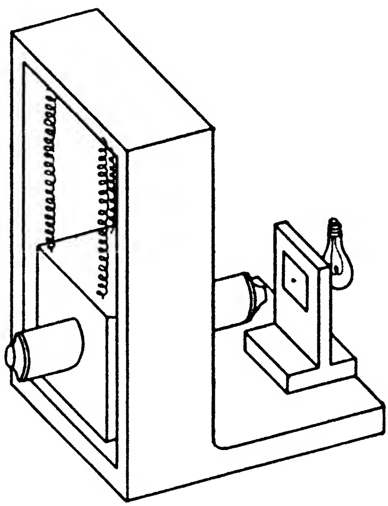


FIG. 3.5

springs. Through it may be observed the motion of a pinhole placed in an opaque sheet, such as tinfoil, attached to the vibrating body and having a light source placed behind it. The pinpoint of light will appear as a closed curve, or Lissajous figure, describing the motion of the body being observed. By the proper design of the block and spring system, the microscope motion may be made negligible.*

A similar device, known as a *torsiograph,** may be used to measure the amplitude of torsional vibrations. A relatively heavy

* T. C. Rathbone, "Turbine Vibration and Balancing," *Trans. A.S.M.E.* 1929, APM-51-23; S. Timoshenko, *Vibration Problems in Engineering*, Van Nostrand, New York, 1937, pp. 443–452; W. K. Wilson, *Practical Solution of Torsional Vibration Problems*, Chap. 8, Wiley, New York, 1940.

hollow cylindrical mass is separated from a shaft by means of a weak spiral spring, as shown schematically in Fig. 3.6. If no vibrations are present in the shaft, the torsiograph rotates as a unit with it; but when vibrations occur, there will be relative motion between the shaft and the rotating mass, which continues to rotate with practically constant angular velocity. These amplitudes may be recorded. The principle of operation and analysis of results follow the procedure described above for lateral vibrations.

Example

Determine the scale of the springs for a vibrometer with a block weighing 20 lb. so that the difference in the reading on the dial

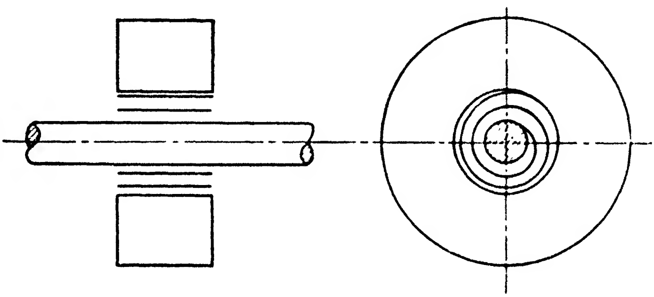


Fig. 3.6

indicator and that of the vibrating machine will not be greater than 3 per cent when the machine vibrates with a frequency of 1,000 cpm. If the machine amplitude is 0.003 in., determine the corresponding dynamic load on the springs of the vibrometer. Assume that the machine has a pure vertical motion.

To satisfy the given conditions, the ratio of x_{r_0}/x_{s_0} must equal -1.03 (negative because the value of r is greater than 1).

Substituting this value in Eq. (3.9),

$$\frac{x_{r_0}}{x_{s_0}} = \frac{r^2}{1 - r^2} = -1.03.$$

Then,

$$-1.03 + 1.03r^2 = r^2,$$

and

$$r^2 = 34.35 = \frac{\omega^2}{\omega_n{}^2}$$

The impressed frequency is $\omega = \frac{f}{60} 2\pi = \frac{1{,}000}{60} 2\pi = 104.7$ rad per sec.

Since $\omega_n{}^2 = k/m$,

$$k = \frac{m\omega^2}{r^2} = \frac{20}{386} \times \frac{104.7^2}{34.35} = 16.5 \text{ lb per in.}$$

If $x_{s_0} = 0.003$ in. and $x_{r_0}/x_{s_0} = -1.03$, the stretch of the spring is $x_{r_0} = -1.03 \times 0.003 = -0.00309$ in. (the negative sign merely indicating the phase displacement), and the dynamic load on the springs is $kx_{r_0} = 16.5 \times 0.00309 = 0.051$ lb.

3.5 *CRITICAL SPEED OF A SINGLE DISK ON A SHAFT*

An interesting application of forced vibrations with damping neglected is that of a single disk mounted on a vertical shaft to eliminate gravitational forces.

In Fig. 3.7, let point O be the center of the disk through which the shaft center line passes, and let G be its center of gravity. Because of slight variations in the disk density, these points will not usually coincide, but will be separated by a distance e, which is constant. Let x be the deflection of the shaft at the disk, which is due to centrifugal action, and let k be the spring scale of the shaft at the disk.

When the shaft rotates at an angular velocity of ω, an impressed or centrifugal force is set up, which may be considered to act at the center of gravity G, and deflects the shaft at the disk a distance x. Hence, the force equals $m(x + e)\omega^2$. This force is balanced by the restoring force of the shaft, which equals kx. For equilibrium,

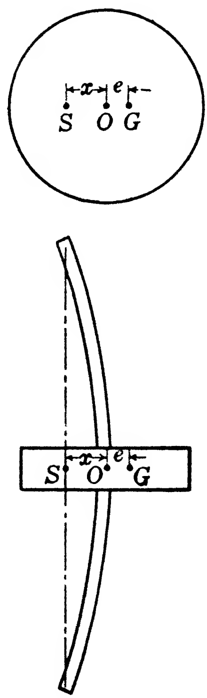


FIG. 3.7

$$m(x + e)\omega^2 - kx = 0,$$

and

$$mx\omega^2 - kx = -me\omega^2.$$

Dividing both sides by $m\omega_n{}^2$,

$$x\left(\frac{\omega^2}{\omega_n{}^2} - \frac{k}{m\omega_n{}^2}\right) = -e\,\frac{\omega^2}{\omega_n{}^2}.$$

But $\omega_n^2 = k/m$. Then,

$$x\left(\frac{\omega^2}{\omega_n^2} - 1\right) = -e\frac{\omega^2}{\omega_n^2},$$

$$\frac{x}{e} = \frac{r^2}{1 - r^2}. \tag{3.10}$$

The form of Eq. (3.10) is the same as that of Eq. (3.9) and may be illustrated by Fig. 3.4 if x/e is substituted for x_{r_0}/x_{s_0}.

An analysis of the physical action that takes place as the speed of rotation is increased is interesting. At all times there are two forces acting radially on the disk: namely, the centrifugal force at the center of gravity and the spring, or restoring force, at the geometric center.

For speeds below the critical, the center of gravity G is always outside the geometric center O of the disk. The relative motion between these two points is that G is rotating about O. The geometric center of the disk is deflected a distance x from the bearing center line S so as to achieve equilibrium between the centrifugal force $m(e + x)\omega^2$ and the shaft restoring force kx. As the rotative speed, or speed ratio r, becomes higher, the amplitude x of the disk also increases.

At the critical speed ($r = 1$), the amplitude x of the disk is infinite, as shown by Eq. (3.10), and the system is said to be in "indifferent equilibrium."

When the speed is above the critical, the amplitude x is negative by Eq. (3.10), and the center of gravity G lies inside the geometric center of the disk O. For extremely high speeds the value of x/e approaches -1, which means that x approaches $-e$ and the disk tends to rotate about its center of gravity. Therefore, the disk has little or no centrifugal force acting upon it.

The effect of damping on the action at the critical speed on a single disk will be considered in Sec. 5.3.

EXAMPLE

A small high-speed steam turbine has a single disk weighing 15 lb mounted at the mid-point of a $\frac{3}{8}$ in. diameter steel shaft. The bearing span is 15 in. Owing to slight manufacturing inaccuracies, the center of gravity of the disk is 0.001 in. from the center of rotation. If the turbine rotates 3,600 rpm, determine the amplitude of the steady-state forced vibration, the dynamic force transmitted to the bearings, and the stress in the shaft due to the dynamic load if the shaft weight is neglected.

From Fig. 2.6, the static deflection of the disk is

$$\delta_{st} = \frac{WL^3}{48EI} = \frac{15 \times (15)^3 \times 64}{48 \times 30(10^6) \times \pi(\frac{3}{8})^4} = 0.0362 \text{ in.}$$

The natural frequency, or critical speed, of the system, by Eq. (2.11), is

$$f_n = \frac{187.5}{\sqrt{\delta_{st}}} = \frac{187.5}{\sqrt{0.0362}} = 987 \text{ cpm.}$$

The speed ratio is $r = f/f_n = 3{,}600/987 = 3.65$ and $r^2 = 13.33$.

Applying Eq. (3.10), in which $e = 0.001$ in., the shaft deflection becomes

$$x = e\frac{r^2}{1 - r^2} = 0.001\frac{13.33}{1 - 13.33} = -0.001081 \text{ in.}$$

(The negative sign indicates that the center of gravity has moved in toward the center of rotation.)

The dynamic bearing load may be found on the basis of either the centrifugal force or the shaft-deflection force.

The centrifugal force acting on the disk is

$$m(x + e)\omega^2 = \frac{15}{386}(-0.001081 + 0.001)\left(\frac{3{,}600}{60}2\pi\right)^2 = 0.449 \text{ lb,}$$

which is also the load transmitted to the bearings.

The spring scale of the shaft is $k = W/\delta_{st} = 15/0.0362 = 415$ lb per in., and since the shaft deflection is $x = 0.001081$ in., the reaction at the bearings is $kx = 0.001081 \times 415 = 0.449$ lb.

From Fig. 2.6, the maximum moment on the shaft is $M = FL/4$, and the force F is the force due to centrifugal action, or 0.449 lb; hence, $M = 0.449 \times \frac{15}{4} = 1.68$ in.-lb.

The shaft stress due to the vibration is

$$s = \frac{Mc}{I} = \frac{M32}{\pi d^3} = \frac{1.68 \times 32}{\pi(\frac{3}{8})^3} = 324 \text{ psi.}$$

PROBLEMS

In the following problems, unless otherwise specified, it should be assumed that the motion is one of a single degree of freedom and takes place in a vertical direction.

3.1. The total weight of a 900-rpm generator is 2,500 lb. The armature weighs 800 lb, and its center of gravity is 0.015 in. from the axis of rotation. When the generator was mounted directly upon a heavy foundation, severe vibration forces were transmitted to the building. It is proposed to mount the generator on eight steel springs to reduce this transmitted force to one hundredth of the impressed force. Assume that the foundation is rigid, and calculate the required spring scale and the maximum dynamic load on each spring. Assume that the springs are so placed that each takes an equal share of the load. *Ans.* 71.16 lb per in.; 0.345 lb.

3.2. A reciprocating pump weighing 1,000 lb runs at 300 rpm. The forces transmitted to the rigid foundation on which it is bolted are objectionable. It is desired to reduce these forces to one tenth of their present value by mounting the pump on steel springs at each of the four corners of the base plate. Assume that the weight is equally divided among the four springs, and determine the scale of each. *Ans.* 58 lb per in.

3.3. What would be the percentage of the induced force transmitted to the foundation if the speed of the pump of Prob. 3.2 drops to 200 rpm and the springs designed in Prob. 3.2 are used?

Ans. 25.6 per cent.

3.4. A radio set weighing 100 lb is located in an airplane cabin which vibrates with an amplitude of 0.004 in. at an engine speed of 2,000 rpm. Determine the scale of four steel springs required to reduce the amplitude of the set to 0.0002 in. Assume that each spring takes an equal share of the weight, and calculate the maximum total load for which each spring must be designed.

Ans. 135 lb per in.; 25.57 lb.

3.5. If the speed of the engine in Prob. 3.4 drops to 1,700 rpm, but the cabin amplitude remains 0.004 in., what would be the amplitude of the radio set and the maximum total load on each spring? Assume that the mount is the one designed in Prob. 3.4.

Ans. 0.0006 in., 25.62 lb.

3.6. Owing to local resonance, the floor of a building vibrates vertically with a frequency of 1,500 cpm and an amplitude of 0.002 in. Except for damage to the instruments on an instrument board located at this spot, the condition is not too objectionable. It is decided to isolate the board, which has a weight of 200 lb, with four steel springs that are to reduce this amplitude to 0.0002 in. What should be the scale of each spring, if the weight is divided equally among them? *Ans.* 290 lb per in.

3.7. What are the amplitude, maximum velocity, and maximum acceleration of a vibrating structure if a vibrometer attached to it records a relative amplitude of 0.0018 in.? The static deflection of the vibrometer weight is $\frac{3}{4}$ in., and the structure vibrates with a frequency of 110 cpm.

Ans. 0.0052 in., 0.06 in. per sec, 0.69 in. per sec², respectively.

3.8. A motor weighing 50 lb runs at 1,750 rpm and has an unbalanced centrifugal force of 10 lb in the armature. The motor is mounted on isolators having little damping. The spring scale of the isolators in the vertical plane is 300 lb per in., and in the horizontal plane it is 2,500 lb per in. Determine the force transmitted to the foundation in each plane and the resultant motion of the center of gravity of the motor.

Ans. $F_v = 0.75$ lb, $F_h = 13.5$ lb; $x_{o_v} = 0.0025$ in., $x_{o_h} = 0.0054$ in.; ellipse.

3.9. It is desired to measure the maximum acceleration of a machine part which vibrates violently with a frequency of 700 cpm. An accelerometer (vibrometer with a stiff spring) is attached to it, and the total travel of the pointer on the dial indicator is found to be 0.326 in. If the accelerometer has a 1-lb block and a spring scale of 90 lb per in., what are the maximum amplitude and maximum acceleration of the part? *Ans.* 0.896 in.; 4,800 in. per sec².

3.10. A torsiograph having a natural frequency of 200 cpm is placed on a 1 in. diameter steel shaft 3 ft long rotating at 2,000 rpm. The total relative twist between the shaft and mass of the torsiograph is recorded as 1.42 deg (relative amplitude of 0.71 deg). What are the amplitude of the shaft vibration and the corresponding maximum shaft stress? $\left(\text{Note that } s_s = \dfrac{Tc}{I_p} \text{ and } \theta = \dfrac{TL}{GI_p} \text{ rad}\right)$.

Ans. 0.703 deg; 2,050 psi.

3.11. Repeat the example of Sec. 3.5 for a rotational speed of 1,000 rpm. *Ans.* 0.038 in., 15.8 lb, 11,400 psi.

3.12. A motor weighing 125 lb is mounted on a simple beam that has a spring scale at that point of 200 lb per in. The motor armature weighs 25 lb and has an eccentricity of 0.003 in. What will be the amplitude of vibration of the motor when it runs 1,760 rpm?

Ans. 0.00061 in.

Chapter 4

DAMPED FREE VIBRATIONS—SINGLE DEGREE OF FREEDOM

4.1 *INTRODUCTION*

The cases considered in the two preceding chapters neglect damping, which, while it may be reasonably close to actual conditions for many applications, is not in true agreement with the laws of nature. Observation shows that any free vibration will die out in time; hence, the presence of a damping force should be recognized.

The damping action may be due to (a) the rubbing of two surfaces together, as in the case of poorly riveted joints or bearing friction; (b) resistance to the motion of the fluid in which the system vibrates, such as air, oil, or water; or (c) the internal friction in the member being vibrated.

The frictional resistance between dry surfaces is generally considered to be independent of the rubbing speed, although the coefficient is much greater for static friction than for kinetic; it is more a function of the two materials and their surface roughness. This type of resistance is generally referred to as Coulomb, or constant, damping and will be considered in Sec. 4.6.

The resistance between lubricated surfaces or bodies moving through a fluid at low velocity is directly proportional to the speed. This type of action occurs in perfectly lubricated bearings, dashpots, or bodies moving with relatively low velocity through air, oil, or water. The action is known as *viscous damping* and may be evaluated from the expression

$$F = c\frac{dx}{dt}, \tag{4.1}$$

where F = resisting force in pounds;

$\frac{dx}{dt}$ = the velocity of the body in inches per second;

c = the damping factor, or resistance in pounds, when the velocity is 1 in. per sec; the units thus are pound-seconds per inch.

Bodies moving with high velocity through a fluid have a resistance generally considered to be proportional to the square of the velocity.

When a member is subjected to a rapid repeated load, the stress-strain or load-strain diagram will show a hysteresis loop, the area of which represents the amount of energy absorbed because of internal friction per cycle. This energy is transformed into heat and hence acts as a damping agent.

Any of the types of damping mentioned above may be replaced by an equivalent viscous damping that has the same energy absorption per cycle.* The solution of problems having viscous damping will therefore be considered in detail in this chapter.

4.2 *EQUATION OF MOTION WITH VISCOUS DAMPING*

Figure 4.1 represents a system subject to free vibrations with viscous damping, consisting of a mass m suspended on a spring having a scale k. The dashpot connecting the mass and the ground provides viscous damping with a factor c. This dashpot opposes the motion of the mass with a force $c\frac{dx}{dt}$, as given by Eq. (4.1).

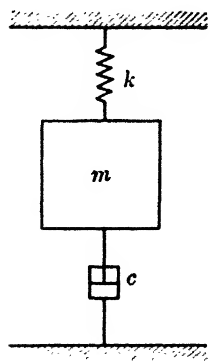


FIG. 4.1

If the mass is considered to be a free body displaced downward a distance x from the equilibrium position and moving downward with a velocity $\frac{dx}{dt}$, the equation of motion based upon the dynamic forces acting on it becomes

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0. \tag{4.2}$$

A solution of this equation is

$$x = e^{pt}, \tag{4.3}$$

* L. S. Jacobsen, "Steady Forced Vibration as Influenced by Damping," *Trans. A.S.M.E.*, 1930, APM-52-15.

where e = the base of the natural logarithms = 2.718, p = a constant, and t = time.

Differentiating Eq. (4.3),

$$\frac{dx}{dt} = pe^{pt}, \qquad \frac{d^2x}{dt^2} = p^2e^{pt},$$

and substituting these expressions in Eq. (4.2),

$$mp^2e^{pt} + cpe^{pt} + ke^{pt} = 0,$$

and

$$e^{pt}(mp^2 + cp + k) = 0.$$

Since e^{pt} is always positive, Eq. (4.3) will be a solution of the differential equation if the expression in the parenthesis equals zero, that is, if

$$mp^2 + cp + k = 0. \tag{4.4}$$

As Eq. (4.4) is a quadratic expression in p, there are two values of p to be considered;

$$p_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}, \tag{4.5}$$

and

$$p_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}. \tag{4.6}$$

Hence, $x = e^{p_1t}$ and $x = e^{p_2t}$ may both be solutions of Eq. (4.2). A general solution is found by combining the two; thus,

$$x = Ae^{p_1t} + Be^{p_2t}, \tag{4.7}$$

in which A and B are arbitrary constants.

There are three conditions of p_1 and p_2 that must be considered in connection with Eq. (4.7). The first is when $\left(\frac{c}{2m}\right)^2$ is equal to k/m, which is known as the critical damping condition and is discussed in Sec 4.3. The second is when $\left(\frac{c}{2m}\right)^2$ is greater than k/m, which is the overdamped condition and considered in Sec. 4.4. The third and most common case is the underdamped condition when $\left(\frac{c}{2m}\right)^2$ is less than $\frac{k}{m}$. This is discussed in Sec. 4.5.

4.3 CRITICAL DAMPING

The critical damping condition seldom occurs in practice; but it is of importance in that it forms a measure of the amount of damping in the system. The damping factor for this case carries the subscript c, and the ratio of the actual damping factor to the critical, or c/c_c, is a measure of the relative amount of damping in a given system.

With critical damping, $\left(\dfrac{c}{2m}\right)^2 = \dfrac{k}{m}$ and the terms under the radical in Eqs. (4.5) and (4.6) equal zero, so that $p_1 = p_2 = -\dfrac{c_c}{2m}$. Hence, Eq. (4.7) may be written

$$x = Ce^{pt} = Ce^{-(c/2m)t}. \tag{4.8}$$

In the usual case of a free vibration, the mass is displaced a

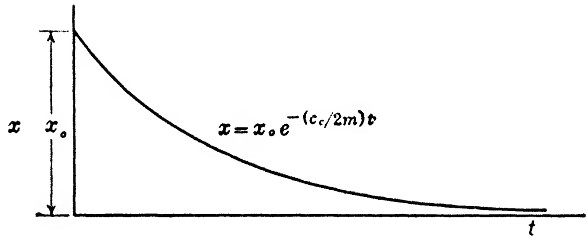


FIG. 4.2

distance x_o and at zero time is released; therefore, when $t = 0$, $x = x_o$. This condition may be used to evaluate the constant C in Eq. (4.8); since then $x_o = C \times 1$ or $C = x_o$, and Eq. (4.8) becomes

$$x = x_o e^{pt} = x_o e^{-(c_c/2m)t}. \tag{4.9}$$

The resulting curve of the motion is shown in Fig. 4.2, in which the displacement x is plotted against time t.

Since $\left(\dfrac{c_c}{2m}\right)^2 = \dfrac{k}{m}$,

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{mk} = 2m\omega_n = \frac{2k}{\omega_n}. \tag{4.10}$$

4.4 OVERDAMPING

Overdamping is associated with a relatively large damping factor c and is specified by the condition that $(c/2m)^2$ is greater than

k/m, or that the ratio of c/c_c is greater than 1. The mass moves slowly back to the equilibrium position rather than vibrating about it, and the motion is said to be *aperiodic*.

Since $(c/2m)^2$ is greater than k/m, the radical term is always less than $c/2m$; hence, the values of p_1 and p_2 are always real but negative.

Again considering the usual case of a free vibration where the mass is displaced a distance x_o and released when time is zero with no initial velocity, the conditions used to evaluate the constants A and B are

$$x = x_o \qquad \text{and} \qquad \frac{dx}{dt} = 0, \qquad \text{when } t = 0.$$

Differentiating Eq. (4.7),

$$\frac{dx}{dt} = Ap_1e^{p_1t} + Bp_2e^{p_2t}.$$

Substituting the conditions $dx/dt = 0$ when $t = 0$,

$$0 = Ap_1 \times 1 + Bp_2 \times 1. \tag{4.11}$$

Substituting $x = x_o$ when $t = 0$ in Eq. (4.7),

$$x_o = A \times 1 + B \times 1 \qquad \text{or} \qquad A = x_o - B.$$

Substituting this expression in Eq. (4.11),

$$0 = (x_o - B)p_1 + Bp_2 \qquad \text{or} \qquad B = \frac{p_1}{p_1 - p_2}x_o,$$

and

$$A = x_o - B = \frac{-p_2}{p_1 - p_2}x_o.$$

Equation (4.7) may now be written

$$x = \frac{-p_2}{p_1 - p_2}x_oe^{p_1t} + \frac{p_1}{p_1 - p_2}x_oe^{p_2t}$$

$$= \frac{x_o}{p_1 - p_2}(-p_2e^{p_1t} + p_1e^{p_2t}), \tag{4.12}$$

which is an expression of the displacement x with time t for the usual type of free vibration.

The solid lines of Fig. 4.3 show the actual shapes of these curves for various damping ratios c/c_c, starting in each case with $x_o = 1$.

If the damping factor is infinite ($c = \infty$), the mass is locked in place, and the curve is a horizontal line. As the amount of damping is reduced, the mass travels back to the equilibrium position faster; and in the limiting case of critical damping ($c/c_c = 1$), the return is most rapid.

4.5 *UNDERDAMPING*

When the damping factor is small, that is, when $(c/2m)^2$ is less than k/m or the ratio c/c_c is less than 1, the system is said to be

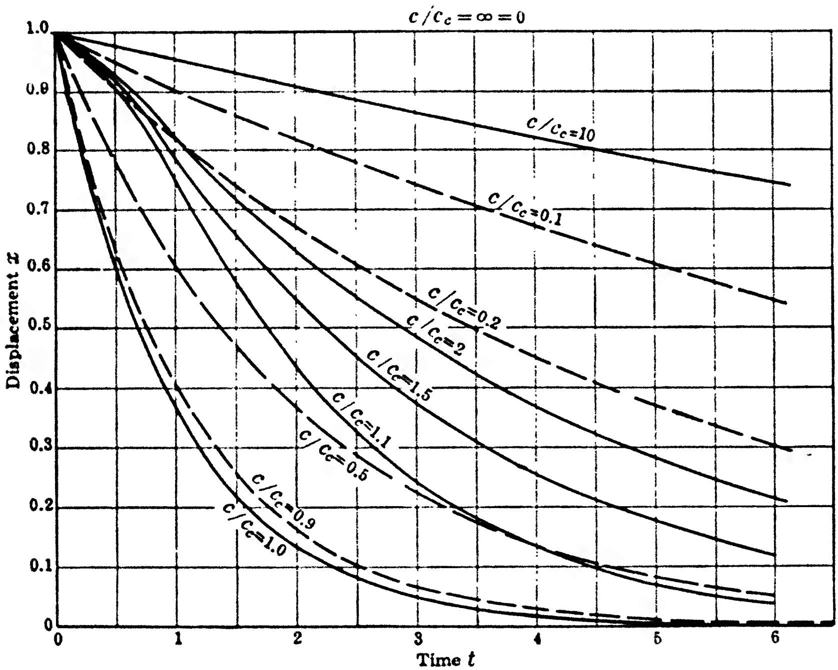


FIG. 4.3

underdamped. The mass vibrates with a decreasing amplitude until it finally comes to rest at the equilibrium position.

Since $(c/2m)^2$ is less than k/m, the exponents p_1 and p_2 are complex, and it becomes convenient to introduce the term j, which equals $\sqrt{-1}$. Then

$$p_1 = -\left(\frac{c}{2m}\right) + j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2},$$

and

$$p_2 = -\left(\frac{c}{2m}\right) - j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}.$$

It will be seen shortly that $\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \omega_{nd}$, which is the damped natural circular frequency. Equation (4.7) may then be written

$$x = Ae^{p_1 t} + Be^{p_2 t}.$$

Or, since

$$e^{-a+b} + e^{-a-b} = e^{-a}(e^b + e^{-b}),$$

$$x = e^{-(c/2m)t}(Ae^{j\omega_{nd}t} + Be^{-j\omega_{nd}t}). \tag{4.13}$$

It can be shown* that $e^{ja} = \cos a + j \sin a$, and therefore, Eq. (4.13) may be written

$$\begin{aligned} x &= e^{-(c/2m)t}[A(\cos \omega_{nd}t + j \sin \omega_{nd}t) + B(\cos \omega_{nd}t - j \sin \omega_{nd}t)] \\ &= e^{-(c/2m)t}[(A + B) \cos \omega_{nd}t + (jA - jB) \sin \omega_{nd}t]. \end{aligned} \tag{4.14}$$

This equation is made up of two multiplying factors; the second is a vibratory simple harmonic motion and the first an exponential similar to that of the overdamped condition. The resultant curve of the motion then is a sine or cosine curve in which the amplitude decreases according to the expression $e^{-(c/2m)t}$, as shown in Fig. 4.4.

* This proposition is based on a Maclaurin's series expansion (see any calculus text, for example Granville, Smith, and Langley, pp. 354–365):

$$\begin{aligned} e^{ja} &= 1 + ja - \frac{a^2}{2!} - j\frac{a^3}{3!} + \frac{a^4}{4!} + j\frac{a^5}{5!} - \cdots \\ &= \left(1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots\right) + j\left(a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \cdots\right). \end{aligned}$$

But also by Maclaurin,

$$\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots$$

and

$$\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \cdots;$$

hence

$$e^{ja} = \cos a + j \sin a.$$

The dashed line represents the limiting value of the amplitude, and this boundary becomes flatter with smaller values of c. These bounding lines for the upper sides of the curves are shown dashed in Fig. 4.3 for various ratios of c/c_c as labeled. They are put on the same figure as the overdamped curves for comparison. If there is no damping (if $c/c_c = 0$), the vibration will not die out; hence, the bounding line is horizontal. As the ratio c/c_c approaches unity (or as c approaches critical damping) as the limiting case, the vibrations die out most rapidly as shown there.

Equation (4.14) can be simplified by considering the second term alone in the usual case of a free vibration starting with a displace-

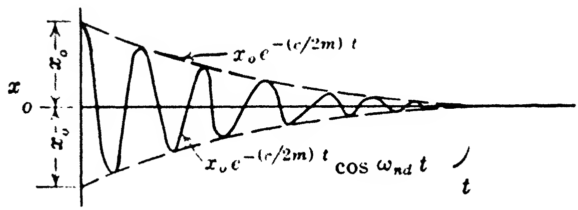


FIG. 4.4

ment of x_o and zero velocity at zero time, that is, $x = x_o$ and $dx/dt = 0$ when $t = 0$. If these conditions are substituted in

$$x = (A + B)\cos \omega_{nd}t + (jA - jB)\sin \omega_{nd}t$$

and in its derivative with respect to time t

$$\frac{dx}{dt} = -(A + B)\omega_{nd}\sin \omega_{nd}t + (jA - jB)\omega_{nd}\cos \omega_{nd}t,$$

the constants are evaluated as

$$A + B = x_o \qquad \text{and} \qquad jA - jB = 0.$$

Equation (4.14) then becomes

$$x = x_o e^{-(c/2m)t}\cos \omega_{nd}t. \tag{4.15}$$

From Eq. (4.14) or (4.15), it is apparent that the damped circular frequency of the vibration is ω_{nd} or $\sqrt{\dfrac{k}{m} - \left(\dfrac{c}{2m}\right)^2}$. The undamped circular frequency is $\omega_n = \sqrt{\dfrac{k}{m}}$; hence, it may be seen that the damped frequency is less than the undamped. The ratio between them in terms of the damping ratio c/c_c may be found with the aid of Eq. (4.10), $c_c^2 = 4mk$:

$$\frac{\omega_{nd}}{\omega_n} = \frac{\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}{\sqrt{\frac{k}{m}}} = \sqrt{1 - \frac{c^2}{4m^2}\frac{m}{k}} = \sqrt{1 - \left(\frac{c}{c_c}\right)^2}. \tag{4.16}$$

This equation is plotted in Fig. 4.5, and it may be seen that damping has little effect on the frequency until the c/c_c ratio becomes fairly large (greater than about 0.5).

It is desirable to investigate the manner in which the amplitude decreases in underdamped vibrations and provide some means of

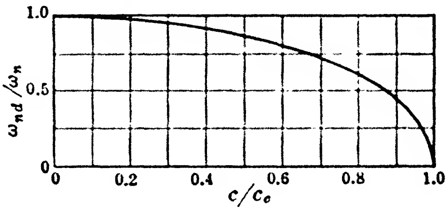


Fig. 4.5

measuring the rate of reduction. These matters are a function of the bounding exponential term of Eqs. (4.14) and (4.15).

Let t_n be the time elapsed from the start of the free vibration to any peak of amplitude x_n, and let t_{n+1} be the time elapsed from the start of the free vibration to the succeeding peak of amplitude x_{n+1}. Since the time elapsed between the two peaks is equal to the period

$$t_{n+1} = t_n + \frac{2\pi}{\omega_{nd}}.$$

From the exponential term of Eq. (4.15),

$$x_n = x_o e^{-(c/2m)t_n};$$

$$\begin{aligned} x_{n+1} &= x_o e^{-(c/2m)t_{n+1}} \\ &= x_o e^{-(c/2m)(t_n + 2\pi/\omega_{nd})} \\ &= x_o e^{-(ct_n/2m) - (c2\pi/2m\omega_{nd})}. \end{aligned}$$

Taking the ratio of x_n/x_{n+1},

$$\frac{x_n}{x_{n+1}} = \frac{x_0 e^{-(c/2m)t_n}}{x_0 e^{-(c/2m)t_n-(c/2m)(2\pi/\omega_{nd})}}$$

$$= e^{-(c/2m)t_n+(c/2m)t_n+(c/2m)(2\pi/\omega_{nd})}$$

$$= e^{\pi c/m\omega_{nd}}. \tag{4.17}$$

From this equation it is obvious that the amplitudes decrease in a constant geometric ratio, which is a function of c, m, and ω_{nd}. The term $\dfrac{\pi c}{m\omega_{nd}}$ is called the *logarithmic decrement* and designated by the Greek letter δ. Hence, $\dfrac{x_n}{x_{n+1}} = e^{\delta}$; and $\delta = \log_e \dfrac{x_n}{x_{n+1}}$.

The logarithmic decrement is a convenient method of measuring the rate of decay of a free vibration. It may be expressed in dimensionless form with the aid of Eq. (4.10), since $c_c^2 = 4mk$, and $mk = c_c^2/4$.

$$\delta = \frac{\pi c}{m\omega_{nd}} = \frac{\pi c}{m\sqrt{\dfrac{k}{m} - \dfrac{c^2}{4m^2}}} = \frac{\pi c}{\sqrt{\dfrac{m^2k}{m} - \dfrac{c^2m^2}{4m^2}}} = \frac{\pi c}{\sqrt{\dfrac{c_c^2}{4} - \dfrac{c^2}{4}}}$$

$$= \frac{\pi c}{\dfrac{c_c}{2}\sqrt{1 - \left(\dfrac{c}{c_c}\right)^2}} = \frac{\dfrac{2\pi c}{c_c}}{\sqrt{1 - \left(\dfrac{c}{c_c}\right)^2}}. \tag{4.18}$$

Equation (4.18) may be solved for c/c_c in terms of the logarithmic decrement, which gives the equation

$$\frac{c}{c_c} = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}. \tag{4.19}$$

EXAMPLE

A body weighing 25 lb is suspended from a spring with a scale $k = 10$ lb per in. A dashpot is attached between the weight and the ground, which has a resistance of 0.1 lb at a velocity of 2 in. per sec. Determine (a) the natural frequency of the system; (b) the critical damping factor; (c) the ratio of successive amplitudes; (d) amplitude 10 cycles later if the mass is initially displaced $\frac{3}{4}$ in. and released.

(a) The damping factor is $c = F/v = 0.1/2 = 0.05$ lb-sec per in. The damped natural circular frequency is

$$\omega_{nd} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{\frac{10 \times 386}{25} - \left(\frac{0.05 \times 386}{2 \times 25}\right)^2}$$

$$= 12.42 \text{ rad per sec.}$$

The damped cyclic frequency is

$$f_{nd} = \frac{\omega_{nd}}{2\pi} = \frac{12.42}{2\pi} = 1.98 \text{ cps} = 118.8 \text{ cpm.}$$

(b) The critical damping factor is

$$c_c = 2\sqrt{mk} = 2\sqrt{\frac{25 \times 10}{386}} = 1.61 \text{ lb-sec per in.}$$

(c) The logarithmic decrement is $\delta = \dfrac{\pi c}{m\omega_{nd}} = \dfrac{\pi 0.05 \times 386}{25 \times 12.42} =$ 0.1955, and also equals $\log_e \dfrac{x_{o_1}}{x_{o_2}}$.

Hence,

$$\frac{x_{o_1}}{x_{o_2}} = e^{\delta} = e^{0.1955} = 1.216;$$

and

$$\frac{x_{o_2}}{x_{o_1}} = 1/1.216 = 0.822.$$

(d) The period is

$$\tau = \frac{2\pi}{\omega_{nd}} = \frac{2\pi}{12.42} = 0.506 \text{ sec.}$$

The time required for 10 cycles is $10 \times \tau = 10 \times 0.506 =$ 5.06 sec.

The amplitude reduction in 10 cycles is

$$e^{-(c/2m)t} = e^{-[(0.05 \times 386)/(2 \times 25)]5.06} = e^{-1.953} = 0.1417.$$

The amplitude at the end of 10 cycles is $x_{10} = \frac{3}{4} \times 0.1417 =$ 0.106 in.

An alternate method of finding this is to raise the reduction ratio to the 10th power and multiply it by the original amplitude; thus,

$$x_{10} = \tfrac{3}{4}(0.822)^{10} = 0.106 \text{ in.}$$

4.6 CONSTANT OR COULOMB DAMPING

The second type of damping to be considered is one in which the resisting force is constant and independent of the velocity. This condition is approximated when dry surfaces are rubbing and may be represented by Fig. 4.6. The constant friction force F always acts opposite to the direction of motion and is given by

$$F = \mu W, \tag{4.20}$$

where μ is the coefficient of friction.

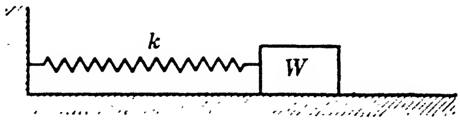


FIG. 4.6

An analysis of the motion may be obtained from work and energy considerations. Assume the weight is initially displaced a distance x_o and released. It will travel through the mid-position and come to rest at a distance $x_{1/2}$ from the mid-position (see Fig. 4.7). As the resisting force F is constant during this half cycle of motion, the work done or energy absorbed is $(x_o + x_{1/2})F$. The initial potential energy contained in the system is $\frac{1}{2}kx_o^2$; and the potential energy at the end of the half cycle is $\frac{1}{2}kx_{1/2}^2$. From these facts the energy equation may be written,

$$\tfrac{1}{2}kx_o^2 - \tfrac{1}{2}kx_{1/2}^2 = F(x_o + x_{1/2}).$$

Simplifying,

$$\tfrac{1}{2}k(x_o^2 - x_{1/2}^2) = F(x_o + x_{1/2}).$$

Then,

$$x_o - x_{1/2} = \frac{2F}{k} \qquad \text{and} \qquad x_{1/2} = x_o - \frac{2F}{k}.$$

Let x_1 be the amplitude at the end of the first cycle, and by a similar process

$$\tfrac{1}{2}kx_{1/2}^2 - \tfrac{1}{2}kx_1^2 = F(x_{1/2} + x_1),$$

and

$$x_1 = x_{1/2} - \frac{2F}{k}.$$

From the above it is apparent that the amplitude decreases at the constant rate of $2F/k$ per half cycle, or $4F/k$ per full cycle; the decay curve is that shown in Fig. 4.7, where the boundary lines are straight.

The resisting frictional force F always opposes the motion of the weight. When the weight is displaced to the left and is moving in that direction, the force equation, considering it to be a free body, is

$$m\frac{d^2x}{dt^2} + kx + F = 0,$$

which may be written

$$m\frac{d^2x}{dt^2} + k\left(x + \frac{F}{k}\right) = 0. \tag{4.21}$$

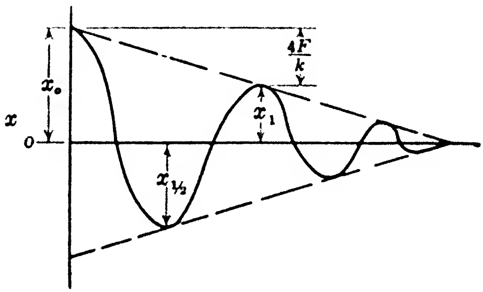


FIG. 4.7

Letting $x + \frac{F}{k} = z$ and noting that

$$\frac{d^2z}{dt^2} = \frac{d^2}{dt^2}\left(x + \frac{F}{k}\right) = \frac{d^2x}{dt^2},$$

Eq. (4.21) may be written

$$m\frac{d^2z}{dt^2} + kz = 0,$$

or

$$\frac{d^2z}{dt^2} = -\left(\frac{k}{m}\right)z.$$

From Eqs. (2.7) and (2.9), it may be seen that this equation corresponds to a free vibration without damping, and $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

When the weight is displaced to the left and is moving to the right, the force equation becomes

$$m\frac{d^2x}{dt^2} + kx - F = 0,$$

which may be written

$$m \frac{d^2x}{dt^2} + k\left(x - \frac{F}{k}\right) = 0. \tag{4.22}$$

Letting $x - F/k = z'$ and noting that

$$\frac{d^2z'}{dt^2} = \frac{d^2}{dt^2}\left(x - \frac{F}{k}\right) = \frac{d^2x}{dt^2},$$

Equation (4.22) may be written

$$m \frac{d^2z'}{dt^2} + kz' = 0,$$

and

$$\frac{d^2z'}{dt^2} = -(k/m)z'.$$

Again from Eqs. (2.7) and (2.9), it is seen that the natural frequency is $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$.

For any position or direction of motion of the weight, the equation of motion is given by either Eq. (4.21) or (4.22). Therefore, the frequency of the weight will always be the same as the undamped natural frequency, and $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$. The motion of the body, however is not simple harmonic, and the shape of the displacement-time curve changes each half cycle.

When a body vibrates with viscous damping, it theoretically never comes to rest, since the amplitude is reduced in a geometric recession rather than a fixed amount. When the velocity becomes quite low, the viscous action breaks down, and the type of damping changes from viscous to constant. Applications of pure Coulomb damping are rare, but the action that brings the body to rest at the end of a free vibration is generally of this type.

Example

A weight of 90 lb slides on a dry surface (Fig. 4.6). The coefficient of friction μ between the weight and surface is 0.2. If the weight is initially released with zero velocity when the spring is stretched 3 in., and the spring scale is 32 lb per in., determine (a) the amplitude of the weight at the end of the first cycle; (b) the position at which the weight will stop, measured from the unstressed spring position.

(a) Friction force is $F = \mu W = 0.2 \times 90 = 18$ lb. The reduction in amplitude during one cycle is

$$x_o - x_1 = \frac{4F}{k} = 4 \times \frac{18}{32} = 2\tfrac{1}{4} \text{ in.}$$

Amplitude at end of first cycle is

$$x_1 = x_o - \frac{4F}{k} = 3 - 2\tfrac{1}{4} = \tfrac{3}{4} \text{ in.}$$

(b) The position at which the weight will come to rest may be found by energy considerations. The initial potential energy is

$$\tfrac{1}{2}kx_o^2 = \tfrac{1}{2} \times 32 \times 3^2 = 144 \text{ in.-lb.}$$

The amplitude at the end of the first half cycle is

$$x_{1_2} = x_o - \frac{2F}{k} = 3 - 1\tfrac{1}{8} = 1\tfrac{7}{8} \text{ in.}$$

The energy absorbed or work done during the first cycle is

$$\begin{aligned} E &= F(x_o + x_{1_2}) + F(x_{1_2} + x_1) \\ &= 18(3 + 1\tfrac{7}{8}) + 18(1\tfrac{7}{8} + \tfrac{3}{4}) \\ &= 87.75 + 47.25 \\ &= 135 \text{ in.-lb.} \end{aligned}$$

The energy remaining in the system at the end of the first cycle is equal to the initial energy minus the work done, that is, $144 - 135 = 9$ in.-lb.

Let x_s be the distance from the unstressed spring position that the weight stops. The energy available at the end of the first cycle equals the potential energy left in the spring when the weight has come to rest plus the energy absorbed or work done in friction from the end of the cycle until the motion stops. Thus,

$$E = \tfrac{1}{2}kx_s^2 + F(x_1 - x_s) = \tfrac{1}{2} \times 32x_s^2 + 18(\tfrac{3}{4} - x_s) = 9.$$

Then,

$$16x_s^2 - 18x_s + 4.5 = 0,$$

and

$$x_s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{18 \pm \sqrt{18^2 - 4 \times 16 \times 4.5}}{32}$$

$$= 0.75 \text{ or } 0.375 \text{ in.}$$

Obviously, the first answer is not the physically true one, since the spring force acting on the block then would be $x_s k = \frac{3}{4} \times 32 =$ 24 lb, which would be greater than the resisting frictional force of 18 lb. The block, therefore, as a free body would not be in equilibrium, and motion would have to take place. The second answer of $\frac{3}{8}$ in. is the actual value.

PROBLEMS

4.1. It is desired to find the properties of an isolator which operates with viscous damping. If a weight of 100 lb is attached to it, a static deflection of $\frac{1}{2}$ in. occurs. The weight is then deflected downward $\frac{1}{4}$ in. and released. After three complete cycles, the amplitude is found to be 0.1 in. Determine (a) the logarithmic decrement; (b) the damping factor ratio c/c_c; (c) the damping factor c; and (d) the frequency of the vibration.

Ans. (a) 0.3055; (b) 0.0485; (c) 0.7; (d) 265 cpm.

4.2. Repeat the example of Sec. 4.5 with the change that the resistance is 0.01 lb at a velocity of 2 in. per sec. The rest of the data remain unchanged.

Ans. (a) 118.9 cpm; (b) 1.61; (c) 0.981; (d) 5.1 sec, 0.617 in.

4.3. A block weighing 75 lb is mounted on an isolator having viscous damping that deflects $\frac{1}{4}$ in. under the weight. When the block is vibrating freely, it is observed that the amplitudes in inches at the end of successive cycles are 0.301, 0.232, and 0.179. Determine (a) the logarithmic decrement; (b) the damping-factor ratio c/c_c; (c) the damping factor c; (d) the frequency of the vibration.

Ans. (a) 0.26; (b) 0.0412; (c) 0.628; (d) 376 cpm.

4.4. The disk of a torsional pendulum has an inertia of 0.4 in.-lb-sec² and is immersed in a viscous fluid. The brass shaft [$G = 6(10^6)$] attached to it is $\frac{1}{4}$ in. in diameter and 8 in. long. When the pendulum is vibrating freely, it is observed that the amplitudes in degrees at the ends of successive cycles are 3, 2.01, and 1.35.

Determine (a) the logarithmic decrement; (b) the damping factor c; (c) the frequency of the vibration.

Ans. (a) 0.4; (b) 1.36; (c) 256 cpm.

4.5. A torsional pendulum has a disk having an inertia of 15 in.-lb-sec², and the spring scale of the shaft is 25 in.-lb per rad. The disk is immersed in a viscous fluid which gives a damping torque of 0.3 in.-lb-sec per rad. Determine (a) the natural frequency of the pendulum; (b) the critical damping factor c_c; (c) the displacement after 2 sec if the disk is initially displaced 10 deg and released; (d) the logarithmic decrement.

Ans. (a) 12.3 cpm; (b) 38.8; (c) 8.3 deg; (d) 0.0486.

4.6. Given the following values of logarithmic decrement for

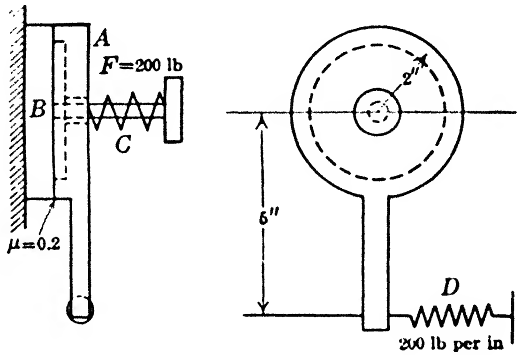


FIG. 4.8

various materials, determine the corresponding c/c_c factors: steel = 0.003; rubber = 0.23; wood = 0.04; concrete = 0.08.

Ans. Steel = 0.00048; rubber = 0.0364; wood = 0.0064; concrete = 0.0127.

4.7. A weight of 35 lb slides back and forth on a dry surface due to the action of a spring having a scale of 100 lb per in. After making four complete cycles, its amplitude is 3 in. What is the average coefficient of friction between the two surfaces if the original amplitude was 5 in.? How much time has elapsed during the four cycles? *Ans.* 0.357; 0.755 sec.

4.8. The apparatus shown in Fig. 4.8 consists of a hollow disk A pressed against plate B by spring C, which exerts a force of 200 lb. The mean radius of the disk A is 2 in., and the coefficient of friction between A and B is 0.2. An arm attached to A is connected to spring D at a radius of 5 in. The scale of spring D is 200 lb per in.

If A is displaced 3 deg from the equilibrium position and released, determine (a) the angular amplitude at the end of the first half cycle; (b) the angular displacement at which disk A comes to rest measured from the position of zero stress in spring D.

Ans. (a) 1.163 deg; (b) 0.670 deg.

4.9. A 20-lb weight attached to a spring having a scale of 12 lb per in. vibrates on a dry surface having a kinetic coefficient of friction of 0.25. If the weight is initially released with zero velocity when the spring is stretched 1 in., determine the position at which it will come to rest measured from the unstressed spring position. *Ans.* 0.167 in.

Chapter 5

DAMPED FORCED VIBRATIONS—SINGLE DEGREE OF FREEDOM

5.1 *INTRODUCTION*

The case of damped forced vibrations is of practical value (a) to determine the action taking place and transmissibility with isolators made of materials having a relatively high damping factor; (b) to understand completely the phenomenon of critical speeds, which is affected by damping; and (c) to interpret the performance of accelerometers that employ large amounts of damping. These topics are considered in this chapter together with a discussion of the various materials used in commercial isolators.

5.2 *EQUATION OF MOTION*

An elastic system subjected to an impressed force and retarded by viscous damping is represented by Fig. 5.1. The mass m is suspended on a spring of scale k and is acted on by the harmonic impressed force $F_o \cos \omega t$. A dashpot is connected between the mass and the ground and provides a damping action whose factor is c.

Assuming that the mass is displaced a distance x and is moving with a velocity dx/dt, the equation of motion may be obtained by considering the mass as a free body and securing equilibrium of the forces acting upon it as in the previous cases. This equation is

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F_o \cos \omega t. \tag{5.1}$$

The mathematical solution of this equation is rather complex, and a clearer understanding of the action may be obtained by vector diagrams. If it is assumed that the mass moves with simple harmonic motion, all the forces may be represented as rotating vectors.

An examination of Eqs. (1.1) to (1.3) reveals the fact that it is possible to differentiate a vector producing simple harmonic motion by multiplying it by the circular frequency ω and moving it forward 90 deg. Each differentiation in these equations is ω times greater than the previous value, and the change in the trigonometric term from cosine to minus sine to minus cosine indicates a rotation of the vector in the direction of motion.

In Fig. 5.2, let the displacement of the mass be represented by the dashed line x_o downward. The spring force is kx, and it acts

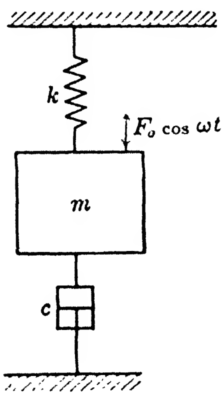


FIG. 5.1

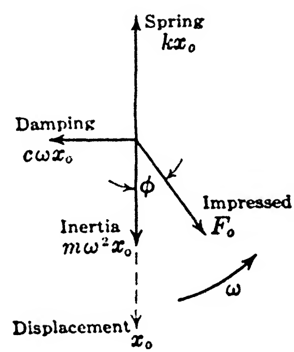


FIG. 5.2

in a direction opposite to the displacement, that is, upward. It may be represented as a vector of length kx_o. The damping force $c\frac{dx}{dt}$ is the first derivative of the displacement and acts 90 deg ahead of the spring force. The length of this vector is $c\omega x_o$. The inertia force $m\frac{d^2x}{dt^2}$ is the second derivative of the displacement and acts 180 deg ahead of the spring force, or downward. Its length is $m\omega^2 x_o$. The fourth vector of the diagram is the impressed force $F_o \cos \omega t$, and it must be placed to secure equilibrium. To satisfy this requirement, the fourth vector must be placed ϕ deg ahead of the displacement as shown in Fig. 5.2. The value of the angle ϕ may be obtained by summing up the vertical and horizontal forces as follows:

$$\Sigma F_v = kx_o - m\omega^2 x_o - F_o \cos \phi = 0,$$

$$\cos\phi = x_o\left(\frac{k - m\omega^2}{F_o}\right). \tag{5.2}$$

$$\Sigma F_h = c\omega x_o - F_o \sin\phi = 0,$$

$$\sin\phi = \frac{c\omega x_o}{F_o}. \tag{5.3}$$

Hence,

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{c\omega}{k - m\omega^2}. \tag{5.4}$$

Since $k = m\omega_n^2$ and, by Eq. (4.10), $c_c = 2m\omega_n$,

$$\tan\phi = \frac{2\left(\dfrac{c}{c_c}\right)\left(\dfrac{\omega}{\omega_n}\right)}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} = \frac{2\left(\dfrac{c}{c_c}\right)r}{1 - r^2}. \tag{5.5}$$

It is apparent from Eq. (5.5) that the phase angle ϕ depends upon the amount of damping c/c_c and the frequency of the impressed force ω/ω_n, or r. This relation is shown graphically in Fig. 5.3. It may be observed that if there is no damping ($c/c_c = 0$), the angle snaps from 0 to 180 deg instantaneously at the natural frequency ($r = 1$). As the amount of damping increases, the change becomes more uniform and gradual, as shown in the figure.

An expression for the amplitude of the mass at a given impressed frequency can be found by substituting Eqs. (5.2) and (5.3) in the trigonometric relation $\sin^2\phi + \cos^2\phi = 1$, which gives

$$\frac{c^2\omega^2 x_o^2}{F_o^2} + \frac{x_o^2(k - m\omega^2)^2}{F_o^2} = 1.$$

Collecting terms and solving for x_o,

$$x_o = \frac{F_o}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}}. \tag{5.6}$$

This equation may be put in dimensionless form by introducing the terms $x_{st} = F_o/k$; $k = m\omega_n^2$; and $c_c = 2m\omega_n$. Then we obtain

$$\frac{x_o}{x_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + \left(2\dfrac{c}{c_c}r\right)^2}}. \tag{5.7}$$

Thus, the amplitude of the mass is also a function of the amount of damping (c/c_c) and the frequency of the impressed force ω/ω_n, or r. The relation is shown graphically in Fig. 5.4 for various damping

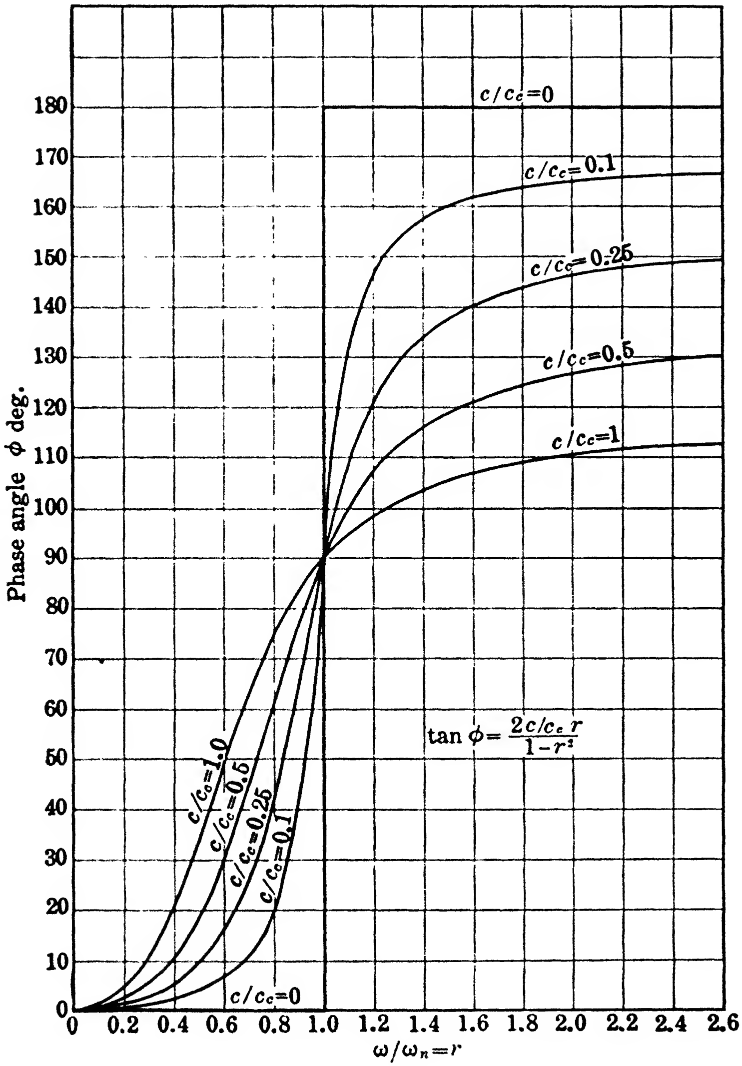


FIG. 5.3

ratios. It will be observed there that even small amounts of damping reduce the amplitude of the mass greatly near the resonance point; and, as might be expected, the greater the damping, the lower the amplitude of the mass.

As pointed out in Sec. 3.2, the ratio of x_o/x_{st} is known as the *magnification factor*, since it is the ratio of the amplitude of the mass

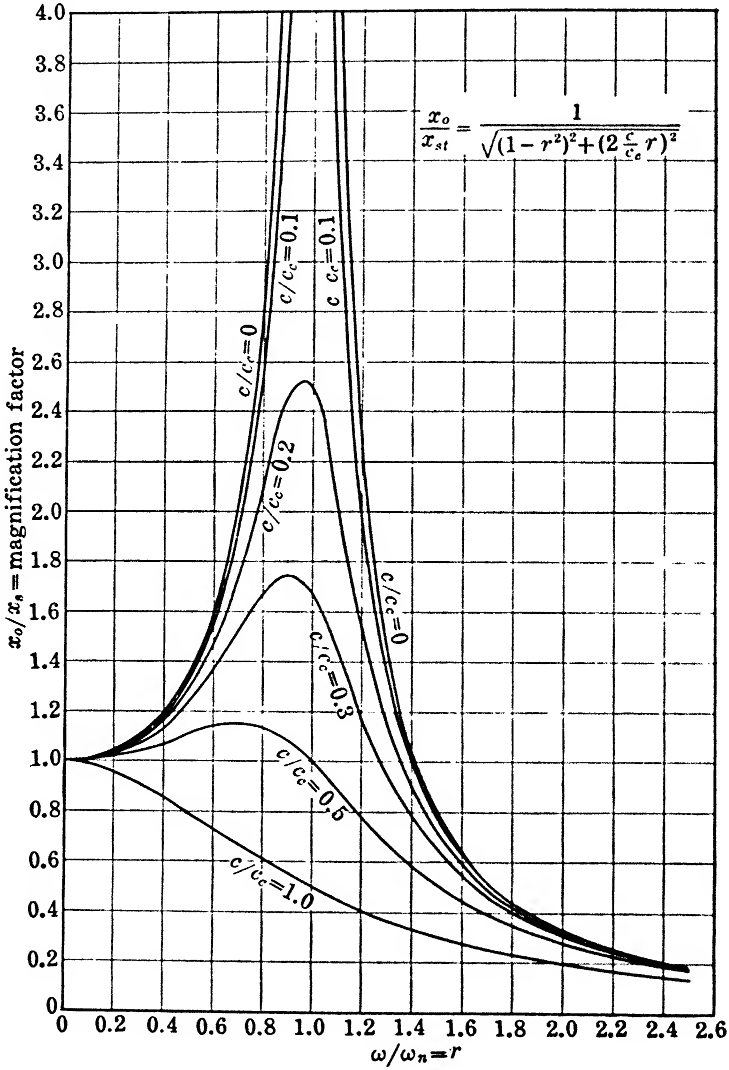


FIG. 5.4

under the dynamic impressed load to that which would be obtained if the impressed force were static.

When the impressed frequency is low, the velocity and acceleration of the mass are small; hence, the inertia and damping forces are almost negligible. The result is that the impressed force approximately balances the spring force, and the phase angle is close to zero, as shown in Fig. 5.5a. The motion of the mass is then about equal to the static deflection, that is, $x_o/x_{st} \approx 1$.

As the impressed frequency is increased, the mass has a higher velocity and acceleration, and the inertia and damping forces therefore increase. As the inertia force increases, the vertical component of the impressed force F_o is reduced; and owing to the increased damping force, the horizontal component of the impressed force F_o must increase to maintain equilibrium. The net effect is an increase in the phase angle ϕ, as shown in Fig. 5.5b, and an increase in the amplitude of the mass x_o, that is, x_o/x_{st}.

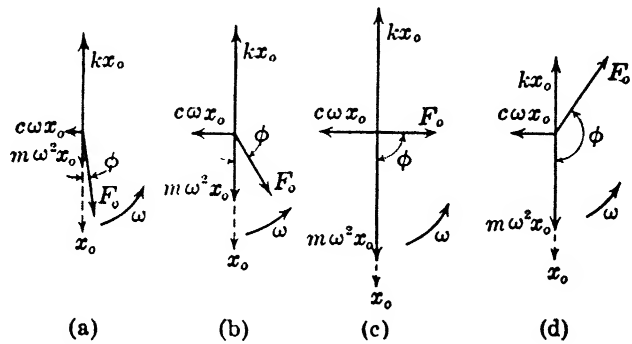


FIG. 5.5

At the natural frequency, the inertia and spring forces are in balance; hence, the impressed force F_o has no vertical component, that is, $\phi = 90$ deg. The entire impressed force acts to balance the damping force, as shown in Fig. 5.5c, and the amplitude of the mass is a maximum; that is, x_o/x_{st} is large.

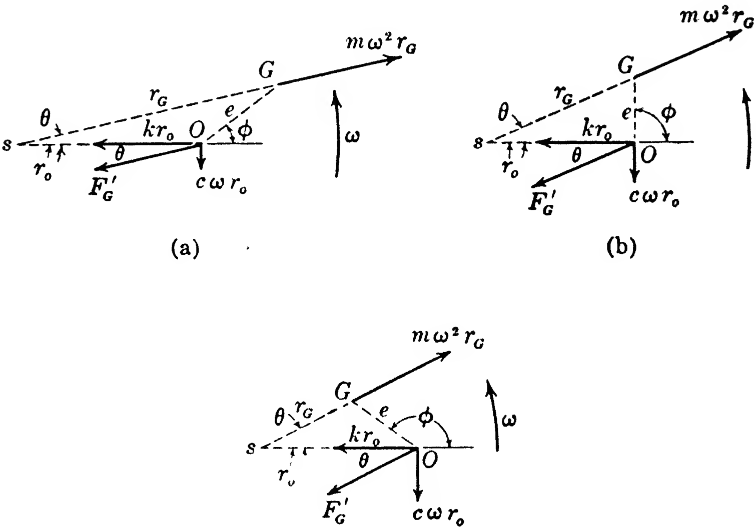
For frequencies above the natural, the inertia force is greater than the spring force, with the result that the vertical component of the impressed force must act upward to maintain equilibrium. The phase angle ϕ is then greater than 90 deg, as shown in Fig. 5.5d. For extremely high impressed frequencies, this angle approaches 180 deg. Since the impressed force acts with the spring force to balance the inertia force, the amplitude of the mass decreases, as shown in Fig. 5.4.

The cases of forced vibrations with Coulomb and other types of damping are taken up in S. Timoshenko, *Vibration Problems in Engineering*, Van Nostrand, New York, 1937, pp. 57–62; and in

J. P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill, New York, 1947, Chap. 8.

5.3 *CRITICAL SPEED OF A SINGLE DISK ON A SHAFT**

The explanation of the action that takes place as a mass is subjected to forced vibrations with viscous damping may be clearer if it is applied to a single disk as it passes through the critical speed. This case is similar to that given in Sec. 3.5, except that now damping is considered.



(c)

FIG. 5.6

Again let O be the geometric center of the disk through which the vertical shaft center line passes, and let G be the center of gravity (Fig. 3.7). These two points are separated by the distance e, owing to the nonhomogeneity of the material. Then let r_o be the deflection of the disk measured to the geometric center, and let r_G be the deflection measured to the center of gravity. In both cases the distances are measured from the center line of the bearings, which is point S. The three parts of Fig. 5.6 show the location of these points and the forces acting upon them when the disk is rotating (a) below, (b) at, and (c) above the critical speed.

* Kimball and Hull, "Vibration Phenomena of a Loaded Unbalanced Shaft," *Trans. A.S.M.E.*, 1925, p. 673.

When the disk rotates below the critical speed (Fig. 5.6a), there is a centrifugal force acting on the center of gravity, which equals $m\omega^2 r_G$. The shaft is deflected an amount r_o and, hence, exerts a restoring force tending to straighten the shaft, which equals kr_o. There is also a damping force that is proportional to the velocity $(r_o\omega)$; it acts at the geometric center of the disk O and equals $c\omega r_o$. In order to balance these forces, the center of gravity swings around through the angle ϕ. The force F_G', shown on the figure, represents the resultant of the spring and damping forces and is equal in magnitude to the centrifugal force and parallel to it. This force acts at the angle θ with the line SO, which is the same angle θ at which the centrifugal force acts with the line SO. There is an unbalanced torque acting on the system, which is equal to the perpendicular distance between F_G' and the centrifugal force, times the centrifugal force or F_G'. It is also equal to the damping force $c\omega r_o$ times the distance r_o. This torque represents that required to rotate the disk, and if it is multiplied by the rotational speed in rpm and divided by 63,000, the horsepower required to drive the disk is obtained.

The condition at the critical speed is represented by Fig. 5.6b. The phase angle ϕ is now 90 deg, and the displacement r_o of the shaft is a maximum. The discussion in the preceding paragraph concerning force balance and driving torque applies in this case as well.

Above the critical speed, the angle ϕ increases further with increased speed, until finally the disk tends to rotate about its center of gravity; that is, points G and S tend to come together, and r_G approaches zero. The shaft deflection then approaches the eccentricity e.

EXAMPLE

A single disk weighing 20 lb is mounted between bearings that are 24 in. apart. The horizontal steel shaft is $\frac{1}{2}$ in. in diameter, and the center of gravity of the disk is $\frac{1}{4}$ in. from its geometric center. If the damping constant of the steel shaft is $c = 0.2$ lb-sec per in., draw a rough diagram of the forces acting when the shaft rotates 700 rpm and label their magnitudes. Compare the dead-load stress in the shaft with the stress at the operating speed. What horsepower is required to drive the shaft at this speed?

From Fig. 2.6, the spring scale of the shaft at the disk is

$$k = \frac{W}{y} = \frac{48EI}{L^3} = \frac{48 \times 30(10^6)\pi(\frac{1}{2})^4}{24^3 \times 64} = 320 \text{ lb per in.}$$

The natural circular frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{386 \times 320}{20}} = 78.65 \text{ rad per sec,}$$

and the cyclic frequency is

$$f_n = 60 \times \frac{\omega_n}{2\pi} = 60 \times \frac{78.65}{2\pi} = 752 \text{ cpm.}$$

The impressed frequency is 700 cpm, or $\omega = 73.25$ rad per sec. By Eq. (5.4) the phase angle ϕ is

$$\tan \phi = \frac{c\omega}{(k - m\omega^2)} = \frac{0.2 \times 73.25}{320 - \frac{20}{386} \times (73.25)^2} = 0.349.$$

$$\phi = 19°16'.$$

Since the impressed frequency is below the natural, or critical, frequency, Fig. 5.6a applies to this case. Since F_G' makes the same angle θ with the horizontal as the centrifugal force,

$$\tan \theta = \frac{c\omega r_o}{kr_o} = \frac{c\omega}{k} = \frac{0.2 \times 73.25}{320} = 0.0457.$$

$$\theta = 2°37'.$$

From Fig. 5.7, the distance $V = e \sin \phi = 0.25 \times 0.330 = 0.0825$ in.; the distance $M = \dfrac{V}{\tan \theta} = \dfrac{0.0825}{0.0457} = 1.802$ in.; and $N = e \cos \phi = 0.25 \times 0.944 = 0.236$ in.

$$\text{distance } r_o = M - N = 1.802 - 0.236 = 1.566 \text{ in.}$$

$$\text{distance } r_G = \frac{V}{\sin \theta} = \frac{0.0825}{0.0456} = 1.803 \text{ in.}$$

The various forces may now be found as follows:
Centrifugal force:

$$m\omega^2 r_G = \tfrac{20}{386}(73.25)^2 1.803 = 501 \text{ lb.}$$

Spring force:

$$kr_o = 320 \times 1.566 = 501 \text{ lb.}$$

Damping force:

$$c\omega r_o = 0.2 \times 73.25 \times 1.566 = 23 \text{ lb.}$$

The distances and forces are labeled on Fig. 5.7, which is not drawn to scale.

The stress in the shaft is

$$s = M\frac{c}{I} = \frac{FL}{4}\frac{d}{2}\frac{64}{\pi d^4} = \frac{F \times 24\frac{1}{2}}{4}\frac{2}{\pi\frac{1}{16}}\frac{64}{} = 488.5F.$$

If the shaft weight is neglected, the dead-load stress for a force $F = 20$ lb is $488.5 \times 20 = 9{,}770$ psi. At the operating speed, the force on the shaft is kr_o, that is, the spring force of 501 lb, to which must be added the dead load of 20 lb, making a total of 521 lb. The stress corresponding to this load is $488.5 \times 521 = 254{,}500$ psi. The size of this result demonstrates the tremendous stresses that

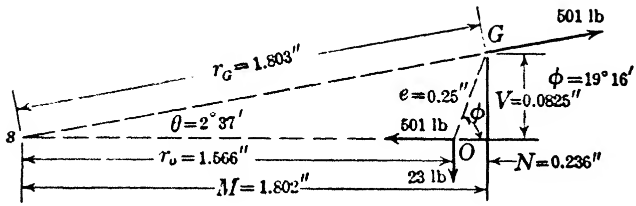


FIG. 5 7

may be built up in a member operating close to resonance. It should be realized, of course, that these stresses will prevail only after the shaft has been run at this speed for some time and "steady-state" conditions have been established. If the shaft speed passes through this value rapidly, however, deflections do not have time to build up, and, consequently, the operation is safe.

The torque required to drive the shaft equals the damping force times the shaft deflection. Thus, $T = c\omega r_o \times r_o = 23 \times 1.566 = 36$ in.-lb. The horsepower required is

$$T\frac{n}{63{,}000} = 36 \times \frac{700}{63{,}000} = 0.4.$$

5.4 RELATIVE MOTION OF BLOCK AND SUPPORT

Another type of forced vibration occurs when the support of a spring-mass system vibrates with simple harmonic motion. The following discussion is similar to that in Sec. 3.4, where damping is neglected. The arrangement shown in Fig. 5.8 includes a dashpot between the mass and support.

The motion of the support is assumed to be harmonic and is $x_s = x_{s_0} \cos \omega t$, and that of the block is $x_b = x_{b_0} \cos \omega t$. The relative motion between the two, or the change in the length of the spring, is $x_r = x_b - x_s$.

The block has the following forces acting upon it: inertia, $m \frac{d^2x_b}{dt^2}$; damping, $c \frac{dx_r}{dt}$; and the spring, kx_r. For equilibrium, the

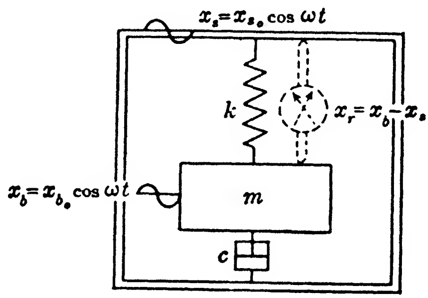


FIG. 5.8

equation of motion is

$$m \frac{d^2x_b}{dt^2} + c \frac{dx_r}{dt} + kx_r = 0. \tag{5.8}$$

But since $x_b = x_r + x_s$, Eq. (5.8) may be written

$$m \frac{d^2x_r}{dt^2} + c \frac{dx_r}{dt} + kx_r = -m \frac{d^2x_s}{dt^2}. \tag{5.9}$$

As the motion of the support is assumed to be simple harmonic,

$$x_s = x_{s_0} \cos \omega t,$$

and

$$\frac{d^2x_s}{dt^2} = -x_{s_0}\omega^2 \cos \omega t.$$

Substituting this expression in Eq. (5.9), we obtain

$$m \frac{d^2x_r}{dt^2} + c \frac{dx_r}{dt} + kx_r = mx_{s_0}\omega^2 \cos \omega t,$$

which is similar in form to Eq. (5.1) if $mx_{s_0}\omega^2$ is taken to be the

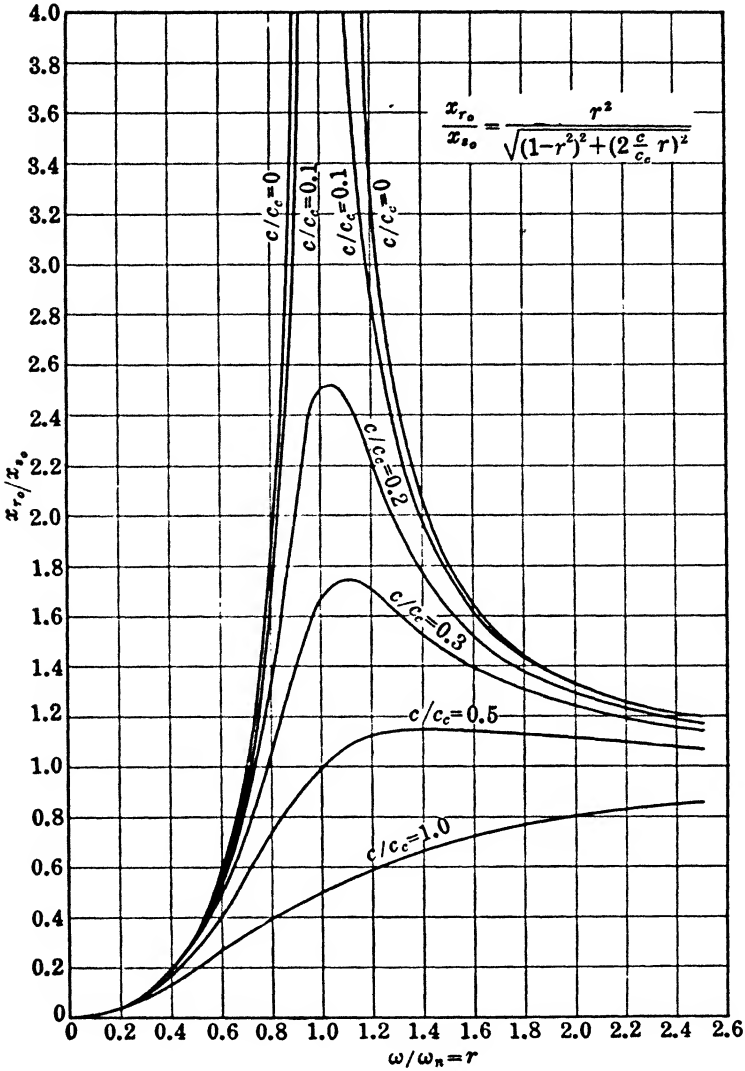


FIG. 5.9

impressed force (F_o) and x_r replaces x. The x_{st} term in the solution of Eq. (5.1), as used in Eq. (5.7), is $\frac{F_o}{k}$, which is equivalent to

$$\frac{mx_{s_o}\omega^2}{k} = \left(\frac{\omega^2}{\omega_n^2}\right)x_{s_o} = r^2x_{s_o}.$$

Hence, this equation may be written as

$$\frac{x_{r_0}}{x_{s_0}} = \frac{r^2}{\sqrt{(1 - r^2)^2 + \left(2 \frac{c}{c_c} r\right)^2}}. \tag{5.10}$$

Figure 5.9 shows a plot of x_{r_0}/x_{s_0} against the speed ratio r for various amounts of damping.

EXAMPLE

An accelerometer is made by mounting a 10-lb block on rubber isolators that have a combined spring scale of 100 lb per in. and a viscous damping factor ratio c/c_c of 0.2 (Fig. 5.8). The amplitude read on the dial indicator of vibrations occurring at 200 cpm is 0.004 in. What is the maximum acceleration of the member to which the accelerometer is attached?

The undamped natural circular frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100 \times 386}{10}} = 62.2 \text{ rad per sec.}$$

The impressed circular frequency is $\omega = \frac{200}{60} 2\pi = 21$. Hence, the frequency ratio $r = \frac{\omega}{\omega_n} = \frac{21}{62.2} = 0.337$, and $r^2 = 0.1137$.

Applying Eq. (5.10),

$$\frac{x_{r_0}}{x_{s_0}} = \frac{r^2}{\sqrt{(1 - r^2)^2 + \left(2 \frac{c}{c_c} r\right)^2}} = \frac{0.004}{x_{s_0}}$$

$$= \frac{0.1137}{\sqrt{(1 - 0.1137)^2 + (2 \times 0.2 \times 0.337)^2}}.$$

Solving for the amplitude of the member, $x_{s_0} = 0.0345$ in. The maximum acceleration is $a_s = x_{s_0}\omega^2 = 0.0345 \times 21^2 = 15.21$ in. per sec.^{2}

5.5 *TRANSMISSIBILITY*

All isolators have some damping; for those made of organic materials, such as cork, felt, and rubber, the amount is appreciable and should be considered.

The generated or impressed force is transmitted to the supporting structure in two ways. The part that passes through the springs equals kx_o, whereas the remainder of the transmitted force passes through the dashpot, or damping portion of the isolator, and equals $c\omega x_o$. Since these two forces act at right angles to each other, the resultant transmitted force to the foundation is $x_o\sqrt{k^2 + (c\omega)^2}$. The impressed force is F_o; hence, the transmissibility t, which is the ratio of the transmitted force to that impressed, is

$$t = x_o \frac{\sqrt{k^2 + (c\omega)^2}}{F_o}.$$

Substituting the value of x_o from Eq. (5.6) in this equation gives

$$t = \frac{F_o}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} \frac{\sqrt{k^2 + (c\omega)^2}}{F_o}$$

$$= \sqrt{\frac{k^2 + (c\omega)^2}{(c\omega)^2 + (k - m\omega^2)^2}}. \tag{5.11}$$

This equation may be put in dimensionless form by introducing the relations $k = m\omega_n^2$; $c_c = 2m\omega_n$; and $\omega/\omega_n = r$; thus,

$$t = \sqrt{\frac{1 + \left(2\frac{c}{c_c}r\right)^2}{(1 - r^2)^2 + \left(2\frac{c}{c_c}r\right)^2}} \tag{5.12}$$

Figure 5.10 shows a plot of transmissibility against the speed ratio r for various damping factor ratios c/c_c based upon Eq. (5.12). It may be observed from this figure that damping is helpful in isolating forces only when the ratio r, or ω/ω_n, is less than $\sqrt{2}$, or 1.414. For ratios greater than this, increased damping permits slightly more force to be transmitted to the foundation and, hence, is not beneficial. For speed ratios less than 1.414, however, the transmitted force is greatly reduced with increased damping.*

5.6 COMMERCIAL ISOLATORS

For effective vibration isolation, it is extremely important that the isolators be correctly designed and applied for if they are not, the forces and motion may be magnified rather than reduced.

* S. Rosenzweig, "Theory of Elastic Engine Supports," *Trans. A.S.M.E.*, January, 1939, pp. 31–36.

Two basic requirements of any isolator are that there be no solid connection between the unit and its supporting structure through which sound may be conducted, and that provision be made to

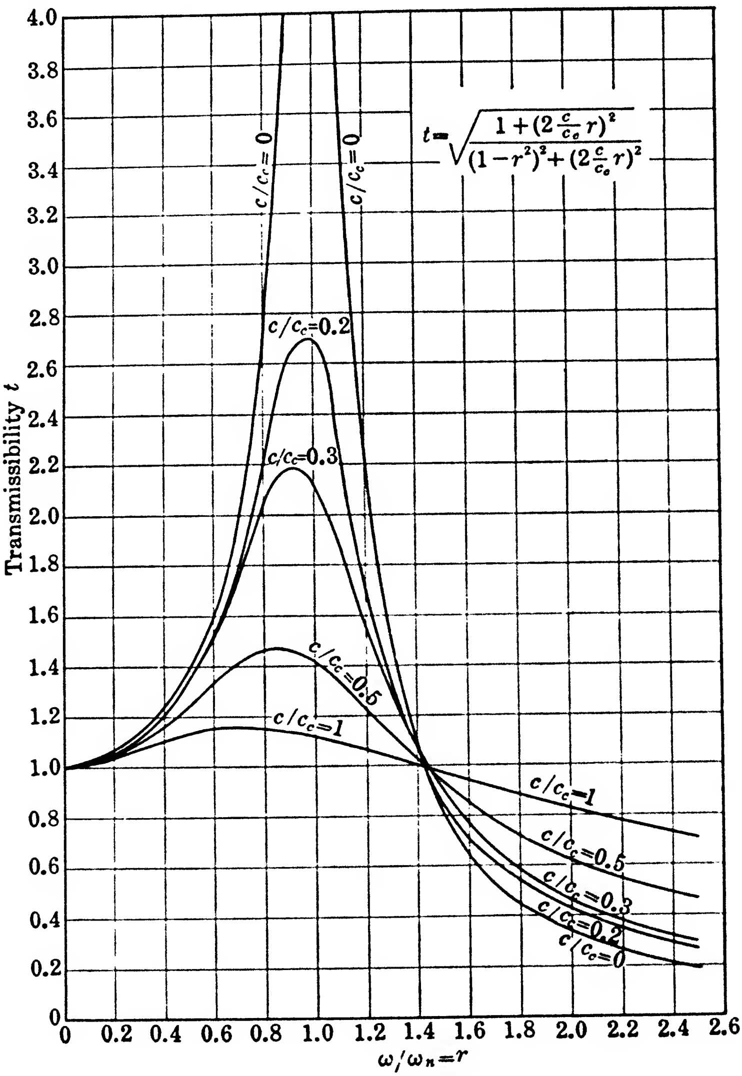


FIG. 5.10

hold the isolator together in the event that the damping material should fail.

As given in a great many manufacturers' bulletins, commercial isolators are selected on the basis of their static deflection when the machine is mounted on them. Thus, from Fig. 5.10 for $c/c_c = 0$ or from Fig. 3.2 it may be determined that for a desired transmissibility the value of f_n should be, say, 750 cpm, which by Eq. (2.11) corresponds to a static deflection of $\frac{1}{16}$ in. The weight of the body to be isolated is divided by the number of supports to determine the dead load on each, and the manufacturers' bulletins are consulted to select an isolator that will have a static deflection of $\frac{1}{16}$ in. at that static load. This procedure assumes that the load always acts directly along the axis of the isolator, which is not generally the case.

Any rigid body has six degrees of freedom as illustrated by Fig. 1.1, and for vibrating machinery it is very unusual to have the entire motion take place in one direction of translation only. Usually dynamic couples (as outlined in Sec. 10.1) are present to produce rocking motions or horizontal vibrations as well as vertical displacements.

To secure satisfactory isolation, isolators should be selected with static deflections that give safe values of the frequency ratio r for each of the six degrees of freedom. This involves the determination of the mass moments of inertia of the machine about the three major axes of rotation.*

The four common materials used for vibration isolators are steel springs, rubber, cork, and felt. The properties of these materials will be compared briefly.

A. *Steel Springs*

Steel springs have very little damping. There is more damping with leaf springs, because of the friction between the leaves, than there is with helical springs, but in either case it is commonly considered to be negligible. They have a high sound transmissibility, although this may be reduced by mounting the springs on pads of felt, cork, or rubber. In addition, steel springs are not affected by the presence of oil or water or by extreme temperatures, and they are quite uniform in their properties. The organic materials are liable to vary with their processing and the conditions under which

* A good discussion of the location of isolators may be found in J. N. Macduff, "Isolation of Vibration in Spring Mounted Apparatus," *Product Eng.*, July, 1946, pp. 106, 159; August, 1946, p. 154.

they are used. The spring scale, however, can be designed with good accuracy for any desired value.

An examination of Fig. 5.10 shows that spring scales are particularly satisfactory for speed ratios of r greater than 1.414, since c/c_c is close to zero. On the other hand, for speed ratios less than 1.414, such as are met in marine applications, for example, where the impressed frequency ω or f is quite low, they are not suitable. For such applications one of the organic materials, which have higher damping factors, would be more satisfactory.

B. *Rubber*

Many commercial isolators are made of rubber, which, like water, is very nearly incompressible and, hence, is generally used in shear rather than in compression to secure greater flexibility (lower spring scale). The stress in the rubber is generally kept between 40 and 70 psi. The properties vary widely* with the load, the temperature, the shape of the piece, and the impressed frequency. For higher temperatures, the stress must be reduced to avoid excessive creep and deterioration. Rubber is generally not considered satisfactory for temperatures above 125 to 150 deg, and since oil and gasoline attack rubber, isolators made of it cannot be used in the presence of these substances.

Rubber has a low sound transmissibility. It also has a low modulus of elasticity for low loads which increases with the load, making it particularly good for light loads and high-frequency vibrations. The stress-strain curve is not a straight line, so care must be exercised to get the correct modulus for a given loading. The stiffness varies widely with the frequency and the value when vibrating may be more than twice as great as static values.

C. *Cork*

Cork is generally used either in compression or in compression and shear. It is not perfectly elastic, being more flexible at high loads; and its properties change with the frequency. In this latter respect it is similar to rubber. Considerable pressure must

* J. F. Downie, Smith, "Rubber Mountings," *Trans. A.S.M.E.*, March, 1938, A-13; and "Rubber Springs on Shear Loading," *Trans. A.S.M.E.*, December, 1939, A-159.

be exerted on cork before the load-deflection rate, or modulus of elasticity, is reduced. Generally it is placed beneath a large concrete block to obtain this result. The recommended pressures for satisfactory performance lie between 1,000 and 3,000 psi.

Cork is filled with air cells; and when it is under load, this air is compressed. When the load is removed, the air expands and restores the cork body to its original shape.

D. *Felt*

Felt is used in the form of small compression pads, which are placed under concrete or steel bases, rather than a large single pad; or

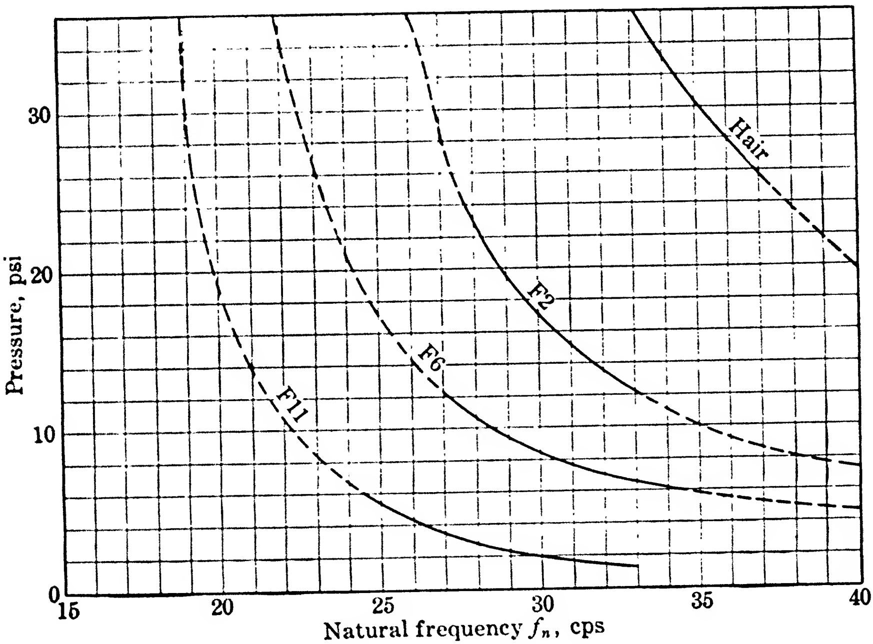


FIG. 5.11

as compression disks under the legs or corners of machines. It has a high damping factor and thus is particularly satisfactory for low speed ratios r. Tests show the optimum thickness of the pads or disks to be 1 in.

The American Felt Company* recommends a design procedure illustrated in Fig. 5.11. The curves shown there are based upon test

* American Felt Co., "Vibration Isolation with Felt," *Data Sheet No. 10.*

results and give the natural frequency in cycles per second for various pressures placed on 1-in. thicknesses of various SAE grades of felt. The recommended portions of the curves are shown in solid lines, the remainder being dashed.

If damping is neglected, the transmissibility t from Sec. 3.3 for speeds above resonance is

$$t = \frac{-1}{1 - \frac{f}{f_n}^2},$$

where f is the impressed frequency and f_n the natural frequency. Solving this equation for the natural frequency,

$$f_n = \sqrt{\frac{f^2 t}{1 + t}}. \tag{5.13}$$

When a suitable transmissibility t has been selected (generally 20 per cent or 0.2 is acceptable for this), and the load to be supported and the impressed frequency in cycles per second are known the natural frequency may be found from Eq. (5.13). By using the solid curves of Fig. 5.11 when possible, the SAE grade of 1-in.-thick felt and the pressure may be found. The area of the felt to be used is then found by dividing the load by the pressure.

Example

A machine operating at 3,600 rpm weighs 150 lb. It is desired to calculate the area of four felt disks to be placed under the legs to give a transmissibility of 20 per cent. What grade and area of 1-in.-thick felt should be used, assuming the weight to be equally divided among the four legs?

The impressed frequency $f = \frac{3{,}600}{60} = 60$ cps. Then from Eq. (5.13), the natural frequency is

$$f_n = \sqrt{\frac{f^2 t}{1 + t}} = \sqrt{\frac{60^2 \times 0.2}{1 + 0.2}} = 24.5 \text{ cps.}$$

From Fig. 5.11, the recommended grade of felt is found to be SAE F-11 (SAE F-6 could be used, but it is not in the recommended range), and the corresponding pressure p is 6 psi.

The area of 1-in.-thick felt under each of the four legs is

$$A = \frac{W}{4p} = \frac{150}{4 \times 6} = 6.25 \text{ sq in.}$$

PROBLEMS

5.1. A weight of 200 lb is suspended on a spring having a scale of 100 lb per in. and is acted on by a harmonic force of 8 lb at the undamped natural frequency. The damping may be considered to be viscous with a factor c of 0.5 lb-sec per in. Determine (a) the undamped natural frequency; (b) the amplitude of the weight; (c) the phase angle ϕ; (d) the force transmitted to the foundation; (e) the damped natural frequency; (f) the force transmitted to the foundation if there were no damping.

Ans. (a) 133 cpm; (b) 1.152 in.; (c) 90 deg; (d) 115.7 lb; (e) 132.5 cpm; (f) infinite.

5.2. Repeat Prob. 5.1, assuming that the frequency of the impressed force is 500 cpm. The remainder of the data are unchanged. Compare the answers obtained for the two problems.

Ans. (a) 133 cpm; (b) 0.00607 in.; (c) 178°52′; (d) 0.63 lb; (e) 132.5 cpm; (f) 0.61 lb.

5.3. Repeat the example of Sec. 5.3 for a rotational speed of (a) 752 rpm; (b) 800 rpm. Compare the results with those of the example.

Ans. (a) $\phi = 90$ deg; $r_G = 5.10$; $r_o = 5.09$; $F_G = 1{,}630$; $F_s = 1{,}626$; $F_d = 80$; 4.85 hp; 800,000 psi. (b) $\phi = 158°40'$; $r_G = 1.74$; $r_o = 2.00$; $F_G = 631$; $F_s = 630$; $F_d = 33.6$; 0.825 hp; 318,000 psi.

5.4. What would be the maximum acceleration if the vibration occurred at 2,000 cpm in the example of Sec. 5.4? The remainder of the data given are unchanged. *Ans.* 162 in. per sec².

5.5. A delicate balance weighing 10 lb must be installed on a bench in a shop. The bench vibrates at 1,760 cpm with an amplitude of 0.001 in. It is proposed to use rubber isolators that have a damping-factor ratio c/c_c of 0.01 and a combined spring scale under load of 40 lb per in. (a) What will be the amplitude of the balance? (b) If steel springs having the same spring scale but negligible damping were used, what would be the amplitude?

Ans. (a) 0.000048; (b) 0.000047.

5.6. A radio set weighing 40 lb must be isolated from vibrations of 0.002 in. amplitude occurring at 500 cpm. The set is mounted on four isolators, each having a spring scale of 160 lb per in. and a damping factor of 2 lb-sec per in. (a) What is the amplitude of vibration of the radio? (b) What is the dynamic load on the isolators due to the vibration?

Ans. (a) 0.00304 in.; (b) 0.1654 lb per isolator.

5.7. A printing press weighs 1,000 lb per loading point and operates with a disturbing frequency of 1,450 cpm. It is proposed to use an isolator having a static deflection under the given load of $\frac{1}{8}$ in. and a damping factor ratio c/c_c of 0.1. If the disturbing force is 500 lb, determine (a) the amplitude of motion of the press; (b) the phase angle of the motion; and (c) the force transmitted to the foundation. *Ans.* (a) 0.0096 in.; (b) 175.18 deg; (c) 88 lb.

5.8. A machine having a total weight of 2,500 lb is mounted on isolators having a combined stiffness k of 9,500 lb per in. A piston weighing 60 lb moves up and down harmonically in the machine with a stroke of 18 in. and a speed of 500 cpm. The amplitude of motion of the unit is 0.403 in. (a) Assume that the damping is viscous, and find the value of the damping factor c. (b) What force is transmitted to the foundation? (c) If there were no damping in the isolators ($c = 0$), what would be the amplitude of motion of the unit? What force would be transmitted to the foundation in part (c)? (e) If the piston is suddenly stopped, determine the frequency of the damped vibration of the unit. (f) What is the frequency of the machine before the piston is stopped? (g) What suggestions could you make to improve the effectiveness of the isolators for this installation? Give reasons for your answers.

Ans. (a) 90.6 lb-sec per in.; (b) 4,290 lb; (c) 0.465 in.; (d) 4,420 lb; (e) 360 cpm; (f) 500 cpm.

5.9. The disk of a torsional pendulum has a mass moment of inertia J of 2 in.-lb-sec² and is connected to the foundation with a steel shaft 30 in. long and $\frac{1}{2}$ in. in diameter. When a harmonic torque of 50 in.-lb is applied to the disk with a frequency of 500 cpm, its amplitude of vibration is found to be 0.7 deg after steady-state conditions have been attained. (a) Assume that viscous damping is present, and find the value of the damping factor c. (b) What is the frequency of the disk in cycles per minute? (c) How much of the impressed torque is transmitted to the foundation? (d) If the impressed torque is suddenly stopped, what would be the frequency of the disk in cycles per minute?

Ans. (a) 53.2 in.-lb-sec per rad; (b) 500 cpm; (c) 0.9, or 45 in.-lb; (d) 309 cpm.

5.10. A machine weighing 520 lb has a disturbing force acting upon it at a frequency of 5,400 cpm. What grade and area of felt should be used to isolate 85 per cent of the impressed force?

Ans. 40 sq in. of SAE F-2 felt.

Chapter 6

UNDAMPED VIBRATIONS—TWO DEGREES OF FREEDOM

6.1 *INTRODUCTION*

The vibration problems considered thus far have been confined to those with a single degree of freedom. Many systems are of the type of two degrees of freedom; hence, it is desirable to investigate cases of this nature. Units having the driver and driven machines separated by a gear are applications of this condition, for example, turbine-driven pumps, aircraft engine and propeller with gear reduction between them, lighting sets on ships, and the like.

Dynamic vibration absorbers may be attached to systems with a single degree of freedom to eliminate or reduce the vibration there. These systems are then transformed into systems of two degrees of freedom with forced vibrations and, hence, will be considered in this chapter.

Cases involving damping with two degrees of freedom are beyond the scope of this book because of their complexity. Information concerning them may be found in J. P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill, New York, 1947, or W. K. Wilson, *Practical Solution of Torsional Vibration Problems*, Wiley, New York, 1940.

6.2 *FREE LATERAL VIBRATIONS*

Figure 6.1 illustrates a system having two degrees of freedom in a vertical plane. The two masses may vibrate in the same direction, or their motion may be in opposite directions. Each of these modes of vibration has its particular natural frequency, the former occurring at the lower frequency, and the latter at the higher frequency.

Assume that the two masses in the figure vibrate with simple harmonic motion of different amplitudes x_{o_1} and x_{o_2}, but at the same frequency, and that at a given instant the masses are both displaced downward by amounts x_1 and x_2. If x_2 is less than x_1, the middle spring will be compressed a distance $x_1 - x_2$. The accelerations of the masses are d^2x_1/dt^2 and d^2x_2/dt^2.

Considering the upper mass as a free body, the inertia and spring forces acting on it must be in equilibrium. Thus,

$$m_1 \frac{d^2x_1}{dt^2} + k_1x_1 + k_2(x_1 - x_2) = 0,$$

$$m_1 \frac{d^2x_1}{dt^2} + x_1(k_1 + k_2) - k_2x_2 = 0. \tag{6.1}$$

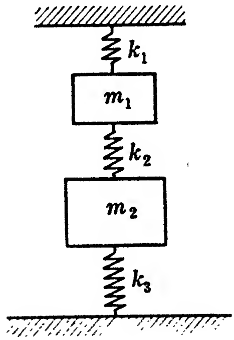


Fig. 6.1

The forces acting upon the lower mass must also be in equilibrium. Thus,

$$m_2 \frac{d^2x_2}{dt^2} + k_3x_2 - k_2(x_1 - x_2) = 0,$$

$$m_2 \frac{d^2x_2}{dt^2} + x_2(k_2 + k_3) - k_2x_1 = 0. \tag{6.2}$$

Since the motion is assumed to be simple harmonic, the terms x_1 and x_2 are given by

$$x_1 = x_{o_1} \cos \omega t, \qquad x_2 = x_{o_2} \cos \omega t;$$

and

$$\frac{d^2x_1}{dt^2} = -x_{o_1}\omega^2 \cos \omega t, \qquad \frac{d^2x_2}{dt^2} = -x_{o_2}\omega^2 \cos \omega t.$$

Substituting these expressions in Eqs. (6.1) and (6.2),

$$[-m_1x_{o_1}\omega^2 + x_{o_1}(k_1 + k_2) - k_2x_{o_2}] \cos \omega t = 0,$$

$$[-m_2x_{o_2}\omega^2 + x_{o_2}(k_2 + k_3) - k_2x_{o_1}] \cos \omega t = 0.$$

Since $\cos \omega t$ will vary from zero to 1, the terms in the brackets must equal zero if these equations are to be valid at all instants of time. Hence,

$$x_{o_1}(-m_1\omega^2 + k_1 + k_2) - x_{o_2}k_2 = 0, \tag{6.3}$$

$$x_{o_1}k_2 + x_{o_2}(m_2\omega^2 - k_2 - k_3) = 0. \tag{6.4}$$

The ratio of the amplitudes from Eq. (6.3) is

$$\frac{x_{o_1}}{x_{o_2}} = \frac{k_2}{-m_1\omega^2 + k_1 + k_2}, \tag{6.5}$$

and from Eq. (6.4)

$$\frac{x_{o_1}}{x_{o_2}} = \frac{k_2 + k_3 - m_2\omega^2}{k_2}. \tag{6.6}$$

Equating Eqs. (6.5) and (6.6) gives

$$\frac{k_2}{-m_1\omega^2 + k_1 + k_2} = \frac{k_2 + k_3 - m_2\omega^2}{k_2}.$$

or

$$k_2^2 = (-m_1\omega^2 + k_1 + k_2)(k_2 + k_3 - m_2\omega^2).$$

Expanding and collecting terms,

$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3}{m_2}\right)\omega^2 + \left(\frac{k_1k_2 + k_2k_3 + k_1k_3}{m_1m_2}\right) = 0, \tag{6.7}$$

which is a quadratic equation in ω^2. Solving for ω^2,

$$\omega^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2 + k_3}{2m_2} \pm \sqrt{\left(\frac{k_1 + k_2}{2m_1} + \frac{k_2 + k_3}{2m_2}\right)^2 - \frac{k_1k_2 + k_2k_3 + k_1k_3}{m_1m_2}}. \tag{6.8}$$

To illustrate the action taking place in a system of this type, the following example is given.

Example

Determine the two natural frequencies of vertical vibration of a system similar to that shown in Fig. 6.1, given the following data: $W_1 = 2$ lb, $W_2 = 1$ lb; $k_1 = 20$ lb per in.; $k_2 = 10$ lb per in.; $k_3 = 20$ lb per in. Also determine the ratio of the amplitudes of the motion of W_1 to W_2 for the two modes of vibration.

$$m_1 = \frac{W_1}{g} = \frac{2}{386} = 0.00518,$$

$$m_2 = \frac{W_2}{g} = \frac{1}{386} = 0.00259.$$

$$\omega_n^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2 + k_3}{2m_2} \pm \sqrt{\left(\frac{k_1 + k_2}{2m_1} + \frac{k_2 + k_3}{2m_2}\right)^2 - \frac{k_1k_2 + k_2k_3 + k_1k_3}{m_1m_2}} \tag{6.8}$$

$$= \frac{20 + 10}{0.01036} + \frac{10 + 20}{0.00518} \pm \sqrt{\left(\frac{20 + 10}{0.01036} + \frac{10 + 20}{0.00518}\right)^2 - \frac{10 \times 20 + 10 \times 20 + 20 \times 20}{0.00518 \times 0.00259}}.$$

$$\omega_n^2 = 4{,}690 \text{ or } 12{,}690.$$

$$\omega_n = 68.5 \text{ or } 112.8 \text{ rad per sec.}$$

$$f_n = 655 \text{ or } 1{,}077 \text{ cpm.}$$

$$\frac{x_{o_1}}{x_{o_2}} = \frac{k_2}{-m_1\omega_n^2 + k_1 + k_2}. \tag{6.5}$$

First mode:

$$\frac{x_{o_1}}{x_{o_2}} = \frac{10}{-0.00518 \times 4{,}690 + 20 + 10}$$

$$= +1.75.$$

Second mode:

$$\frac{x_{o_1}}{x_{o_2}} = \frac{10}{-0.00518 \times 12{,}690 + 20 + 10}$$

$$= -0.28.$$

6.3 *FREE TORSIONAL VIBRATIONS*

As a disk vibrates torsionally, an inertia torque equal to $J\alpha$ is developed. If the disk vibrates harmonically with an amplitude of β, the maximum value of the acceleration α is $\beta\omega^2$, where ω is the circular frequency of the vibration. Hence, the maximum inertia torque, which occurs when the displacement is a maximum, is $T = J\beta\omega^2$.

The three-disk two-shaft system shown in Fig. 6.2 is an example of a system of two degrees of freedom, since there are two manners, or modes, in which the disks may vibrate; that is, one end disk may oscillate in a direction opposite to the other two, or the middle disk may oscillate against the two end ones. Consequently, there will be two natural frequencies corresponding to the two modes of free vibration. Expressions for these frequencies will now be developed.

As disk 1 oscillates with an amplitude β_1, the maximum torque developed is $T_1 = J_1\beta_1\omega^2$. This torque is taken by shaft 1 and results in a relative twist between disks 1 and 2 of $\Delta\beta_{1\text{-}2} = \dfrac{T_1}{k_{t_1}} = \dfrac{J_1\beta_1\omega^2}{k_{t_1}}$. The amplitude of disk 2 equals $\beta_2 = \beta_1 - \Delta\beta_{1\text{-}2}$, and the corresponding torque T_2 developed in the second disk is $T_2 = J_2\beta_2\omega^2$. The torque acting on shaft 2 is the algebraic sum of the torques

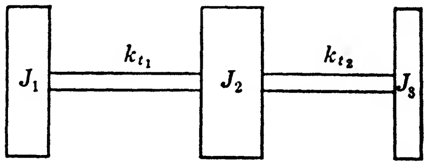


FIG. 6.2

developed by disks 1 and 2; thus, $T_{2\text{-}3} = T_1 + T_2$. The relative twist between disks 2 and 3 is $\Delta\beta_{2\text{-}3} = \dfrac{T_{2\text{-}3}}{k_{t_2}} = \dfrac{T_1 + T_2}{k_{t_2}}$; and the amplitude of disk 3 is $\beta_3 = \beta_2 - \Delta\beta_{2\text{-}3}$. The torque developed by disk 3 is $T_3 = J_3\beta_3\omega^2$.

Since the vibration is a free one, no external torque is added to the system; hence, the sum of the inertia torques must equal zero, that is, $T_1 + T_2 + T_3 = 0$. The exact magnitude of the various amplitudes is independent of the frequencies; therefore, the above discussion applies for any value of amplitude of disk 1, or β_1.

The following expressions, in terms of β_1, ω^2, and the known quantities in the system, follow the discussion given above; so they are listed without further explanation.

$$T_1 = J_1\beta_1\omega^2. \tag{6.9}$$

$$\Delta\beta_{1\text{-}2} = \frac{T_1}{k_{t_1}} = \frac{J_1\beta_1\omega^2}{k_{t_1}},$$

$$\beta_2 = \beta_1 - \Delta\beta_{1\text{-}2} = \beta_1 - \frac{J_1\beta_1\omega^2}{k_{t_1}}$$

$$= \beta_1\left(1 - \frac{J_1\omega^2}{k_{t_1}}\right). \tag{6.10}$$

$$T_2 = J_2\beta_2\omega^2 = J_2\beta_1\omega^2\left(1 - \frac{J_1\omega^2}{k_{t_1}}\right). \tag{6.11}$$

$$T_{2\text{-}3} = T_1 + T_2 = J_1\beta_1\omega^2 + J_2\omega^2\beta_1\left(1 - \frac{J_1\omega^2}{k_{t_1}}\right)$$

$$= \beta_1\omega^2\left(J_1 + J_2 - \frac{J_1J_2\omega^2}{k_{t_1}}\right).$$

$$\Delta\beta_{2\text{-}3} = \frac{T_{2\text{-}3}}{k_{t_2}} = \frac{1}{k_{t_2}}\left[\beta_1\omega^2\left(J_1 + J_2 - \frac{J_1J_2\omega^2}{k_{t_1}}\right)\right]$$

$$= \beta_1\omega^2\left(\frac{J_1 + J_2}{k_{t_2}} - \frac{J_1J_2\omega^2}{k_{t_1}k_{t_2}}\right).$$

$$\beta_3 = \beta_2 - \Delta\beta_{2\text{-}3} = \beta_1\left(1 - \frac{J_1\omega^2}{k_{t_1}}\right) - \beta_1\omega^2\left(\frac{J_1 + J_2}{k_{t_2}} - \frac{J_1J_2\omega^2}{k_{t_1}k_{t_2}}\right)$$

$$= \beta_1\left[1 - \frac{J_1\omega^2}{k_{t_1}} - \frac{(J_1 + J_2)\omega^2}{k_{t_2}} + \frac{J_1J_2\omega^4}{k_{t_1}k_{t_2}}\right]. \tag{6.12}$$

$$T_3 = J_3\beta_3\omega^2 = J_3\omega^2\beta_1\left[1 - \frac{J_1\omega^2}{k_{t_1}} - \frac{(J_1 + J_2)\omega^2}{k_{t_2}} + \frac{J_1J_2\omega^4}{k_{t_1}k_{t_2}}\right]. \tag{6.13}$$

Adding Eqs. (6.9), (6.11), and (6.13) gives the total torque for the system, which must equal zero. Thus,

$$T_1 + T_2 + T_3 = J_1\beta_1\omega^2 + J_2\omega^2\beta_1\left(1 - \frac{J_1\omega^2}{k_{t_1}}\right) +$$
$$J_3\omega^2\beta_1\left[1 - \frac{J_1\omega^2}{k_{t_1}} - \frac{(J_1 + J_2)\omega^2}{k_{t_2}} + \frac{J_1J_2\omega^4}{k_{t_1}k_{t_2}}\right]$$
$$= \beta_1\omega^2\left[J_1 + J_2 - \frac{J_1J_2\omega^2}{k_{t_1}} + J_3 - \frac{J_1J_3\omega^2}{k_{t_1}} - \right.$$
$$\left.\frac{J_1J_3\omega^2}{k_{t_2}} - \frac{J_2J_3\omega^2}{k_{t_2}} + \frac{J_1J_2J_3\omega^4}{k_{t_1}k_{t_2}}\right]$$
$$= 0.$$

Since β_1 and ω^2 must be real numbers, the expression in the bracket must equal zero to satisfy the equation; hence,

$$\frac{J_1J_2J_3}{k_{t_1}k_{t_2}}\omega^4 - \left(\frac{J_1J_2 + J_1J_3}{k_{t_1}} + \frac{J_1J_3 + J_2J_3}{k_{t_2}}\right)\omega^2 + (J_1 + J_2 + J_3) = 0. \tag{6.14}$$

This is a quadratic equation in terms of ω^2. Solving for ω^2,

$$\omega^2 = \frac{1}{2}\left[\frac{k_{t_2}}{J_3} + \frac{k_{t_2}}{J_2} + \frac{k_{t_1}}{J_2} + \frac{k_{t_1}}{J_1}\right] \pm$$

$$\sqrt{\left(\frac{J_1J_2 + J_1J_3}{k_{t_1}} + \frac{J_1J_3 + J_2J_3}{k_{t_2}}\right)^2\left(\frac{k_{t_1}k_{t_2}}{2J_1J_2J_3}\right)^2 - \frac{(J_1 + J_2 + J_3)k_{t_1}k_{t_2}}{J_1J_2J_3}} \tag{6.15}$$

Two values of ω^2 are obtained from this equation. The lower value represents the natural frequency for a single node, that is, one end disk oscillating against the other two disks; whereas the upper represents the natural frequency with two nodes, that is, the middle disk oscillating against the two end disks.

It is frequently desirable to know the relative magnitude of amplitude of the disks and to locate the position of the nodes. For the middle disk, the ratio of β_2/β_1 from Eq. (6.10) is

$$\frac{\beta_2}{\beta_1} = 1 - \frac{J_1\omega^2}{k_{t_1}}; \tag{6.16}$$

whereas for disk 3, from Eq. (6.12),

$$\frac{\beta_3}{\beta_1} = 1 - \frac{J_1\omega^2}{k_{t_1}} - \frac{(J_1 + J_2)\omega^2}{k_{t_2}} + \frac{J_1J_2\omega^4}{k_{t_1}k_{t_2}}. \tag{6.17}$$

It may be simpler, however, to find the ratio of β_2/β_3, which is,

$$\frac{\beta_2}{\beta_3} = 1 - \frac{J_3\omega^2}{k_{t_2}}. \tag{6.18}$$

Since the torque along a given shaft length is constant, the deflection curve between any two disks is a straight line. If the amplitude of disk 1 is taken to be 1 rad, the amplitudes of the other disks in radians are given by Eqs. (6.16) to (6.18). Since the deflection curves consist of a series of straight lines, the location of the nodes can be found by proportion. Thus, in Fig. 6.3, part (a) represents two possible deflection curves for the lower frequency, and part (b) shows the deflection curve for the higher frequency. It may be seen from these that

$$\frac{L_3}{L_2} = \frac{\beta_3}{\beta_2 - \beta_3} \qquad \text{or} \qquad L_3 = \frac{\beta_3 L_2}{\beta_2 - \beta_3}, \tag{6.19}$$

$$\frac{L_4}{L_1} = \frac{\beta_1}{\beta_1 - \beta_2} \qquad (6.20)$$

or

$$L_4 = \frac{\beta_1 L_1}{\beta_1 - \beta_2}, \qquad (6.21)$$

where L_1 = distance between disks 1 and 2,
L_2 = distance between disks 2 and 3,
L_3 = distance from node to disk 3,
L_4 = distance from node to disk 1.

EXAMPLE

The 250-kw lighting set used on some cargo ships may be reduced to a three-mass system having the following inertias

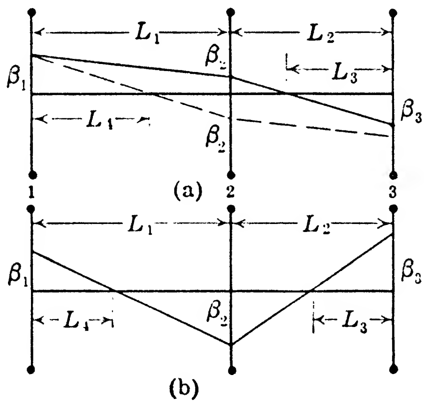


FIG. 6.3

expressed in in.-lb-sec² referred to the generator speed: turbine 1,880, gears 404, generator 320. The equivalent shaft lengths corrected to the generator speed and a 10 in. diameter steel shaft are turbine to gears 685 in., gears to generator 700 in. (a) Determine the natural frequencies of the unit. (b) Assume the turbine amplitude to be 1 rad, and find the amplitudes of the other masses at each frequency. (c) Locate the positions of the nodes in the system at each frequency.

Let J_1 = 1,880, J_2 = 404, J_3 = 320.

The spring scale is $k_t = \frac{G\pi d_e^4}{32L_e}$; hence,

$$k_{t_1} = \frac{12(10^6)\pi(10^4)}{32 \times 685} = 17.2(10^6) \text{ in.-lb per rad.}$$

$$k_{t_2} = \frac{12(10^6)\pi(10^4)}{32 \times 700} = 16.83(10^6) \text{ in.-lb per rad.}$$

(a) The natural frequencies may be found from Eq. (6.14); thus,

$$\frac{J_1J_2J_3}{k_{t_1}k_{t_2}}\omega^4 - \left[\frac{J_1J_2 + J_1J_3}{k_{t_1}} + \frac{J_1J_3 + J_2J_3}{k_{t_2}}\right]\omega^2 + (J_1 + J_2 + J_3) = 0;$$

$$\frac{1{,}880 \times 404 \times 320}{17.2 \times 16.83(10^{12})}\omega^4 - \left[\frac{1{,}880 \times 404 + 1{,}880 \times 320}{17.2(10^6)} + \frac{1{,}880 \times 320 + 404 \times 320}{16.83(10^6)}\right]\omega^2 + (1{,}880 + 404 + 320) = 0;$$

$$0.84(10^{-6})\omega^4 - 0.1227\omega^2 + 2604 = 0.$$

From the above

$$\omega^2 = 0.0259(10^6) \text{ or } 0.12(10^6).$$

$$\omega = 161 \text{ or } 346.5 \text{ rad per sec,}$$

and

$$f_n = 1{,}540 \text{ or } 3{,}315 \text{ cpm.}$$

(b) The amplitude at mass J_2 is

$$\beta_2 = 1 - \frac{J_1\omega^2}{k_{t_1}}. \tag{6.16}$$

For the single node frequency:

$$\beta_2 = 1 - \frac{1{,}880 \times 0.0259(10^6)}{17.2(10^6)} = -1.83 \text{ rad.}$$

For the double node frequency:

$$\beta_2 = 1 - \frac{1{,}880 \times 0.12(10^6)}{17.2(10^6)} = -12.12 \text{ rad.}$$

The amplitude at J_3 is

$$\beta_3 = 1 - \frac{J_1\omega^2}{k_{t_1}} - \frac{(J_1 + J_2)\omega^2}{k_{t_2}} + \frac{J_1J_2\omega^4}{k_{t_1}k_{t_2}}. \tag{6.17}$$

For the single node frequency:

$$\beta_3 = 1 - \frac{1{,}880 \times 0.0259(10^6)}{17.2(10^6)} - \frac{(1{,}880 + 404)0.0259(10^6)}{16.83(10^6)} + \frac{1{,}880 \times 404[0.0259(10^6)]^2}{17.2 \times 16.83(10^{12})} = -3.6 \text{ rad.}$$

For the double node frequency:

$$\beta_3 = 1 - \frac{1{,}880 \times 0.12(10^6)}{17.2(10^6)} - \frac{(1{,}880 + 404)0.12(10^6)}{16.83(10^6)} + \frac{1{,}880 \times 404[0.12(10^6)]^2}{17.2 \times 16.83(10^{12})} = +9.48 \text{ rad.}$$

The alternative method of finding the amplitude of mass 3 is

$$\frac{\beta_2}{\beta_3} = 1 - \frac{J_3\omega^2}{k_{t_2}}. \tag{6.18}$$

For the single node frequency:

$$\frac{\beta_2}{\beta_3} = 1 - \frac{320 \times 0.0259(10^6)}{16.83(10^6)} = 0.508,$$

$$\beta_3 = -\frac{1.83}{0.508} = -3.6 \text{ rad.}$$

For the double node frequency:

$$\frac{\beta_2}{\beta_3} = 1 - \frac{320 \times 0.12(10^6)}{16.83(10^6)} = -1.28,$$

$$\beta_3 = -\frac{12.12}{-1.28} = +9.48 \text{ rad.}$$

(c) Node positions may then be found.

For the single node frequency:

$$L_4 = \frac{\beta_1 L_1}{\beta_1 - \beta_2} = \frac{1 \times 685}{1 + 1.83} = 242 \text{ in.} \tag{6.21}$$

For the double node frequency:

$$L_3 = \frac{\beta_3 L_2}{\beta_2 - \beta_3} = \frac{9.48 \times 700}{-12.12 - 9.48} = 307 \text{ in.} \tag{6.19}$$

$$L_4 = \frac{\beta_1 L_1}{\beta_1 - \beta_2} = \frac{1 \times 685}{1 + 12.12} = 52.2 \text{ in.} \tag{6.21}$$

6.4 *DYNAMIC VIBRATION ABSORBER**

Occasionally it may be found that a system having a single degree of freedom has imposed upon it a disturbing force whose frequency coincides with, or is close to, its natural frequency. In such cases the natural frequency of the system could be altered by changing its stiffness or its mass. If this is not convenient or desirable, a dynamic vibration absorber may be attached to the mass.

The absorber consists of a mass-and-spring combination that has the same natural frequency as that of the disturbing force or the natural frequency of the original single degree of freedom system. Its effect is to transform the original system into one having two degrees of freedom, so that the new system will have two natural frequencies, one of which is above, and the other below, the original value. The amplitude of the original system at the impressed frequency will be zero if the design is properly carried out.

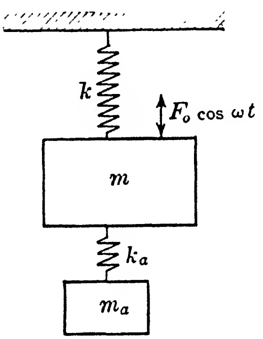


Fig. 6.4

Absorbers may be designed for either lateral or torsional systems. The theory will be developed here for the lateral system shown schematically in Fig. 6.4. The original single degree of freedom system consists of the spring having a scale k and mass m on which acts the impressed force $F_o \cos \omega t$. To this are added the absorber mass m_a and its spring, which has a scale k_a.

Following an analysis of the various forces acting on each of the masses, similar to that used in Sec. 6.2 for any impressed frequency ω, the following equations are obtained:

$$m \frac{d^2x}{dt^2} + x(k + k_a) - k_a x_a = F_o \cos \omega t, \tag{6.22}$$

$$m_a \frac{d^2x_a}{dt^2} + k_a(x_a - x) = 0. \tag{6.23}$$

Assuming that the motion of the masses is simple harmonic gives

$$x = x_o \cos \omega t, \qquad x_a = x_{o_a} \cos \omega t,$$

* Ormondroyd and Den Hartog, "The Theory of the Dynamic Vibration Absorber," *Trans. A.S.M.E.*, 1928, APM-50-7.

and

$$\frac{d^2x}{dt^2} = -x_o\omega^2 \cos \omega t, \qquad \frac{d^2x_a}{dt^2} = -x_{o_a}\omega^2 \cos \omega t.$$

Substituting these values in Eqs. (6.22) and (6.23) gives

$$[-m\omega^2 x_o + x_o(k + k_a) - k_a x_{o_a}] \cos \omega t = F_o \cos \omega t,$$

$$[-m_a\omega^2 x_{o_a} + k_a(x_{o_a} - x_o)] \cos \omega t = 0,$$

which may be written

$$x_o(-m\omega^2 + k + k_a) - k_a x_{o_a} = F_o, \tag{6.24}$$

$$-x_o k_a + x_{o_a}(k_a - m_a\omega^2) = 0. \tag{6.25}$$

As was stated previously, the natural frequency of the absorber is made equal to that of the main mass; hence, $\omega_n^2 = \frac{k}{m} = \frac{k_a}{m_a}$. If the ratio of the impressed frequency to the natural frequency of these parts of the entire system is designated as r, then $r^2 = \frac{\omega^2}{\omega_n^2} = \omega^2 \frac{m}{k} = \omega^2 \frac{m_a}{k_a}$. Also let $x_{st} = \frac{F_o}{k}$, where x_{st} is the deflection of mass m when the force F_o is applied to it as a static load. Then dividing Eq. (6.24) by k gives

$$x_o\left(\frac{-m\omega^2}{k} + 1 + \frac{k_a}{k}\right) - \left(\frac{k_a}{k}\right) x_{o_a} = \frac{F_o}{k},$$

$$x_o\left(-r^2 + 1 + \frac{k_a}{k}\right) - \left(\frac{k_a}{k}\right) x_{o_a} = x_{st}. \tag{6.26}$$

Equation (6.25) may be written as follows:

$$x_o = x_{o_a}\left(\frac{k_a}{k_a} - m_a \frac{\omega^2}{k_a}\right) = x_{o_a}(1 - r^2). \tag{6.27}$$

Combining Eqs. (6.26) and (6.27),

$$x_{o_a}\left[(1 - r^2)\left(1 - r^2 + \frac{k_a}{k}\right) - \frac{k_a}{k}\right] = x_{st},$$

$$\frac{x_{o_a}}{x_{st}} = \frac{1}{(1 - r^2)\left(1 - r^2 + \frac{k_a}{k}\right) - \frac{k_a}{k}}. \tag{6.28}$$

$$x_o\left[\left(1 - r^2 + \frac{k_a}{k}\right) - \frac{k_a}{k}\frac{1}{1 - r^2}\right] = x_{st},$$

$$\frac{x_o}{x_{st}} = \frac{1 - r^2}{(1 - r^2)\left(1 - r^2 + \frac{k_a}{k}\right) - \frac{k_a}{k}}. \tag{6.29}$$

If the impressed frequency is the same as the natural frequency of the absorber or of the main system, $r = 1$, then the amplitude of the main mass becomes zero by Eq. (6.29). From Eq. (6.28), it may be observed that then $x_{o_a} = -x_{st}\frac{k}{k_a} = \frac{-F_o}{k_a}$, that is, the force acting on the absorber spring equals F_o. Hence, when the impressed

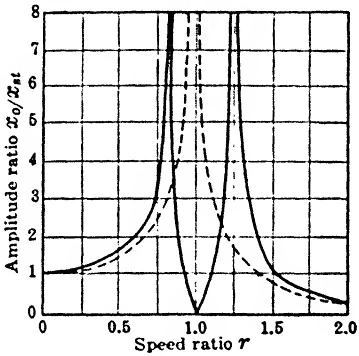


Fig. 6.5

frequency equals that of the main mass, the main mass does not move and the absorber spring must be designed to carry the impressed force.

Since the natural frequency of the main mass and that of the absorber are identical, $\frac{k}{m} = \frac{k_a}{m_a}$. This may be written as $\frac{m_a}{m} = \frac{k_a}{k}$ and this ratio designated as μ. It should be realized that μ is the ratio of the absorber mass or weight to that of the main mass or weight.

It may be observed that the denominators on the right side of Eqs. (6.28) and (6.29) are identical. When they equal zero, the amplitudes x_o and x_{o_a} are infinite, that is, a resonant condition in the entire system is obtained.

Inserting the μ term in the expression for the denominator and equating it to zero gives

$$(1 - r^2)(1 - r^2 + \mu) - \mu = 0. \tag{6.30}$$

This equation may be expanded to

$$r^4 - (2 + \mu)r^2 + 1 = 0,$$

which is a quadratic equation in r^2. Hence,

$$r^2 = \frac{\omega^2}{\omega_n^2} = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}}. \tag{6.31}$$

The impressed frequencies ω found from Eq. (6.31) correspond to the two resonant frequencies of the entire system, whereas ω_n

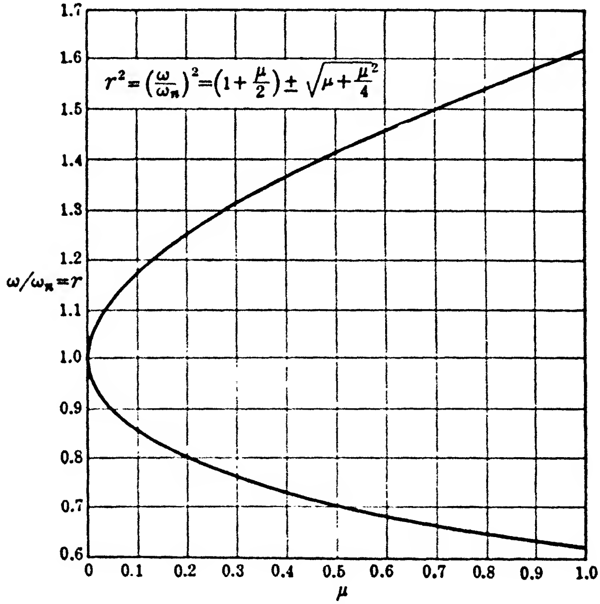


FIG. 6.6

corresponds to the resonant condition of the main system without the absorber. This relation is illustrated in Fig. 6.5 where the ratio of x_o/x_{st} is plotted against the ratio r, or ω/ω_n. The dashed curve represents the relation between these two variables when the absorber is not used and is the same as Fig. 3.2, whereas the full-line curve shows the relation with the absorber in place. It will be noticed that in the latter case there are two peaks of resonant conditions rather than one and that x_o/x_{st} or the amplitude of the main mass is zero at $r = 1$, which is the frequency of the impressed force.

If the size of the absorber mass is quite small, it will have relatively small effect and the two new resonant frequencies will be quite close to the original one. Figure 6.6 shows a plot of the speed ratio r against the ratio of the mass sizes m_a/m, or μ based upon Eq. (6.31). For the absorber to be effective, the new frequencies should be at least 20 per cent away from the impressed frequency. Hence, the speed ratio r should be at least 0.8 or 1.2. It should be observed that the curve is not symmetrical about a horizontal line or $r = 1$. When $\mu = 0.2$, the ratio r is close to 0.8 and 1.25. If the two frequencies are to be at least 20 per cent away from the impressed frequency, the lower value will control, and the mass or weight of the absorber must be 20 per cent of that of the main mass or weight.

EXAMPLE

A foundry table weighs with the mold 800 lb. The sand is packed in vertically with a vibrator having a frequency of 400 cpm.

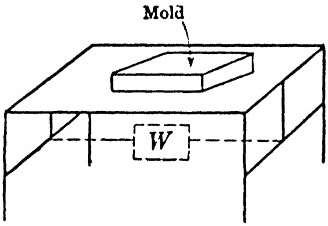


FIG. 6.7

Under these conditions resonance occurs, and the table "walks" around. It is proposed to eliminate this condition by placing a simple beam 5 ft long made of 1 in. diameter steel shafting between the crosspieces (shown dashed in Fig. 6.7) and attaching a weight at its mid-point. What weight is required? What will be the natural frequencies of the table after the weight is installed? Assume that the crosspieces do not deflect.

Spring scale of the beam is

$$k = \frac{48EI}{L^3} = \frac{48 \times 30(10^6)}{60^3} \frac{\pi 1^4}{64} = 328 \text{ lb per in.}$$

The natural frequency of the absorber must be the same as that of the vibrator, namely, 400 cpm, which corresponds to $\omega = 41.85$ rad per sec. Since $\omega_a{}^2 = \dfrac{k_a}{m_a}$, the absorber mass is $m_a = \dfrac{k_a}{\omega_a{}^2} =$

$\frac{328}{41.85^2} = 0.1875$ lb-sec² per in., and the absorber weight is $W_a = m_a g = 0.1875 \times 386 = 72.5$ lb.

The mass ratio $\mu = \frac{m_a}{m} = \frac{72.5}{800} = 0.0906$.

From Eq. (6.31), the frequency ratio squared is

$$r^2 = \frac{\omega^2}{\omega_n{}^2} = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}}$$

$$= \left(1 + \frac{0.0906}{2}\right) \pm \sqrt{0.0906 + \frac{(0.0906)^2}{4}} = 0.7413 \text{ or } 1.3493.$$

The value of r, or $\frac{\omega}{\omega_n}$, $= 0.861$ or 1.162, and the new natural frequencies of the table are $f_n = 400(0.861$ or $1.162) = 344$ or 465 cpm.

The same principles may be applied to torsional systems having a single degree of freedom, and the curves of Figs. 6.5 and 6.6 apply.

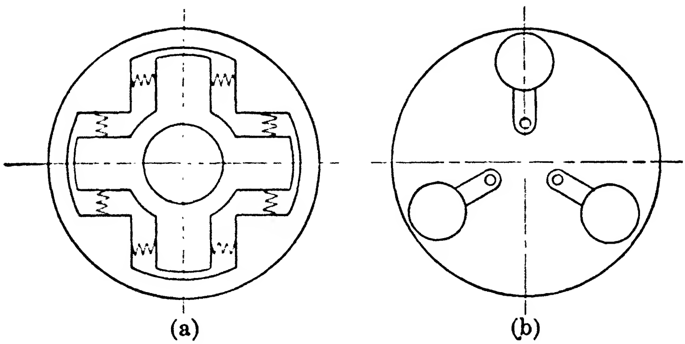


FIG. 6.8

The necessary changes are that the ordinates of Fig. 6.5 are θ_o/θ_{st} rather than x_o/x_{st}, and μ of Fig. 6.6 is J_a/J rather than m_a/m.

The addition of a length of shafting with a mass may be too bulky to be practical; so, many devices have been used to replace it. Only two will be described here, but others may be found in the literature* of the subject.

* W. K. Wilson, *Practical Solution of Torsional Vibration Problems*, Wiley, New York, 1940, Chaps. 10 and 11; Den Hartog and Ormondroyd,

The torsional absorber shown in Fig. 6.8a consists of a ring attached to one of the disks of the original system. A mass is connected to this ring by means of springs. If no vibration is present, the entire unit rotates at a constant speed. When oscillations occur in the system, the mass tends to continue to rotate at constant speed, so that the springs are deflected and it acts as an absorber. Thus the lateral springs replace the length of shafting.

A pendulum-type absorber is shown schematically in Fig. 6.8b, in which pendulums are attached to one of the disks. In its action, this absorber is similar to the one just described, except that the restoring force acting on the masses is due to centrifugal force rather than springs. Thus it is possible to design the system to act at any rotational speed rather than at just one particular value.

PROBLEMS

6.1. Repeat the example in Sec. 6.2, assuming that the bottom spring is removed, so that $k_3 = 0$, but the rest of the system is unchanged.

Ans. $f_n = 420$ cpm; $\dfrac{x_{o_1}}{x_{o_2}} = +0.5$; $f_n = 840$; $\dfrac{x_{o_1}}{x_{o_2}} = -1.0$.

6.2. Derive expressions for the natural circular frequencies of a symmetrical three-mass system in which $k_1 = k_3 = k$; $k_2 = k'$; and $m_1 = m_2 = m$. *Ans.* $\omega = \sqrt{\dfrac{k}{m}}$ or $\sqrt{\dfrac{k + 2k'}{m}}$.

6.3. Determine the two natural frequencies and the relative amplitudes of a symmetrical system similar to that shown in Fig. 6.1. The sizes of the various parts are as follows: $W_1 = W_2 = 19.3$ lb; $k_1 = k_3 = 100$ lb per in.; $k_2 = 200$ lb per in.

Ans. 428 and 955 cpm.

6.4. Two steel cantilever springs are 1 in. wide, $\frac{1}{16}$ in. thick, and 3 in. long. They are connected at their free ends by a weak coil spring having a scale of 1 lb per in. At the free ends of the cantilevers, weights of 2 lb each are attached. Neglecting the masses of the springs, determine the natural frequencies of the system. *Ans.* 1,095 and 1,110 cpm.

"Torsional Vibration Dampers," *Trans. A.S.M.E.*, 1930, APM-52-13; W. A. Tuplin, *Torsional Vibration*, Wiley, New York, 1934, Chap. 13.

6.5. A three-mass system similar to that shown in Fig. 6.2 has inertias in in.-lb-sec² of $J_1 = 2{,}073$; $J_2 = 1{,}036$; $J_3 = 1{,}554$. The equivalent lengths in inches referred to a steel shaft of 3 in. diameter are $L_{e_1} = 20$; $L_{e_2} = 30$. Determine the natural frequencies, and locate the positions of the nodes for the two modes of vibration.

Ans. 443 cpm, $L_3 = 28.54$ in.; 947 cpm, $L_3 = 6.53$ in., $L_4 = 4.64$ in.

6.6. The inertia of a certain airplane propeller is $J_1 = 100$, the equivalent inertia of the moving parts of a radial engine is $J_2 = 15$, and attached to the same shaft is a supercharger having a $J_3 = 3$. All inertia units are in in.-lb-sec². The corresponding spring scales are $k_{t_1} = 5.57(10^6)$ in.-lb per rad, and $k_{t_2} = 0.375(10^6)$ in.-lb per rad. Determine the two natural frequencies of the system.

Ans. 3,300 and 6,480 cpm.

6.7. In the example in Sec. 6.3, it is desired to place the node at the gears for the lower frequency by changing the equivalent length of shafting between the gears and the generator. There is to be no change in the mass moments of inertia. Determine (a) the value of k_{t_2} and L_{e_2} required; (b) the resulting frequencies; and (c) the corresponding node positions and amplitudes of the masses, assuming $\beta_1 = +1$.

Ans. (a) $k_{t_2} = 2.925\ (10^6)$, $L_{e_2} = 4{,}025$ in.; (b) 915 and 2,320 cpm; (c) $\beta_2 = -0.003$ and -5.47, $\beta_3 = -5.91$, and $+0.95$; $L_3 = 4{,}025$ in. for the first mode; $L_3 = 597$ in., $L_4 = 106$ in. for the second mode.

6.8. If, in the example in Sec. 6.4, it is desired to have the nearest resultant frequency of the table 20 per cent from the impressed frequency, what size weight should be placed on what diameter shaft? *Ans.* 160-lb weight on shaft of 1.22 in. diameter.

6.9. A jig used to size coal contains a screen that reciprocates with a frequency of 600 cpm. The body of the jig has a weight of 500 lb, and its natural frequency is 600 cpm. A dynamic vibration absorber weighing 125 lb is mounted on the jig body. What should be the scale of the spring connecting the two members, and what are the resulting frequencies?

Ans. 1,280 lb per in., $f_n = 470$ and 765 cpm.

6.10. In a chemical plant a double-acting single-cylinder reciprocating pump discharges into a pipe line. When the pump speed is 178 rpm, the fluid pulsations coincide with the natural frequency of a section of the line. A cantilever is clamped to the line and a weight of 50 lb attached to act as a dynamic vibration absorber. With this arrangement, the resulting natural frequencies of the system were found to be 310 and 409 cpm. Because of plant conditions it is desirable to have these frequencies below 250 and above 500 cpm. What size of weight should be used, and for what spring scale should the cantilever be designed?

Ans. 340 lb, $k = 1{,}222$ lb per in.

6.11. A marine engine having an inertia of 15 in.-lb-sec² is connected to a three-bladed propeller having an inertia of 10 in.-lb-sec² by means of a shaft having a scale of 200,000 in.-lb per rad. Since the natural frequency is close to the disturbing frequency, an absorber of the type shown in Fig. 6.8a is attached to the engine mass. The inertia of the swinging weight is 3 in.-lb-sec². Each of the eight springs has a scale of 500 lb per in., and they act at an effective radius of 5 in. Calculate the equivalent spring scale of the absorber and determine the natural frequencies of the system.

Ans. 100,000 in.-lb per rad; 1,460 and 2,205 cpm.

6.12. A torsional pendulum has an inertia J of 10 in.-lb-sec² and a spring scale k_t of 2,000 in.-lb per rad. It is subjected to torque impulses of 150 cpm. If it is desired to keep the natural frequency at least 25 per cent from the impressed frequency, determine the inertia and spring scale required for the absorber.

Ans. $J = 3.41$ in.-lb-sec², $k_{t_a} = 682$ in.-lb per rad.

Chapter 7

MULTIMASS TORSIONAL SYSTEMS

7.1 *INTRODUCTION*

Most of the systems subjected to periodic torques are of the multimass type, especially if the inertia of the shafting is considered. In many cases, it is sufficiently accurate to group these masses to estimate natural frequencies; but in other cases, it is necessary to make a more accurate determination. The Holzer method, discussed in the next section, describes the procedure for the latter case.

There are two general types of multimass systems: the *in-line* and the *branched*, both of which can be handled by the Holzer method, as described in this chapter.

7.2 *THE HOLZER METHOD*

The Holzer* method of determining the natural frequencies of multimass systems is based on the principle stated in Sec. 6.3; namely, that the sum of the inertia torques developed in a system because of the vibration must equal zero if the vibration is free. Equations (6.14) and (6.15), which apply to a three-disk system, are rather complex. When more than three disks make up the system, it is more convenient to determine the natural frequencies with the aid of a table.

The principal equations of Sec. 6.3 may be summarized in general terms for disk n in a system as follows:

Torque T_n developed by disk n

$$T_n = J_n \beta_n \omega^2. \tag{7.1}$$

* H. Holzer, *Die Berechnung der Drehschwingungen*, Springer, 1921.

Total torque ΣT_n acting on shaft n

$$\Sigma T_n = \omega^2 \sum_1^n J\beta. \tag{7.2}$$

Angle of twist $\Delta\beta_{n,n-1}$ in shaft n

$$\Delta\beta_{n,n-1} = \frac{\sum_1^n T_n}{k_{t_n}}. \tag{7.3}$$

Amplitude β_n of disk n

$$\beta_n = \beta_{n-1} - \frac{\sum_1^n J\omega^2\beta}{k_{n-1}}. \tag{7.4}$$

A natural frequency f_n or ω_n of the system and an amplitude of 1 rad for the first disk are assumed. The amplitudes and inertia torques for each disk in turn are calculated with the aid of Eqs. (7.1) to (7.4), as shown in the following example. If the algebraic sum of the inertia torques equals zero, the correct frequency was assumed; if not, additional assumptions must be made until this condition is satisfied. The procedure for estimating proper frequency assumptions will be considered in the next section.

EXAMPLE

A four-cylinder engine with a flywheel runs at 1,200 rpm and is connected to a generator by a flexible coupling. The mass moments of inertia in in.-lb-sec² are as follows: cylinders, $J_1 = J_2 = J_3 = J_4 = 0.55$; flywheel $J_5 = 25.6$; coupling hub $J_6 = 0.5$; generator $J_7 = 8.75$. The equivalent lengths referred to a 3 in. diameter steel shaft in inches are the following: cranks $L_{e_1} = 5$; $L_{e_2} = 5.5$; $L_{e_3} = L_{e_4} = 5$; flexible coupling $L_{e_5} = 45.5$; hub to generator $L_{e_6} = 41$. Determine the lowest natural frequency with the aid of a Holzer table.

Before setting up the table, it is desirable to convert equivalent lengths to spring scales. Since $k_t = \dfrac{G\pi d_e^4}{32L_e}$ and $d_e = 3$ in., $k_t = \dfrac{12(10^6)\pi 3^4}{32L_e} = \dfrac{95.6(10^6)}{L_e}$. The values of the spring scales then are $k_{t_1} = k_{t_3} = k_{t_4} = 19.12(10^6)$; $k_{t_2} = 17.37(10^6)$; $k_{t_5} = 2.1(10^6)$; and $k_{t_6} = 2.33(10^6)$ in.-lb per rad.

Table 7.1 may now be set up by listing the numbers of the items in the first column (1, 2, 3, etc.) and in column 2 the values of inertia as given. In column 7, the values of spring scales just calculated are listed. A value of $\omega_n^2 = 0.165(10^6)$ will be assumed as the natural frequency; hence, the values of J in column 2 are multiplied by $0.165(10^6)$ to give the values of column 3. From here on the table must be worked in successive lines.

The amplitude of the first disk (β_1) is always assumed to be 1 rad (any assumption could be made, but 1 is usual), and this is put in the first line of column 4. Column 5 is obtained by multiplying the values of columns 3 and 4; for the first line this is 0.0908,

TABLE 7.1

(1) Item	(2) J	(3) $J\omega^2(10^6)$	(4) β	(5) $J\omega^2\beta(10^6)$	(6) $\Sigma J\omega^2\beta(10^6)$	(7) $k_t(10^6)$	(8) $\dfrac{\Sigma J\omega^2\beta}{k_t}$
1	0.55	0.0908	1.0000	0.0908	0.0908	19.12	0.0047
2	0.55	0.0908	0.9953	0.0905	0.1813	17.37	0.0104
3	0.55	0.0908	0.9849	0.0894	0.2707	19.12	0.0142
4	0.55	0.0908	0.9707	0.0881	0.3588	19.12	0.0188
5	25.60	4.2240	0.9519	4.0208	4.3796	2.10	2.0865
6	0.50	0.0825	−1.1346	−0.0937	4.2859	2.33	1.8394
7	8.75	1.4438	−2.9740	−4.2939	−0.0080		

and the same value is placed on the first line of column 6. It represents the total inertia torque acting on section 1 of the shaft. Dividing this by the k_t of the shaft (column 7) gives the angle of twist $\Delta\beta_{12}$ in this section by Eq. (7.3), or 0.0047 rad. From Eq. (7.4), $\beta_2 = \beta_1 - \Delta\beta_{12} = 1.000 - 0.0047 = 0.9953$ rad, which is placed in the second line of column 4. The torque developed by the second disk, as given by Eq. (7.1), is $T_2 = J_2\beta_2\omega^2$, or column 3 times column 4, that is, $0.0908 \times 0.9953 = 0.0905$ rad. The total torque acting on shaft 2, by Eq. (7.2), is $T_2 = \omega^2\sum_1^2 J\beta$, that is, line 1 of column 6 plus line 2 of column 5, or $0.0908 + 0.0905 = 0.1813$ in.-lb. The remainder of the table is carried out in a similar manner with due regard for algebraic signs. In this example, the torque remainder as given by line 7 of column 6 is −0.0080 in.-lb. If the correct frequency had been chosen, this remainder would be zero.

The deflection curve may be plotted along the shaft by using the values of β in column 4 for the various disks, as shown in Fig. 7.1. It will be noted that there is only one node (located between disks 5 and 6). Hence, the value of $\omega_n^2 = 0.165(10^6)$, and $f_n = 3{,}880$ cpm is very close to the first natural frequency. For torsional vibrations, the mode of the vibration is the same as the number of nodes, which means that the first critical has one node, the second two, and so on.

7.3 NOTES ON CALCULATION PROCEDURE

One of the difficulties of the Holzer method is in estimating the frequency to assume. Generally, this is based upon judgment and experience, but a fair trial value can be obtained by grouping

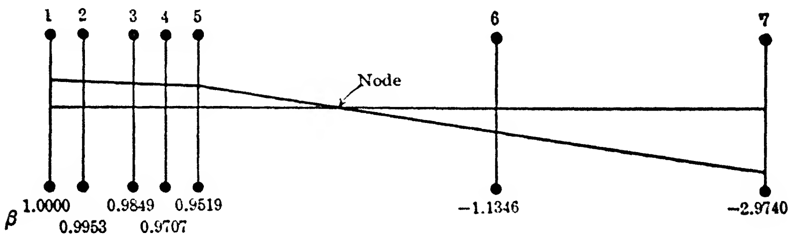


FIG. 7.1

together the masses that have a short equivalent shaft length or large spring scale between them. Thus, the multimass system may be reduced to one having a single or double degree of freedom. The approximate frequency to be assumed may then be found from Eqs. (2.18) or (6.15).

To illustrate, in the example in the previous section the coupling mass may be neglected, since it is small, and the system reduced to a two-mass, single degree of freedom system having $J_1' = J_1 + J_2 + J_3 + J_4 + J_5 = 27.8$; $J_2' = J_7 = 8.75$; and $L_e = L_{e_6} + L_{e_7} = 86.5$ in., or $k_t = 1.105(10^6)$. By Eq. (2.18),

$$f_n = \frac{60}{2\pi}\sqrt{\frac{k_t(J_1 + J_2)}{J_1J_2}} = \frac{60}{2\pi}\sqrt{\frac{1.105(10^6)(27.8 + 8.75)}{27.8 \times 8.75}} = 3{,}890 \text{ cpm}.$$

Then $\omega^2 = 166{,}000$. This approximation is very close to the correct value as shown by the previous section.

If a number of trial frequencies are assumed, it will be found that the torque remainder (corresponding to -0.0080 in the previous example) may be plotted against the assumed frequency f or ω. It will also be found that the curve shape will always have the general characteristics shown in Fig. 7.2, that is, it is always positive below the first frequency (labeled 1 in the diagram), and then alternates between minus and plus for the higher natural frequencies (labeled 2, 3, and so on, in the diagram), although the exact curve shape and the spacing between the criticals or points of zero torque remainder vary with each system. This information is useful in making closer assumptions. For the example, the remainder is slightly negative, and since the first natural frequency is desired, a closer assumption would be one slightly lower than $\omega^2 = 165{,}000$, and perhaps $\omega^2 = 164{,}950$ should be tried. Plotting the torque-

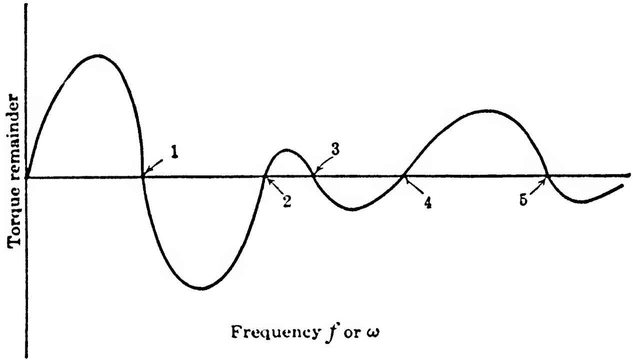


Fig. 7.2

remainder curve may also serve to locate errors for a given trial frequency, since the curve should be smooth.

An impressed torque, which tends to build up large vibration amplitudes, may be due to the firing stroke of an internal-combustion engine, the torque variation as the propeller blades rotate in fluid of varying density, and many similar factors. To avoid resonance conditions, it is desirable that these impulses should not occur at the natural frequency or some whole multiple of it. The *order* of a vibration is the ratio of the natural frequency to that of the operating speed in revolutions per unit time. A *major order* number is one that coincides with the number of torque impulses per revolution or a multiple of it, and all other order numbers are *minors*. Generally whole-order numbers (1, 2, 3, and so on) are majors and must be avoided; but in the case of four-cycle internal-combustion engines, which fire every other revolution, the half-order numbers ($\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, and so on) are also dangerous. When they occur, the natural frequency should be altered by changing the inertia of some of the masses, the stiffness of some of the sections of the shaft, or both.

Nodal drives* are frequently planned in systems having gears. In a nodal drive the properties of the system are so selected and arranged that the nodes are placed at, or close to, the gears for the natural frequency nearest to the running frequency or some harmonic of it. This placement will eliminate the possibility of the gear teeth leaving contact, or at least it will reduce the sudden loads that act on the teeth and tend to increase wear and shorten their life.

7.4 *BRANCHED SYSTEMS*

The procedure and example considered in Sec. 7.2 were based on "in-line" systems, where all the masses were attached to one

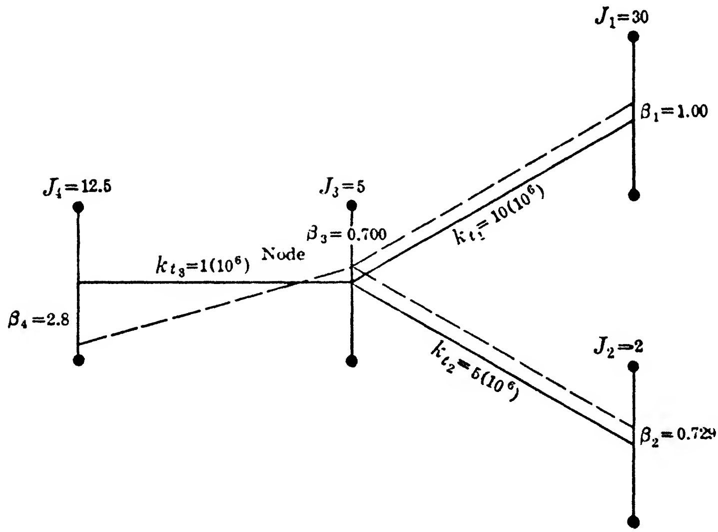


Fig. 7.3

equivalent or actual shaft. A second type of system is the "branched," where two or more shafts operating in "parallel" are connected to a third shaft by means of gears, as shown in Fig. 7.3. This application occurs in steam-turbine-driven marine units having more than one driver, for example, high- and low-pressure steam

* J. H. Smith, "Nodal Arrangements of Geared Drives," *Engineering*, vol. 113, pp. 438, 467, 1922.

turbines driving a common propeller, the rear wheels of automobiles driven by a common drive shaft, and so on.

Problems of this type may be solved with the aid of a Holzer table, although the solution becomes more complex, since the torques and amplitudes must be balanced among the shafts.

The procedure may best be illustrated by a simple example which does not represent any actual machine, but merely illustrates the principles involved.

Example

Given the four-mass branched system shown in Fig. 7.3 with the inertias and spring scales shown there, determine the lowest natural frequency.

It will be observed that the spring scales k_t of the two branches are $10(10^6)$ and $5(10^6)$, which are large compared to that of the main shaft, which is $1(10^6)$. For a preliminary estimate of the natural frequency, the actual system may be approximated by a two-mass system having $J_1' = J_4 = 12.5$, $J_2' = J_1 + J_2 + J_3 = 30 + 2 + 5 = 37$, and the spring scale $k_t = 1(10^6)$. From Eq. (2.18), the lowest natural frequency may be estimated as follows:

$$f_n = \frac{60}{2\pi}\sqrt{\frac{k_t(J_1' + J_2')}{J_1'J_2'}} = 9.55\sqrt{\frac{1(10^6)(12.5 + 37)}{12.5 \times 37}} = 3{,}120 \text{ cpm.}$$

Actually, the lowest natural frequency occurs at 3,020 cpm or at $\omega^2 = 0.1(10^6)$, and this value will be used in working the example.

The general procedure is to assume that the end mass of one of the branches, say J_1, has an amplitude of 1 rad and work along this branch to the common mass of the two branches (J_3 in this example). Then the amplitude of the end mass of the other branch J_2 is assumed to have an amplitude of 1 rad, and the Holzer table is worked along this branch to the common mass J_3. It will be found with these assumptions that the common mass J_3 is vibrating with two different amplitudes, which is obviously impossible. It is then necessary to repeat the calculation of one of the branches, say the lower, with an assumed amplitude of the end mass J_2 such that the common mass J_3 will have the same amplitude when calculated along either branch. This means that the amplitude of J_2 must be changed in the ratio of the two amplitudes for the common mass, that is, $\frac{\beta_{3_{upper}}}{\beta_{3_{lower}}}\beta_{2_{assumed}}$. The revised inertia torques of all the masses on the branches plus that of the common mass are added to find the torque acting on the main shaft (in this example, $T_1 + T_2 + T_3$). Using the final values of the torque and amplitude of the common mass, the main shaft is calculated in the usual manner to see if the total torque of the whole system equals zero. If it does not, a new assumption of frequency must be made and the entire process repeated until that condition is satisfied.

The procedure is illustrated in Table 7.2 for the numerical example.

TABLE 7.2

Branch or shaft	(1) Item	(2) J	(3) (10^6) $J\omega^2$	(4) β	(5) (10^6) $J\omega^2\beta$	(6) (10^6) $\Sigma J\omega^2\beta$	(7) (10^6) k_t	(8) $\frac{\Sigma J\omega^2\beta}{k_t}$
Upper	1	30	3.0	1.000	3.0	3.0	10	0.30
	3	5	0.5	0.700				
Lower	2	2	0.2	1.000	0.2	0.2	5	0.04
	3	5	0.5	0.960				

Mass 3 cannot have the two amplitudes of 0.70 and 0.96 rad at the same time, so it is necessary to recalculate the lower branch assuming a reduced amplitude β_2. The new value of β_2 must be reduced in the ratio of 0.70/0.96 from its former value of 1, which will be 0.729 rad.

Lower	2	2	0.2	0.729	0.1458	0.1458	5	0.029
	3	5	0.5	0.700	0.350			

The value of β_3 is now 0.70 rad when calculated along each branch, and so it is satisfactory. The torque acting on shaft 3 between J_3 and J_4 equals the sum of the inertia torques developed by masses 1, 2 (revised), and 3; or $T_3 = 3.000 + 0.1458 + 0.350 = 3.4958$ in.-lb. For this last step the amplitude β_3 is taken from the previous steps, and the total torque of column 6 is the value of 3.4958 just found.

Main	3			0.700		3.4958	1	3.4958
	4	12.5	1.25	−2.7958	−3.4948	+0.0010		

It will be observed from the last line of Table 7.2 that the total inertia torque developed in the system equals +0.0010 in.-lb; hence, the assumed frequency is nearly correct. The amplitudes β as found in column 4 may be plotted as shown by dashed lines in Fig. 7.3 to locate the position of the node. Since this is the lowest natural frequency, there will be only one node; in this case it is located on the main shaft near the gears (mass 3).

It would be instructive for the reader to work out the cases for the two higher natural frequencies and plot the deflection curves. The values of ω^2 to be used are $1.445(10^6)$ for the second natural frequency and $4.569(10^6)$ for the third. Since there are four masses, there are only three natural frequencies (always one less frequency than the number of masses).

PROBLEMS

7.1. Determine the lowest natural frequency of the equivalent system shown in Fig. 8.4b (page 136). *Ans.* 5,600 cpm.

7.2. Solve Prob. 6.5, by means of a Holzer table for the lower frequency and check the position of the node by means of it.

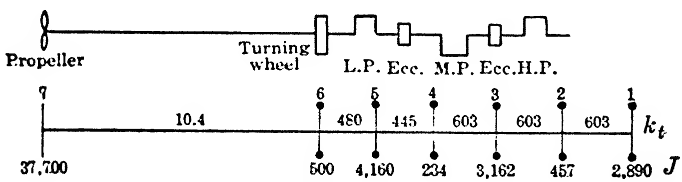


FIG. 7.4

7.3. A three-cylinder marine engine designed to run at 100 rpm has the following inertias expressed in in.-lb-sec²: cylinders, $J_1 = J_2 = J_3 = 2{,}300$; wormwheel, $J_4 = 600$; propeller (including allowance for entrained water), $J_5 = 33{,}200$. The equivalent lengths referred to an 8 in. diameter steel shaft in inches are $L_{e_1} = L_{e_2} = 14.5$; $L_{e_3} = 11$; and $L_{e_4} = 400$. Determine the two lowest natural frequencies of the system. *Ans.* 417 cpm and 3,442 cpm.

7.4. The data shown in Fig. 7.4 are taken from the November, 1943, *Journal of the American Society of Naval Engineers*, page 751, for a marine drive. The propeller inertia has been increased 25 per cent to allow for the entrained water. The units are in.-lb-sec² for the inertias J; and the values of k_t have been divided by (10^6) with the units of in.-lb per rad. Check the value of the lowest natural frequency, which was found to be 324 cpm.

7.5. If the main shaft of the example in Sec. 7.4 has its spring scale changed from $1(10^6)$ to $0.5(10^6)$, but the rest of the system is unchanged, determine the three natural frequencies.

Ans. $\omega_n^2 = 51{,}800$; $1.415(10^6)$; and $4.52(10^6)$.

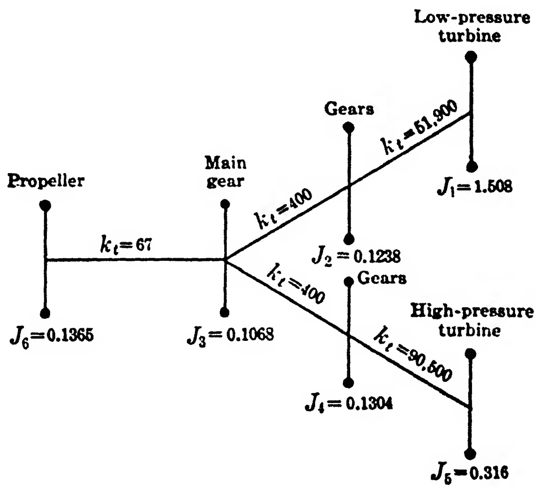


FIG. 7.5

7.6. The equivalent branched system referred to the propeller speed of a marine geared turbine drive is shown diagrammatically in Fig. 7.5. All values of J and k_t have been divided by (10^6). Determine the lowest natural frequency of this system.

Ans. $\omega_n^2 = 432$, or $f_n = 199$ cpm.

7.7. As suggested in Sec. 7.4, determine the two higher natural frequencies of the branched system shown in Fig. 7.3.

Ans. $\omega_2^2 = 1.445(10^6)$; $\omega_3^2 = 4.569(10^6)$.

Chapter 8

EQUIVALENT TORSIONAL SYSTEMS

8.1 *INTRODUCTION*

Before it is possible to analyze a system that is subject to torsional vibration, it is generally necessary to replace the actual parts by an equivalent system consisting of point mass moments of inertia connected by massless springs. This process is usually long and tedious, and for complicated systems it may require the exercise of a considerable amount of judgment based upon experience.

The procedure for handling ordinary cases is given here; for more complicated or advanced problems the reader is referred to the following: W. A. Tuplin, *Torsional Vibration*, Wiley, New York, 1934, Chap. 2; W. K. Wilson, *Practical Solution of Torsional Vibration Problems*, Wiley, New York, 1940, Chap. 3.

8.2 *EQUIVALENT MASS MOMENTS OF INERTIA*

A. *General*

The mass moment of inertia is defined as the product of the mass and the square of the radius of gyration about the axis of rotation; that is, $J = m\bar{r}^2 = \dfrac{W}{g}\bar{r}^2$. Complex shapes may generally be broken down into simple elements, and the total value of the inertia is then the sum of the component parts.

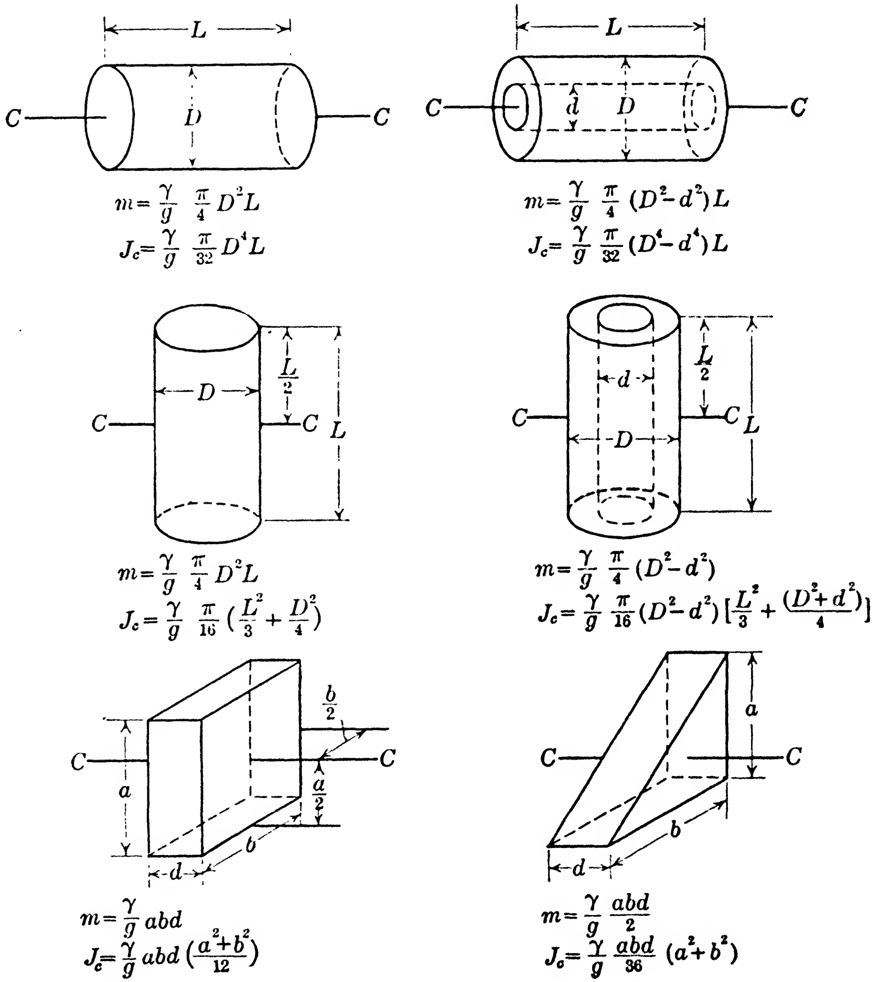
The transfer equation may be used to determine J for elements whose center of gravity does not lie on the axis of rotation. This relationship expressed mathematically is

$$J_o = J_c + mx^2, \tag{8.1}$$

where J_o = mass moment of inertia about the axis of rotation;
J_c = mass moment of inertia about the center of gravity;

m = mass of body or element;
x = distance between the axis of rotation and the center of gravity.

Expressions for the mass m and the mass moment of inertia about the center of gravity J_c for a few common-shaped elements



$$m = \frac{\gamma}{g}\,\frac{\pi}{4}\,D^2L \qquad J_c = \frac{\gamma}{g}\,\frac{\pi}{32}\,D^4L$$

$$m = \frac{\gamma}{g}\,\frac{\pi}{4}\,(D^2-d^2)L \qquad J_c = \frac{\gamma}{g}\,\frac{\pi}{32}\,(D^4-d^4)L$$

$$m = \frac{\gamma}{g}\,\frac{\pi}{4}\,D^2L \qquad J_c = \frac{\gamma}{g}\,\frac{\pi}{16}\left(\frac{L^2}{3}+\frac{D^2}{4}\right)$$

$$m = \frac{\gamma}{g}\,\frac{\pi}{4}\,(D^2-d^2) \qquad J_c = \frac{\gamma}{g}\,\frac{\pi}{16}\,(D^2-d^2)\left[\frac{L^2}{3}+\frac{(D^2+d^2)}{4}\right]$$

$$m = \frac{\gamma}{g}\,abd \qquad J_c = \frac{\gamma}{g}\,abd\left(\frac{a^2+b^2}{12}\right)$$

$$m = \frac{\gamma}{g}\,\frac{abd}{2} \qquad J_c = \frac{\gamma}{g}\,\frac{abd}{36}\,(a^2+b^2)$$

FIG. 8.1

are given in Fig. 8.1. Derivations of these expressions may be found in books on mechanics and are based upon the equation $J = \int \bar{r}^2\,dm$.

The effective mass moment of inertia of shafting to be added to the inertia of the concentrated masses was considered in Sec. 2.5c. The inertia of complicated members may be checked experimentally by oscillating them as pendulums, as discussed in Secs. 2.7b and 2.7d.

B. *Connecting Rods*

Since one end of the connecting rod rotates with the crankpin while the other end reciprocates with the wrist pin or crosshead, it is difficult to derive an exact equation for its equivalent mass in the system. It is customary to divide its total weight between the two ends in proportion to their distances from the center of gravity.

In Fig. 8.2, let G be the center of gravity of the rod of length L, and let L_C and L_R be the distances from point G to the crankpin C and wrist pin R, respectively. The portion of the rod weight W

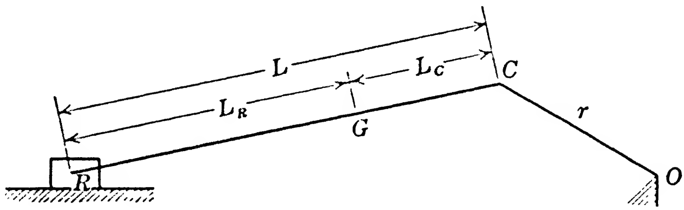


Fig. 8.2

to be placed at the crankpin C can be found by taking moments about point R; thus, $W_C = \dfrac{WL_R}{L}\cdot$ The portion at the wrist pin then is $W_R = \dfrac{WL_C}{L}\cdot$

For estimating or checking purposes it is useful to know that usually W_c is approximately $\frac{2}{3}W$ and W_R is about $\frac{1}{3}W$. If the throw of the crank is r, the mass moment of inertia of the portion of the connecting rod considered as acting at the crankpin W_C is

$$J = W_C \frac{r^2}{g} = \frac{W}{g}\frac{L_R}{L} r^2.$$

C. *Reciprocating Parts*

The inertia of the reciprocating parts of the engine must have an effect on the mass moment of inertia of the crank. When the piston is at the end of its stroke, this effect is negligible. When the crank is perpendicular to the connecting rod, the full inertia effect is transmitted. The general practice is to assume that one half of the inertia acts at the crankpin during the complete cycle. The equivalent mass moment of inertia of the reciprocating parts acting at the crankpin may then be taken as

$$J = \frac{r^2}{2g}(W_R + W_p) = \frac{r^2}{2g}\left(\frac{WL_C}{L} + W_p\right),$$

where W_p is the weight of the piston, piston rod, wrist pin, or crosshead, and the other symbols have the meanings given above.

D. *Crankshafts*

Since crankshafts are made in a great variety of shapes and proportions, it is difficult to give any general equation for calculating their mass moment of inertia. Generally, it is necessary to break them up into a number of elements and find the summation of the individual inertias. Thus, the journals are cylinders rotating about their center of gravity; the crankpins are cylinders rotating about an axis that is at a distance equal to the crank throw from their center of gravity; the side webs and balance weights are blocks rotating away from their center of gravity. The procedure for finding the summation of the individual inertias is illustrated in the example of Sec. 8.5.

8.3 *EQUIVALENT ELASTICITY*

A. *Equivalent Shaft Lengths*

As outlined in Sec. 2.5b, the concept of equivalent shaft lengths is to find the length of a piece of solid shaft of fixed diameter which will have the same flexibility as a stepped shaft or other discontinuous member. To satisfy this condition, the spring scale k_t of the actual and equivalent shafts must be the same. Thus,

$$k_t = \frac{G_a \pi d_a^4}{32 L_a} = \frac{G_e \pi d_e^4}{32 L_e},$$

where the subscripts *a* and *e* refer to the actual and equivalent shafts, respectively.

Solving for the equivalent length,

$$L_e = \frac{d_e^4}{d_a^4}\frac{G_e}{G_a} L_a. \tag{8.2}$$

If the actual shaft is hollow, the equivalent length of the solid reference shaft is

$$L_e = \frac{d_e^4}{D_a^4 - d_a^4} \frac{G_e}{G_a} L_a. \tag{8.3}$$

All shafting is usually made of steel, but occasionally a portion or fitting may be made of cast iron, brass, or some other metal, in which case the modulus-of-elasticity term must be used.

B. *Couplings*

It is difficult to predict the exact equivalent elasticity of couplings, since the sections have abrupt changes in size and the torque must be transmitted through the coupling bolts, whose deflection depends largely on the tolerances.

If the coupling halves are forged integral with the shaft, it is customary to treat each flange as a shaft enlargement having a diameter equal to the bolt-circle diameter and a length equal to the flange thickness.

If the coupling halves are keyed to the shaft, the torque is gradually transferred from the shaft to the hub and from the hub to the flanges. One assumption for this case is that the shaft is unaffected by the hub for a distance equal to one half the coupling length, and that the balance of the coupling length acts as a hollow stepped shaft with the inside diameter equal to that of the shaft and the outside diameter equal to the hub diameter or bolt-circle diameter. This assumption is illustrated in the example in Sec. 8.5.

The action of various types of flexible and hydraulic couplings is quite complex, and the reader should consult the references given in Sec. 8.1 concerning them.

C. *Crankshafts*

A crankshaft has many sudden changes in section, so that it is difficult to derive a satisfactory equation for the equivalent elasticity. In addition, the shape and proportions of various crankshafts vary considerably. A basis for evolving a satisfactory equation would be to consider the equivalent lengths of the various component parts, that is, journal, crankpin, and side webs. Their sum gives the equivalent length of the crank.

Two generally accepted equations that are used to find the equivalent lengths of commercial crankshafts are those due to Carter* and to Wilson†.

The Carter equation for solid journals and crankpins is

$$L_e = d_e^4 \left[\frac{e + 0.8a}{D_J^4} + \frac{0.75b}{D_c^4} + \frac{1.5r}{ac^3} \right]. \tag{8.4}$$

For hollow journals and crankpins the equation is

$$L_e = d_e^4 \left[\frac{e + 0.8a}{D_J^4 - d_J^4} + \frac{0.75b}{D_c^4 - d_c^4} + \frac{1.5r}{ac^3} \right]. \tag{8.5}$$

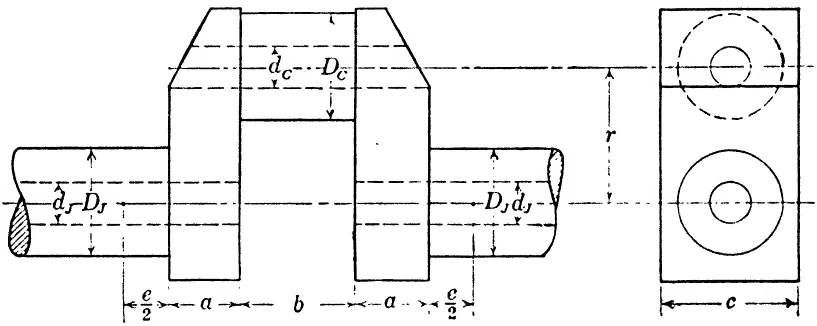


FIG. 8.3

The equation developed by Wilson for hollow journals and crankpins is

$$L_e = d_e^4 \left[\frac{e + 0.4D_J}{D_J^4 - d_J^4} + \frac{b + 0.4D_c}{D_c^4 - d_c^4} + \frac{r - 0.2(D_J + D_c)}{ac^3} \right]. \tag{8.6}$$

For solid crankpins and journals this equation becomes

$$L_e = d_e^4 \left[\frac{e + 0.4D_J}{D_J^4} + \frac{b + 0.4D_c}{D_c^4} + \frac{r - 0.2(D_J + D_c)}{ac^3} \right]. \tag{8.7}$$

The symbols have the meanings shown in Fig. 8.3. The symbol e represents the journal length, and for end cranks this value may have to be assumed. The first term of the equation gives the elasticity of the journal; the second, the elasticity of the crankpin; and the third, the elasticity of the crank webs.

* B. C. Carter, "An Empirical Formula for Crankshaft Stiffness in Torsion," *Engineering*, July 13, 1928, p. 36.

† W. K. Wilson, *Practical Solution of Torsional Vibration Problems*, Wiley, New York, 1940, p. 192.

8.4 GEARED SYSTEMS

Many machine drives employ gears for speed reduction or increase. In those cases the actual system must be corrected for the differences in speed of the component parts; that is, the inertias and spring constants are referred to one speed of rotation n_e. The basis for these transformations is that the potential and kinetic energies of the equivalent system should be the same as those of the actual.

The expression for the kinetic energy of a body vibrating torsionally is $E_k = \frac{1}{2}J\omega^2$, where ω is the circular frequency of the vibration. If the kinetic energy of the actual and equivalent systems is the same, $\frac{1}{2}J_a\omega_a^2 = \frac{1}{2}J_e\omega_e^2$, where the subscripts a and e refer to the actual and equivalent systems, respectively. Hence, $\dfrac{J_e}{J_a} = \left(\dfrac{\omega_a}{\omega_e}\right)^2$. For geared systems, the speed or gear ratio of the shafts n_a/n_e is the same as ω_a/ω_e, and the mass moments of inertia are inversely proportional to the speed ratio squared.

The potential energy developed in a vibrating shaft at any instant equals one half the spring scale of the shaft times the angular displacement squared, that is, $\frac{1}{2}k_t\theta^2$. The angular displacement is proportional to the speed or gear ratio of the shafts n_a/n_e; hence,

$$E_p = \tfrac{1}{2}k_{t_a}\theta_a^2 = \tfrac{1}{2}k_{t_e}\theta_e^2,$$

and

$$\frac{k_{t_e}}{k_{t_a}} = \left(\frac{\theta_a}{\theta_e}\right)^2 = \left(\frac{n_a}{n_e}\right)^2,$$

and the spring scales are inversely proportional to the speed ratio squared. It was shown in the previous section that the equivalent length of the shaft is inversely proportional to the spring scale. It follows, therefore, that $\dfrac{L_e}{L_a} = \left(\dfrac{n_e}{n_a}\right)^2$.

The above transformations may be summarized by the following relationships:

$$\frac{J_e}{J_a} = \frac{k_{t_e}}{k_{t_a}} = \frac{L_a}{L_e} = \left(\frac{n_a}{n_e}\right)^2 = (\text{speed ratio})^2 \tag{8.8}$$

8.5 AN EXAMPLE OF EQUIVALENT TORSIONAL SYSTEMS

To illustrate the principles developed in this chapter, a problem of converting an actual machine to an equivalent system will be worked out in detail.

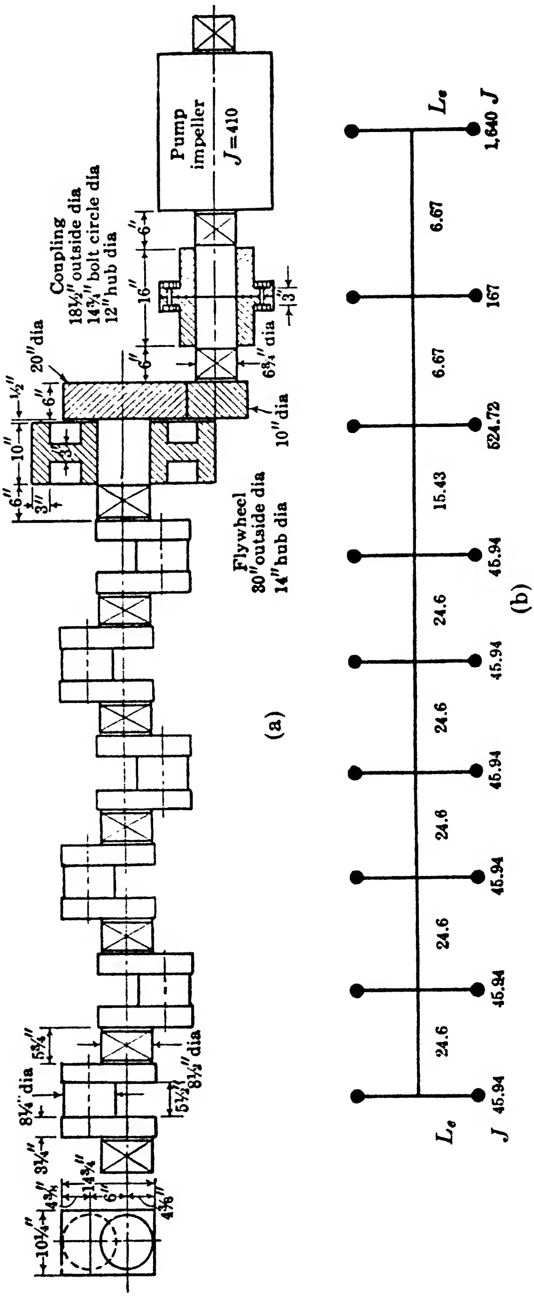

Pump impeller
J=410
Coupling
18½″ outside dia
14¾″ bolt circle dia
12″ hub dia
20″ dia
10″ dia
6¾″ dia
Flywheel
30″ outside dia
14″ hub dia
8½″ dia
8¼″ dia
(a)
(b)
L_e
J
1,640
167
524.72
45.94
6.67
15.43
24.6

FIG. 8.4

A six-cylinder 11 × 12 Diesel engine running at 750 rpm drives a centrifugal pump at 1,500 rpm through a pair of speed-up gears, as shown in Fig. 8.4a. Given the following data, sketch the equivalent system referred to the engine speed and an $8\frac{1}{2}$ in. diameter steel shaft.

The dimensions are given on the sketch (Fig. 8.4a). All six cranks are identical. Each connecting rod weighs 78.6 lb and is 25 in. long. The center of gravity of each rod is 7.84 in. from the crankpin. The weight of each piston assembly is 45.3 lb. The steel gears may be considered as solid cylinders with the outside diameter equal to the given pitch diameters. The flange coupling is made of cast iron and keyed to the two shafts. The pump impeller, including its journals, has an inertia J of 410 in.-lb-sec^2 about its axis.

A. *Mass Moments of Inertia*

Crank. As outlined in Sec. 8.2d, the inertia of each crank will be found by adding the inertias of the component elements as found with the aid of Fig. 8.1 and Eq. (8.1). The component parts of the crank are the crankpin, journal, and side webs. The value of J is first found about the center of gravity and then transferred to the center of rotation.

Crankpin.

$$J_c = \frac{\gamma}{g}\frac{\pi}{32}D^4L = \frac{0.283}{386}\frac{\pi}{32}(8\tfrac{1}{4})^4(5\tfrac{1}{2})$$
$$= 1.833 \text{ in.-lb-sec}^2.$$

$$m = \frac{\gamma}{g}\frac{\pi}{4}D^2L = \frac{0.283}{386}\frac{\pi}{4}(8\tfrac{1}{4})^2(5\tfrac{1}{2})$$
$$= 0.2157 \text{ lb-sec}^2 \text{ per in.}$$

The transfer distance x is the crank throw, which is 6 in.

$$J_o = J_c + mx^2 = 1.833 + 0.2157(6)^2$$
$$= 9.603 \text{ in.-lb-sec}^2.$$

Journal.

$$J_c = \frac{\gamma}{g}\frac{\pi}{32}D^4L = \frac{0.283}{386}\frac{\pi}{32}(8\tfrac{1}{2})^4(5\tfrac{3}{4})$$
$$= 2.16 \text{ in.-lb-sec}^2.$$

The journal rotates about its center of gravity, and therefore, no x correction need be made.

Side Webs. From Figs. 8.1 and 8.4, $a = 14\frac{3}{4}$, $b = 10\frac{1}{4}$, $d = 3\frac{1}{4}$. Then,

$$J_c = \frac{\gamma}{g}\,abd\left(\frac{a^2 + b^2}{12}\right) = \frac{0.283}{386}\,(14\tfrac{3}{4} \times 10\tfrac{1}{4} \times 3\tfrac{1}{4})\left(\frac{(14\frac{3}{4})^2 + (10\frac{1}{4})^2}{12}\right)$$

$$= 9.7 \text{ in.-lb-sec}^2.$$

$$m = \frac{\gamma}{g}\,abd = \frac{0.283}{386} \times 14\tfrac{3}{4} \times 10\tfrac{1}{4} \times 3\tfrac{1}{4}$$

$$= 0.36 \text{ lb-sec}^2 \text{ per in.}$$

The transfer distance x is the distance from the center of gravity of the web to the center of rotation, or 3 in. Then,

$$J_o = J_c + mx^2 = 9.7 + 0.36 \times 3^2 = 12.94 \text{ in.-lb-sec}^2.$$

Since there are two webs per crank, this value is doubled, or 25.88 in.-lb-sec^2.

Connecting Rods and Reciprocating Parts. The total weight of the connecting rod must be divided between the piston and crankpin, as outlined in Sec. 8.2b. The distance L_c is given as 7.84 in., and the rod weight W is 78.6 lb. The portion of the rod acting at the piston then is

$$W_R = \frac{WL_c}{L} = 78.6 \times \frac{7.84}{25} = 24.6 \text{ lb};$$

and that at the crankpin is

$$W_c = W - W_R = 78.6 - 24.6 = 54 \text{ lb}.$$

The mass moment of inertia due to W_c acting at the crankpin radius is

$$J = \frac{W_c r^2}{g} = \frac{54 \times 6^2}{386} = 5.04 \text{ in.-lb-sec}^2.$$

The total reciprocating weight is the weight of the piston assembly plus the portion of the connecting rod acting at the wrist pin; that is,

$$W_P + W_R = 45.3 + 24.6 = 69.9 \text{ lb}.$$

As developed in Sec. 8.2c, it may be assumed that one half of this is effective in producing inertia at the crankpin radius. Then,

$$J = \frac{(W_R + W'_P)r^2}{2g} = \frac{69.9 \times 6^2}{2 \times 386}$$

$$= 3.257 \text{ in. lb-sec}^2.$$

The total inertia that may be considered to act at the center of each crank is the sum of the inertias calculated. Thus,

$$J = 9.603 + 2.16 + 25.88 + 5.04 + 3.257$$

$$= 45.94 \text{ in.-lb-sec}^2.$$

Flywheel and Gears. The flywheel and the two gears are close together and may be considered to act as a single mass.

The inertia of the steel flywheel may be found by subtracting the inertia of the cutout portion from the inertia of a solid disk of 30 in. outside diameter and 10 in. width and then adding to this result the inertia of the shaft extension on each side. Thus,

$$J = \frac{0.283}{386}\frac{\pi}{32}[30^4 \times 10 - (24^4 - 14^4)7 + (8\tfrac{1}{2})^4(6 - \tfrac{1}{2} \times 5\tfrac{3}{4} + \tfrac{1}{2})]$$

$$= 436.5 \text{ in.-lb-sec}^2.$$

The gear may be considered as a solid cylinder having a diameter equal to its pitch diameter, so that

$$J = \frac{\gamma}{g}\frac{\pi}{32}D^4L = \frac{0.283}{386}\frac{\pi}{32}(20)^4 6$$

$$= 69.12 \text{ in.-lb-sec}^2.$$

The pinion may also be considered as a solid cylinder and should have one half the shaft inertia between it and the coupling added to its inertia. Since the speed of the pinion is twice that of the engine to which the equivalent system is referred, it must be multiplied by the speed ratio squared, as given by Eq. (8.8). Thus,

$$J = \frac{\gamma}{g}\frac{\pi}{32}D^4L\left(\frac{n_a}{n_e}\right)^2 = \frac{0.283}{386}\frac{\pi}{32}[10^4 \times 6 + (6\tfrac{3}{4})^4 \times 3]\left(\frac{1{,}500}{750}\right)^2$$

$$= 19.1 \text{ in.-lb-sec}^2.$$

The total inertia of this mass combination is the sum of the three parts.

$$J = 436.5 + 69.12 + 19.1 = 524.72 \text{ in.-lb-sec}^2.$$

Flange Coupling. The coupling, made of cast iron, may be broken into two hollow cylinders if the guard lip is neglected. The steel-shaft inertia must be added to this, including one half of the inertia between the coupling and pinion. Since the pump-impeller inertia includes the journals, this portion of the shaft will not be considered. Again the speed-correction factor of $(1{,}500/750)^2$ must be applied. Cast iron has a specific weight of 0.260 lb per cu in. Thus, we obtain

$$J = \frac{\gamma}{g}\frac{\pi}{32}(D^4 - d^4)L\left(\frac{n_a}{n_b}\right)^2$$

$$= \left\{\frac{0.260}{386}\frac{\pi}{32}[(12^4 - \{6\tfrac{3}{4}\}^4)(16 - 3) + (18\tfrac{1}{2}^4 - 6\tfrac{3}{4}^4)3] + \frac{0.283}{386}\frac{\pi}{32}[(6\tfrac{3}{4})^4(16 + 3)]\right\}\left(\frac{1{,}500}{750}\right)^2$$

$$= 167 \text{ in.-lb-sec}^2.$$

Pump Impeller. The mass moment of inertia of the pump impeller, including its journals, is given as 410 in.-lb-sec²; hence, it is necessary only to apply the speed correction of $(1{,}500/750)^2$ to this amount. Therefore, $J = 1{,}640$ in.-lb-sec².

The calculated inertias are placed on the equivalent system shown in Fig. 8.4b beneath the corresponding points on the actual system.

B. *Equivalent Lengths*

The equivalent lengths of the various parts of the shafting will be referred to the engine speed and a solid steel shaft of $8\frac{1}{2}$ in. diameter; that is, $d_e = 8\frac{1}{2}$.

Crank. From Figs. 8.3 and 8.4 we find the following values:

$$a = 3\tfrac{1}{4}; \qquad b = 5\tfrac{1}{2}; \qquad c = 10\tfrac{1}{4}; \qquad e = 5\tfrac{3}{4};$$

$$D_J = 8\tfrac{1}{2}; \qquad D_C = 8\tfrac{1}{4}.$$

Then by Carter's formula (Eq. 8.4),

$$L_e = d_e^4\left[\frac{e + 0.8a}{D_J^4} + \frac{0.75b}{D_C^4} + \frac{1.5r}{ac^3}\right],$$

and substituting the values given above,

$$L_e = (8\tfrac{1}{2})^4 \left[\frac{5\frac{3}{4} + 0.8 \times 3\frac{1}{4}}{(8\frac{1}{2})^4} + \frac{0.75 \times 5\frac{1}{2}}{(8\frac{1}{4})^4} + \frac{1.5 \times 6}{3\frac{1}{4}(10\frac{1}{4})^3} \right]$$

$$= 26.2 \text{ in.}$$

By Wilson's formula (Eq. 8.7),

$$L_e = d_e^4 \left[\frac{e + 0.4D_J}{D_J^4} + \frac{b + 0.4D_C}{D_C^4} + \frac{r - 0.2(D_J + D_C)}{ac^3} \right],$$

and again substituting the given values,

$$L_e = (8\tfrac{1}{2})^4 \left[\frac{5\frac{3}{4} + 0.4 \times 8\frac{1}{2}}{(8\frac{1}{2})^4} + \frac{5\frac{1}{2} + 0.4 \times 8\frac{1}{4}}{(8\frac{1}{4})^4} + \frac{6 - 0.2(8\frac{1}{2} + 8\frac{1}{4})}{3\frac{1}{4}(10\frac{1}{4})^3} \right]$$

$$= 23 \text{ in.}$$

The average of these two values, which is 24.6 in., will be used for the cranks.

Crank 6 to Flywheel. Between the center line of the last crank (No. 6) and the flywheel, the equivalent length is composed of one half an equivalent crank length (12.3 in.) plus the distance between the center of the journal and the flywheel ($6 - \frac{1}{2} \times 5\frac{3}{4} = 3\frac{1}{8}$ in.), a total of 15.43 in.

Pinion to Coupling Center Line. The method of calculating the equivalent length of a keyed coupling follows that given in Sec. 8.3b. The operation is performed in three parts for the example that we are considering from Fig. 8.4. The first part deals with the steel shaft between the pinion and coupling end plus one half-length of the coupling; the second part considers the coupling as a hollow cast-iron shaft having an outside diameter equal to that of the hub and an inside diameter equal to that of the shaft; and the third part considers the coupling as a hollow cast-iron shaft with an outside diameter equal to the bolt-circle diameter, an inside diameter equal to the shaft diameter, and a length equal to the flange thickness. A correction for material must be made to the last two parts, according to Eq. (8.2), since cast iron has a shearing modulus of elasticity of $7(10^6)$, whereas steel has one of $12(10)^6$. Moreover, since the coupling rotates at a speed greater than the equivalent engine speed, a speed-correction factor of $(750/1{,}500)^2$

must be applied to the entire coupling, according to Eq. (8.8). Then, substituting the values found in Fig. 8.4, we have

$$L_e = \left[\frac{(8\frac{1}{2})^4}{(6\frac{3}{4})^4}\left(6 + \frac{8}{2}\right) + \frac{(8\frac{1}{2})^4}{12^4 - (6\frac{3}{4})^4}\left(8 - \frac{8}{2} - 1\frac{1}{2}\right)\frac{12(10^6)}{7(10^6)} + \frac{8\frac{1}{2}^4}{(14\frac{3}{4})^4 - (6\frac{3}{4})^4}\,1\frac{1}{2}\,\frac{12(10^6)}{7(10^6)}\right]\left(\frac{750}{1{,}500}\right)^2$$

$$= 6.67 \text{ in.}$$

Coupling Center Line to Pump Impeller. An examination of the coupling center line to pump impeller in Fig. 8.4a shows that the dimensions of this section are the same as those between the pinion and coupling center line. Hence, the equivalent length of this section is the same, namely, 6.67 in.

The equivalent lengths just found are shown on Fig. 8.4b beneath the corresponding sections on the actual system of Fig. 8.4a.

PROBLEMS

8.1. Set up the equivalent torsional system at the speed of mass 1 and for a 1 in. diameter steel shaft for the geared drive shown in

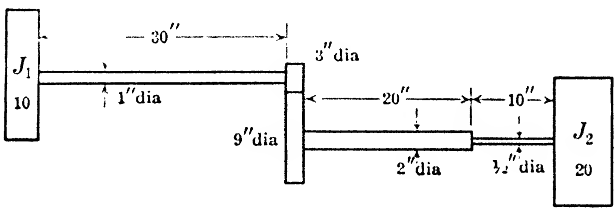


FIG. 8.5

Fig. 8.5. Neglect the mass moment of inertia of the gears and shafts. The shafts are steel. What is the natural frequency of the system?

Ans. $J_1' = 10$; $J_2' = 2.22$; $k_t = 795$ in.-lb per rad; $f_n = 200$ cpm.

8.2. Repeat Prob. 8.1, but assume that the shaft attached to mass 2 is made of cast iron. The rest of the system is made of steel.

Ans. $J_1' = 10$; $J_2' = 2.22$; $k_t = 469$ in.-lb per rad; $f_n = 153$ cpm.

8.3. A single-cylinder steam engine is direct-connected to a generator by means of a cast-iron flange coupling, as shown in Fig. 8.6. From this sketch and the following data determine (a) J of the connecting rod and reciprocating parts; (b) J of the crank; (c) J of the generator armature; (d) the equivalent length of the

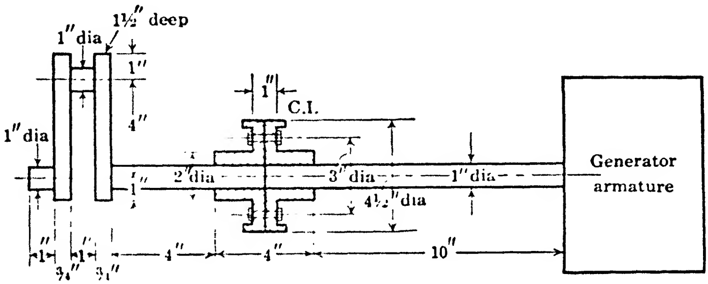


FIG. 8.6

crank based upon Carter's equation; (e) the equivalent length of the shaft and coupling; (f) the equivalent system, neglecting the J of the coupling, but considering its elasticity; (g) the natural frequency of the system in cycles per minute.

Data: The J of the generator armature with its shaft is found by suspending it on three wires that are 14 in. long and located 3 in.

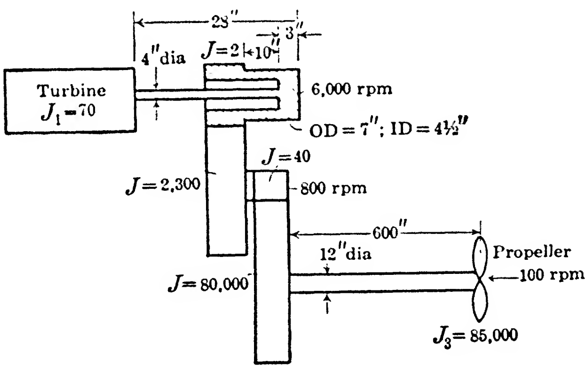


FIG. 8.7

from the axis of rotation. The time required for 50 complete oscillations is 100 sec.

The weight of the generator armature is 20 lb; the connecting-rod length is 20 in.; the distance of the center of gravity of the connecting rod from the crankpin is 8 in.; the connecting-rod weight is 25 lb; the weight of the piston, piston rod, and crosshead is 50 lb.

Ans. (a) 1.865; (b) 0.0807; (c) 1.303; (d) 2.36 in. of 1 in. diameter; (e) 0.0106 in. of 1 in. diameter; (f) $J_1 = 1.945$, $J_2 = 1.303$, $L_e = 19.37$ in. of 1 in. diameter; (g) 2,675 cpm.

8.4. Figure 8.7 illustrates a double-reduction marine drive with a hollow-quill shaft between the turbine and gears to obtain greater flexibility. The units of J given on the sketch are in.-lb-sec^2. (a) Draw an equivalent three-mass system referred to the propeller speed and show on it the corrected J values and equivalent lengths

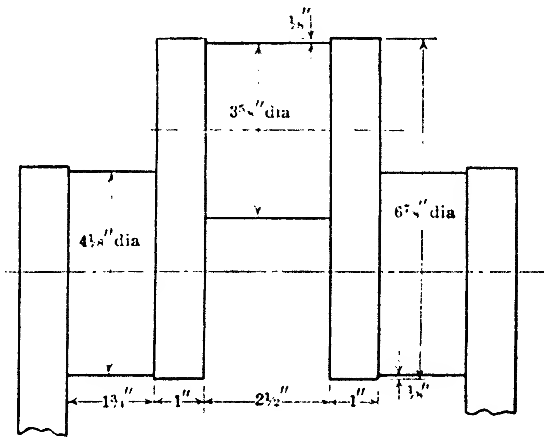


FIG. 8.8

referred to a 1 in. diameter steel shaft. Consider all the gears together as the middle mass, and neglect the J of the shafts. (b) Determine the natural frequencies of the system.

Ans. (a) $J_1' = 252{,}000$; $J_2' = 237{,}000$; $J_3' = 85{,}000$; $k_1 = 40{,}800(10^6)$; $k_2 = 40.7(10^6)$; (b) 213.5 and 5,520 cpm.

8.5. Determine the value of J and L_e for one cylinder of a four-cylinder 4.4 in. by 5.5 in. Diesel engine, one crank of which is shown in Fig. 8.8. Use Wilson's equation for the equivalent length of the crank based on a $4\frac{1}{8}$ in. diameter shaft. The width of the web is $4\frac{1}{4}$ in. Each connecting rod weighs 10.8 lb (1.7 lb at the wrist pin and 9.1 lb at the crankpin). Each piston assembly weighs 6.33 lb. *Ans.* $J = 0.775$ in.-lb-sec^2; $L_e = 14.5$ in.

Chapter 9

MULTIMASS LATERAL SYSTEMS

9.1 *INTRODUCTION*

Although many actual systems may be approximated for engineering purposes by simpler systems having one or two degrees of freedom, a more accurate frequency determination requires that they be treated as multimass systems.

There are also a large number of cases that generally cannot be reduced satisfactorily. Problems of this type include (a) stepped machinery shafts, such as those used in centrifugal machines (pumps, blowers, turbines, motors, and the like); (b) structures, such as foundation steel work, buildings, and bridges; (c) tapered cantilevers, such as steam-turbine blades, airplane propellers and wings; (d) disks and plates; and many other applications.

Approximations must be used in setting up idealized systems of such types as those listed above. The solution of the natural frequencies is based upon a trial-and-error method in which the natural frequency, or some function of it, is assumed and a check made to see if this assumption satisfies the basic requirements of the solution. This checking of the trial assumption may be carried out either graphically or mathematically, as developed in this chapter.

There are three principal methods of solving problems of this nature which will be described, namely, (a) the Rayleigh, (b) the Stodola, and (c) the general. The first two methods are generally simpler and quicker to use if only the lowest natural frequency is desired. It is just as easy to find higher natural frequencies by the last method as it is to find the lowest; but it is necessary to use many mathematical steps, so that the possibility of errors is relatively great.

9.2 *THE RAYLEIGH METHOD*

The first method to be developed for finding natural frequencies is that due to Lord Rayleigh. Raleigh ascertained that all points in the system vibrated with simple harmonic motion and reached the position of maximum deflection simultaneously. His method is based upon equating the maximum kinetic and potential energies, as outlined in Sec. 2.3, to determine the natural frequency. In the case in Sec. 2.3 there was only one mass and, hence, one deflection. In multimass systems there are many masses and, hence, many possible deflection curves. Each possible deflection curve will have a corresponding frequency based upon equating the energies. Rayleigh found that the system always vibrates in a manner that will make the frequency a minimum. Any other assumed deflection curve will give a frequency that is higher than the natural. If many deflection curves are assumed for the system, the values of W and y for each may be used to find a frequency, and the lowest value of these will approximate the true natural frequency.

A more orderly procedure based upon the Rayleigh method, may be summarized as follows:

(a) Assume a deflection curve of the system which is "reasonable." This is generally taken as the deflection curve of the system, considering the masses to act as dead or static loads.

(b) Equate the summations of the maximum potential and kinetic energies of the masses deflected as assumed in (a) to find the circular frequency.

$$\text{maximum potential energy} = \tfrac{1}{2}W_1y_1 + \tfrac{1}{2}W_2y_2 + \cdots,$$

$$\text{maximum kinetic energy} = \frac{1}{2}\frac{W_1}{g}y_1^2\omega_1^2 + \frac{1}{2}\frac{W_2}{g}y_2^2\omega_1^2 + \cdots.$$

Equating the maximum potential and kinetic energies and solving for ω_1^2 gives

$$\omega_1^2 = \frac{\frac{1}{2}W_1y_1 + \frac{1}{2}W_2y_2 + \cdots}{\frac{1}{2}\frac{W_1}{g}y_1^2 + \frac{1}{2}\frac{W_2}{g}y_2^2 + \cdots} = \frac{g\Sigma Wy}{\Sigma Wy^2}. \tag{9.1}$$

(c) The frequency found by part (b) will be somewhat high as the actual deflection curve is due to the inertia forces rather than to the dead or static load. However, the frequency will generally be within 3 per cent of the correct value, which is close enough for engineering purposes. A considerable change in the loadings is required to change the curve shape or natural frequency appreciably, as will be shown in the example in Sec. 9.4.

If greater accuracy is desired, the loads should be taken as the inertia forces, that is, $F = my\omega_1^2$, where ω_1 is the frequency based upon the previous or static-load assumption, and y is the corresponding deflection. The frequency is then found from the following equation:

$$\omega_2^2 = \frac{\frac{1}{2}F_1y_1' + \frac{1}{2}F_2y_2' + \cdots}{\frac{1}{2}m_1y_1'^2 + \frac{1}{2}m_2y_2'^2 + \cdots} = \frac{g\Sigma Fy'}{\Sigma Wy'^2}, \qquad (9.2)$$

where y' is the deflection due to the inertia load F. The potential energy is a function of the inertia force F, whereas the kinetic energy is based upon the mass, or W/g.

Successive approximations may be made by applying Eq. (9.2) until the derived deflection curve agrees with the assumed curve. The process is rapidly convergent for the lowest natural frequency, but the first approximation is generally sufficiently accurate for engineering purposes. The procedure is illustrated in Sec. 9.4.

If there is a case where the process does not converge, or where the deflection curve is difficult to predict, a modification of the Rayleigh method developed by Ritz* may be employed. In this modification the deflection curve may be expressed in terms of a distance along an axis, for example, $y = x^n$, $y = (x - x_o)^n$ and so on. The values of ω^2 or f, obtained from deflection curves based upon assumed values of the exponent n may be plotted against those values of n to determine the minimum frequency, which approximates the true value. This method is used by Stodola in determining frequencies of steam-turbine disks.

In using Eqs. (9.1) and (9.2), the sign of the deflection is always taken to be positive, regardless of the direction of the deflection, since these equations involve energies rather than deflections and energy is not a function of the direction of the displacement. If the deflection curve crosses the bearing centerline, as in the case where the end of the shaft is overhung from the bearings, the static deflections for the first assumption should be taken to act in opposite directions. This condition is illustrated in Sec. 9.5 and Fig. 9.2.

If the member is rotating, the natural frequency is raised because

* S. Timoshenko, *Vibration Problems in Engineering*, Van Nostrand, New York, 1937, pp. 370–376.

of the action of the centrifugal force, which replaces the gravitational, or g, term in the equations and generally exceeds it. This effect is pronounced in airplane propellers and steam-turbine blades rotating at high speeds.

9.3 *THE STODOLA METHOD*

The Stodola method is similar to the Rayleigh method, but it is slightly simpler to use. The procedure is as follows:

(a) Assume a deflection curve of the system that is "reasonable." Generally this is the static deflection curve as in the Rayleigh method.

(b) If the deflection of the masses is y, the inertia loadings will be $my\omega^2$, or $\frac{W}{g}\,y\omega^2$. If the circular frequency squared, which corresponds to the curve of part (a), is assumed to be 386, that is, if $\omega^2 = 386$, then $\omega^2/g = 1$, and the term Wy is the inertia loading of the mass. Note that if $\omega^2 = 386$, the cyclic frequency is $f = 187.5$ cpm.

(c) Assume that the system is loaded with the inertia loads of Wy from part (b), and find the corresponding new deflections y'.

(d) If the assumed deflections y and the derived deflections y' have a constant ratio along the system, the shape of the assumed curve was correct. All that is then needed to make the two curves coincide is to increase the frequency of the assumed curve. Therefore,

$$\frac{f_n}{f} = \sqrt{\frac{y}{y'}}; \qquad \text{or} \qquad f_n = 187.5\sqrt{\frac{y}{y'}}. \tag{9.3}$$

(e) If the ratio of y/y' is not constant throughout the system, the derived curve y' may be used as the next assumption and the above steps repeated. As pointed out before, the first assumption is generally sufficiently close. This procedure is illustrated in the next section.

9.4 *NOTES ON THE RAYLEIGH AND STODOLA METHODS*

Before comparing the Rayleigh and Stodola methods of determining natural frequencies, it is desirable to illustrate their use by means of an example.

EXAMPLE

The deflections of a constant-section, weightless cantilever beam with a single concentrated load as taken from beam tables are given in parts (a) and (b) of Fig. 9.1.

(a) If a cantilever is loaded as shown in part (c) of Fig. 9.1, derive expressions for the deflections y_1 and y_2 in terms of F_1 and F_2. The following data apply: $E = 30(10^6)$ psi, $L = 10$ in., $a = 6$ in., $I = 1$ in.^{4}.

(b) If a weight W_1 of 100 lb is placed at F_1 and a weight W_2 of 200 lb is placed at F_2, determine the natural frequency of the

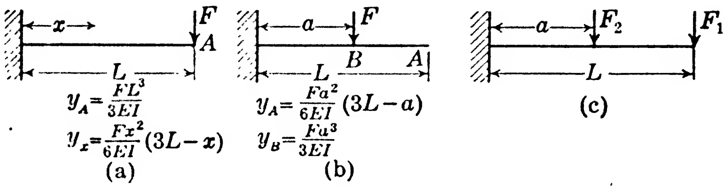


FIG. 9.1

system by the Rayleigh method, using the static deflection for the first assumption.

(c) Repeat part (b), using the Stodola method.

(a) The total deflection at any point on the beam is the sum of the deflections at that point due to each load acting independently. By combining the equations of Fig. 9.1,

$$y_1 = \frac{F_1L^3}{3EI} + \frac{F_2a^2}{6EI}(3L - a),$$

$$y_2 = \frac{F_2a^3}{3EI} + \frac{F_1a^2}{6EI}(3L - a).$$

Substituting the values $E = 30(10^6)$, $L = 10$, $a = 6$, and $I = 1$ gives

$$y_1 = 11.11(10^{-6})F_1 + 4.8(10^{-6})F_2, \tag{9.4}$$

$$y_2 = 4.8(10^{-6})F_1 + 2.4(10^{-6})F_2. \tag{9.5}$$

(b) *Rayleigh method.* For the static deflection let $F_1 = W_1 = 100$ and $F_2 = W_2 = 200$ in Eqs. (9.4) and (9.5). Then,

$$y_1 = 11.11(10^{-6}) \times 100 + 4.8(10^{-6}) \times 200$$

$$= 0.00207 \text{ in.},$$

$$y_2 = 4.8(10^{-6}) \times 100 + 2.4(10^{-6}) \times 200$$
$$= 0.00096 \text{ in.}$$

Applying the deflections just found to Eq. (9.1) gives

$$\omega_1^2 = \frac{g\Sigma Wy}{\Sigma Wy^2} = \frac{386(100 \times 0.00207 + 200 \times 0.00096)}{100 \times 0.00207^2 + 200 \times 0.00096^2}$$
$$= 251{,}300,$$
$$\omega_1 = 501.3 \text{ rad per sec.}$$

Therefore, $f_1 = 4{,}788$ cpm.

For the second assumption, the inertia loads $F = my\omega_1^2$ should be used, where y is the static deflection and ω_1 is the static circular frequency. Then,

$$F_1 = \tfrac{100}{386} \times 0.00207 \times 251{,}300 = 134.7 \text{ lb},$$
$$F_2 = \tfrac{200}{386} \times 0.00096 \times 251{,}300 = 125.0 \text{ lb}.$$

Substituting these values of F in Eqs. (9.4) and (9.5) to obtain the corresponding deflections,

$$y_1' = 11.11(10^{-6}) \times 134.7 + 4.8(10^{-6}) \times 125$$
$$= 0.002097 \text{ in.},$$
$$y_2' = 4.8(10^{-6}) \times 134.7 + 2.4(10^{-6}) \times 125$$
$$= 0.000946 \text{ in.}$$

Since this approximation is the second in which the inertia loading is used, the circular frequency is given by Eq. (9.2) as follows:

$$\omega_2^2 = \frac{g\Sigma Fy'}{\Sigma Wy'^2} = \frac{386(134.7 \times 0.002097 + 125 \times 0.000946)}{100 \times 0.002097^2 + 200 \times 0.000946^2}$$
$$= 250{,}000,$$
$$\omega_2 = 500 \text{ rad per sec.}$$

Therefore, $f_2 = 4{,}775$ cpm.

It is interesting to note that, even though the loading on the beam changes considerably with the two assumptions, there is very little difference in the derived frequencies. Since the second approximation is closer to the true conditions, the frequency is lower, which is in accord with Rayleigh's principle.

For the third approximation, the inertia loads are found by using the second derived deflections and its frequency, that is, $F' = my'\omega_2^2$. Thus,

$$F_1' = \tfrac{100}{386} \times 0.002097 \times 250{,}000$$
$$= 135.8 \text{ lb},$$
$$F_2' = \tfrac{200}{386} \times 0.000946 \times 250{,}000$$
$$= 122.7 \text{ lb}.$$

Since these forces are very close to those used in the second approximation, the corresponding deflections will be little different, and the effect on the frequency is negligible. There is little point in continuing with this calculation, and the frequency of the system therefore is 4,775 cpm, or very slightly below that figure.

(c) *Stodola method.* The static deflection curve will be used for the first approximation of the deflection of the beam. From part (b) the static deflections at the two masses are $y_1 = 0.00207$ in. and $y_2 = 0.00096$ in.

If it is assumed that $\omega^2 = 386$ or $f = 187.5$ cpm, the product Wy is the inertia loading on the beam. Hence,

$$F_1 = W_1y_1 = 100 \times 0.00207 = 0.207 \text{ lb},$$
$$F_2 = W_2y_2 = 200 \times 0.00096 = 0.192 \text{ lb}.$$

The deflections y' caused by these loads are found in Eqs. (9.4) and (9.5). Substituting in these equations, we obtain

$$y_1' = 11.11(10^{-6}) \times 0.207 + 4.8(10^{-6}) \times 0.192$$
$$= 3.221(10^{-6}) \text{ in.},$$
$$y_2' = 4.8(10^{-6}) \times 0.207 + 2.4(10^{-6}) \times 0.192$$
$$= 1.454(10^{-6}) \text{ in.}$$

It is now necessary to see if the ratio of the assumed and derived deflections is constant along the system or beam:

$$\frac{y_1}{y_1'} = \frac{0.00207}{3.221(10^{-6})} = 643,$$

$$\frac{y_2}{y_2'} = \frac{0.00096}{1.454(10^{-6})} = 660.$$

Since these two figures differ, it is desirable to use the derived deflections y' as the basis for calculating the inertia loads for an additional trial, assuming again that the circular frequency squared is $\omega^2 = 386$ and $f = 187.5$ cpm. Thus,

$$F_1' = W_1 y_1' = 100 \times 3.221(10^{-6}) = 322.1(10^{-6}) \text{ lb},$$

$$F_2' = W_2 y_2' = 200 \times 1.454(10^{-6}) = 290.8(10^{-6}) \text{ lb}.$$

The deflections y'' caused by these loads are found in Eqs. (9.4) and (9.5) as follows:

$$y_1'' = 11.11(10^{-6}) \times 322.1(10^{-6}) + 4.8(10^{-6}) \times 290.8(10^{-6})$$
$$= 4{,}974(10^{-12}) \text{ in.},$$
$$y_2'' = 4.8(10^{-6}) \times 322.1(10^{-6}) + 2.4(10^{-6}) \times 290.8(10^{-6})$$
$$= 2{,}247(10^{-12}) \text{ in.}$$

Checking the ratio of the assumed and derived deflections y'/y'' gives

$$\frac{y_1'}{y_1''} = \frac{3.221(10^{-6})}{4{,}974(10^{-12})} = 647.6,$$

$$\frac{y_2'}{y_2''} = \frac{1.454(10^{-6})}{2{,}244(10^{-12})} = 647.9.$$

These two ratios are sufficiently close together to obviate the necessity of an additional trial. The average value of this ratio is 647.8, which will be used to determine the natural frequency with the aid of Eq. (9.3). Thus,

$$f_n = 187.5\sqrt{\frac{y'}{y''}} = 187.5\sqrt{647.8} = 4{,}772 \text{ cpm},$$

which very closely approximates the value of 4,775 cpm found by the Rayleigh method.

If the example illustrating the use of the Rayleigh method is examined carefully, it will be found that for the second and later approximations the frequency used to obtain the inertia loadings is unimportant, since it will cancel out when Eq. (9.2) is applied. (It would be instructive for the reader to check this statement by using a value of ω^2 of say 1,000 for the second approximation.)

Therefore, for assumptions after the first, or static, deflection, the two methods amount to the same procedure.

For complicated systems, such as many masses distributed along a stepped shaft, it is more convenient to use a graphical method for finding the deflections, as will be outlined in Sec. 9.5, or an approximate method involving tabular integration,* rather than the mathematical method just used.

While it is possible to use a generalized form of the Stodola method to determine higher natural frequencies,† usually the general method that will be discussed in Sec. 9.7 will be found to be more practical and direct.

9.5 GRAPHICAL DETERMINATION OF DEFLECTIONS‡

The deflection curves of multimass systems may be determined graphically by the principles of graphic statics, and for many cases this is the simplest procedure. The basic principles are covered in books on strength of materials and mechanics.

The method is illustrated in Fig. 9.2, in which it is required to determine the lowest critical speed of a two-bearing stepped shaft based upon the static deflection. The various parts of Fig. 9.2 are designated by capital letters in parentheses, and these will be referred to throughout the remainder of this section. The scales to which the figure is drawn apply to the original drawing, which was necessarily reduced in printing.

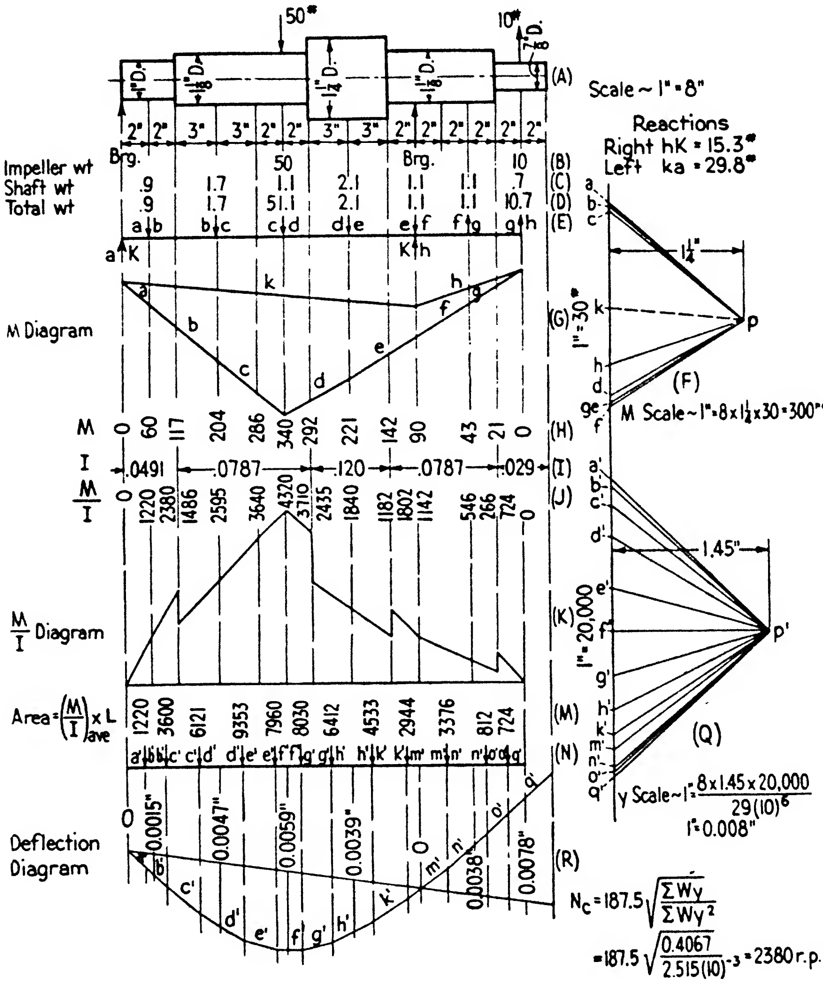
Part (A) shows an overhung stepped shaft, fully dimensioned, drawn to a space scale of 1 in. = 8 in. It has two impellers, one weighing 50 lb and the other 10 lb, located on it. The left bearing is located at the left end of the shaft, and the right bearing is 10 in. in from the right end. The impeller weights are listed under the loads in line (B). The shaft weight is found by dividing the shaft into a number of lengths, calculating the weight of each, and assuming that these loads act at the mid-point of each length. These loads are listed in line (C). The shaft weights are added to the impeller weights to give the total loads in line (D). The loads between the

* A. H. Church, "Difference Calculus Simplifies Computation of Shaft Deflections," *Product Eng.*, January, 1942, pp. 13–15; May, 1942, p. 295.

† N. A. Boukidis and R. J. Ruggiero, "An Iterative Method for Determining Dynamic Deflections and Frequencies," *J. Aeronaut. Sci.*, vol. 11, no. 4, pp. 319–328, October, 1944.

‡ Reprinted in revised form by permission of the publisher from A. H. Church, *Centrifugal Pumps and Blowers*, Wiley, New York, 1944.

bearings act downward, whereas those to the right of the right bearing act upward. Owing to the shaft deflection while whirling at the critical speed, the centrifugal forces will act outward, and the static loading must be taken accordingly.



$$N_c = 187.5\sqrt{\frac{\Sigma Wy}{\Sigma Wy^2}}$$

$$= 187.5\sqrt{\frac{0.4067}{2.515(10)^{-3}}} = 2380 \text{ r.p.m.}$$

FIG. 9.2

These total forces of line (D) are represented vectorially in line (E) and labeled according to Bow's notation. The forces are laid off successively to a scale of 1 in. = 30 lb on the vertical line of part (F) (the distance *ab* represents force *ab*, etc.), and they are laid off in the direction in which they act. A pole p is located $1\frac{1}{4}$ in. from the vertical line, and the radial lines from p to a, b, and so on, are drawn. The pole distance is made $1\frac{1}{4}$ in. so that the moment scale is an even value and the moments may be read off directly with the aid of an engineer's decimal scale. The vertical location of the pole p does not matter, but a neater diagram generally results if it is placed opposite the mid-point of the vertical line.

The moment diagram (G) is now drawn, starting at the left bearing with line a parallel to line pa of part (F), line b parallel to pb, and so on. The lines in part (G) are located directly below the correspondingly labeled spaces of part (E); thus, the d line is located in the d space between the loads cd and de, and the h line in the h space between the loads gh and hk, and so on. The polygon is closed by drawing in the k line between the bearings. A line drawn parallel to this k line on part (F) gives the bearing reactions graphically by the lengths hk and ka. The moment at any point on the shaft is given by the vertical distance between lines k or h and the other lines, to a scale of 1 in. = space scale × force scale × pole distance, so that 1 in. = $8 \times 30 \times 1\frac{1}{4} = 300$ in.-lb. The values of the moment M as scaled from part (G) are listed on line (H).

The moments of inertia I of the various sections are calculated and shown on line (I). The values of M/I are calculated, listed on line (J), and plotted to form an M/I diagram (K). The value of M/I changes instantaneously with a change in shaft diameter.

A conjugate beam is assumed to be loaded with the area under the M/I diagram. These areas are calculated and listed on line (M) and represented by the vectors on line (N). For example, for vector $b'c'$,

$$\text{area} = \left(\frac{M}{I}\right)_{av} L = \left(\frac{1{,}220 + 2{,}380}{2}\right)2 = 3{,}600.$$

These imaginary loads will act at the center of gravity of the areas, which may be approximated with sufficient accuracy. They are laid off successively on the vertical line of part (Q) to a scale of 1 in. = 20,000. A pole p' is assumed 1.45 in. to the right of the vertical line, and the radial lines $p'a'$, $p'b'$, and so on, are drawn. The pole distance is again selected such that the deflection scale is an even value to facilitate reading off deflèctions with an engineer's decimal scale. The vertical location of the pole p' is again immaterial.

The deflection curve (R) is drawn similarly to the moment diagram, that is, line a' is parallel to $p'a'$ in the a' space, b' is parallel to $p'b'$ in the b' space, and so on. As it is assumed that the bearings do not deflect, the closing line crosses the deflection curve at the bearings. The deflection scale is given by multiplying the space scale by the pole distance of part (Q) by the area scale and dividing by the modulus of elasticity E; thus,

$$y \text{ scale } 1 \text{ in.} = \frac{8 \times 1.45 \times 20{,}000}{29{,}000{,}000} = 0.008 \text{ in.}$$

The vertical distance between the closing line and the curve at any point represents the deflection. The deflections under the loads are listed on the deflection diagram.

The critical speed is then found from Eq. (9.1).

W	y	y^2	Wy	Wy^2
0.9	0.0015	2.25(10⁻⁶)	0.0013	2.03(10⁻⁶)
1.7	0.0047	22.09(10⁻⁶)	0.0080	37.55(10⁻⁶)
51.1	0.0059	34.81(10⁻⁶)	0.3015	1,778.80(10⁻⁶)
2.1	0.0039	15.21(10⁻⁶)	0.0082	31.94(10⁻⁶)
1.1	0	0	0	0
1.1	0.0038	14.44(10⁻⁶)	0.0042	15.88(10⁻⁶)
10.7	0.0078	60.84(10⁻⁶)	0.0835	651.00(10⁻⁶)
			0.4067	2,517.20(10⁻⁶)

$$n_c = 187.5 \sqrt{\frac{\Sigma Wy}{\Sigma Wy^2}} = 187.5 \sqrt{\frac{0.4067}{2{,}517.2(10^{-6})}} = 2{,}380 \text{ rpm.}$$

The running speed of the shaft should be at least 20 per cent away from the critical, that is, it should not operate between speeds of 1,900 and 2,860 rpm.

It is not necessary to construct the M/I diagram (K). It was done in Fig. 9.2 to clarify the procedure, but the M/I values are all that are required to get the deflection curve.

9.6 *FACTORS INFLUENCING THE CRITICAL SPEEDS OF SHAFTS**

The critical speed of a shaft is a special case of the natural frequency of a multimass system. There are many factors tending

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to alter its value that operate only in this application. Frequently, the magnitude of the effect of these factors cannot be predicted accurately. When special cases arise in which these factors become important or in which it is necessary to account for discrepancies between test results and calculated values, the references given below in footnotes may be consulted for further details.

A. *Bearing Length*

It is usually assumed that the shaft is simply supported, and this assumption is justified if the bearings are free to oscillate and adjust themselves to the shaft deflections. Often the bearings are fixed and, if long, compared to their diameter, they may tend to create the effect of a fixed support. This condition decreases the shaft deflection and raises the critical speed.

As the shaft deflects, the center of pressure of the oil film is no longer at the bearing center line but tends to move in toward the main mass. A general rule is to move the assumed bearing center "in" toward the main mass about one sixth of the bearing length when fixed bearings are used.*

B. *Gyroscopic Effect of the Impellers*

If the impellers are heavy and have a large diameter, they create a gyroscopic action and resist any change in the direction of their axis. When the shaft begins to whirl, the impeller resists the motion and tends to keep it straight, thus reducing the deflection and raising the critical speed. This effect is greater for impellers nearer to the bearings where the slope of the shaft is greater.†

C. *Bearing Elasticity*

The usual assumption made in calculating the deflection curve is that the bearings are rigid and do not deflect. Actually every bearing will deflect somewhat because of the load on it. This deflection will tend to lower the critical speed, since the deflection is greater than calculated, and may be as much as 25 to 50 per cent. The bearings may deflect more in one direction than another (as is

* See Stodola-Loewenstein, *Steam and Gas Turbines*, McGraw-Hill, New York, p. 430.

† See Stodola-Loewenstein, *Steam and Gas Turbines*, *ibid*, pp. 430–437; S. Timoshenko, *Vibration Problems in Engineering*, Van Nostrand, New York, 1937, p. 290.

true of the pedestal type which is not so rigid horizontally as it is vertically), resulting in two distinct critical speeds for the same shaft. It is very difficult to predict the bearing deflection in advance, but it may be measured in a unit already manufactured to account for discrepancies.*

D. *Shrink Fits of Impeller Hubs*

If the hubs are quite heavy and shrunk tightly on the shaft, they tend to stiffen it and raise the critical speed. It is difficult to predict the amount of the stiffening because of manufacturing uncertainties in the tolerances, but the effect may be appreciable.

The critical speed of a shaft may be placed either above or below the operating speed. If the unit is to operate at high speeds that do not vary widely, the critical speed may be below the operating speed, and the shaft is then said to be flexible. In bringing the shaft up to the operating speed, the critical speed must be passed through; but if this is done rapidly, resonance conditions do not have a chance to build up and no difficulty is experienced. If the operating speed is low or must vary through wide ranges, the critical speed is placed above it and the shaft is said to be "rigid" or "stiff."

In the preliminary design of a machine it may be found that the critical speed is quite close to the operating speed, so that trouble may be expected. In such cases it is necessary to alter the dimensions to keep the critical speed at least 20 per cent away from the operating speed. It is then convenient to know approximately what the effect of design changes will be on the critical speed. This may be found by considering the shaft as one having a single degree of freedom, in which case the critical speed is inversely proportional to the square root of the static deflection. For beams or shafts the static deflection is proportional to $\frac{WL^3}{EI}$; so the critical speed is proportional to $\frac{1}{\sqrt{W}}, \frac{1}{L^{3/2}}, \sqrt{E}, \sqrt{I}$, or d^2. These relations are quite useful in estimating the changes that may be made in a shaft to bring the critical speed to a safe value.

9.7 *GENERAL METHOD*

As mentioned previously, the Rayleigh and Stodola methods are not generally suitable to obtain higher natural frequencies.

* See A. L. Kimball, *Vibration Prevention in Engineering*, Wiley, New York, 1932, p. 72.

Equations and methods have been developed to solve special cases. Recently a general method has been developed by which any natural frequency may be found relatively simply. This method was presented by N. O. Myklestad,* of California Institute of Technology, and M. A. Prohl,† of the General Electric Co., independently within the same year. Both methods are essentially the same; the procedure described here is that due to Prohl.

The method is based upon assuming a frequency and, after working across the shaft or beam, determining a residual function, such as bending moment. If this function is zero, the assumed frequency is a natural one. A remainder curve for the function may be plotted similar to the torque remainder curve of the Holzer method. It is more complicated than the Holzer method since four integrations are involved rather than two, and additional complications arise in dealing with the boundary conditions.

The differential equation at the natural frequency is

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2y}{dx^2}\right) = \mu\omega^2 y, \tag{9.6}$$

where μ is the mass per unit length.

From the elementary beam theory,

$$EI\,\frac{d^2y}{dx^2} = M. \tag{9.7}$$

Hence, Eq. (9.6) becomes

$$\frac{d^2M}{dx^2} = \mu\omega^2 y. \tag{9.8}$$

Since Eq. (9.6) is of the fourth order, four boundary conditions must be satisfied. Any frequency ω that satisfies these four boundary conditions is a natural one. Equations (9.7) and (9.8) may be transformed to permit tabular integration and thus form the basis for constructing the M and y diagrams. Then,

$$\Delta\left(\frac{dy}{dx}\right) = \left(\frac{\Delta x}{EI}\right) M_{ave}. \tag{9.9}$$

* N. O. Myklestad, "A New Method of Calculating Natural Modes of Uncoupled Bending Vibration of Airplane Wings and Other Types of Beams," *J. Aeronaut. Sci.*, April, 1944, pp. 153–162; also *Vibration Analysis*, McGraw-Hill, New York, 1944, pp. 184–214.

† M. A. Prohl, "A General Method for Calculating Critical Speeds of Flexible Rotors," *Trans. A.S.M.E.* September, 1945, A-142.

and

$$\Delta\left(\frac{dM}{dx}\right) = (\mu\omega^2\,\Delta x)y_{ave}, \tag{9.10}$$

where Δx is the length of a given section and M_{ave} and y_{ave} are the average values of bending moment and deflection for that section.

Equation (9.9) states that the change in slope of the deflection curve at a given section equals $\frac{\Delta x}{EI} \times M_{ave}$. Equation (9.10) states that the change in slope of the moment curve at a given section equals $\mu\omega^2\,\Delta x \times y_{ave}$.

For any assumed frequency ω the M and y diagrams can be constructed to satisfy three of the four boundary conditions. By plotting the fourth boundary condition against ω, the natural frequency ω_n will occur when this remainder equals zero.

The actual calculation method is based upon the following series of equations.

The shaft or beam is transformed into a number of point masses connected by weightless springs. The moment diagram has a constant slope in each section, since

$$\frac{dM}{dx} = V. \tag{9.11}$$

The change in shear at a mass is

$$\Delta V = my\omega^2 \tag{9.12}$$

The deflection curve is smooth, since the body is continuous.

Assume that V_o, M_o, θ_o, and y_o are known and refer to Fig. 9.3; then,

$$V_1 = V_o + m_o y_o \omega^2, \tag{9.13}$$

$$M_1 = M_o + V_1(\Delta x)_1, \tag{9.14}$$

$$M = M_o + \frac{M_1 - M_o}{(\Delta x)_1}\,x, \tag{9.15}$$

$$\theta = \frac{1}{(EI)_1}\int_0^x M\,dx + C, \tag{9.16}$$

where C is a constant of integration. When the distance x is zero, the slope θ becomes θ_o; hence, the constant $C = \theta_o$.

Substituting Eq. (9.15) in Eq. (9.16) and integrating,

$$\theta = \frac{1}{(EI)_1}\left[M_o x + \frac{M_1 - M_o}{(\Delta x)_1}\frac{x^2}{2}\right] + \theta_o, \tag{9.17}$$

$$y = \int_0^x \theta\, dx + C', \tag{9.18}$$

where C' is another constant of integration. When the distance x is zero the deflection y becomes y_o; hence, the constant $C' = y_o$.

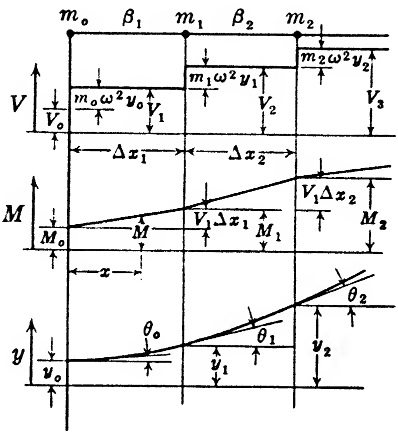

FIG. 9.3

Substituting Eq. (9.17) in Eq. (9.18) and integrating,

$$y = \frac{1}{(EI)_1}\left[M_o\frac{x^2}{2} + \frac{M_1 - M_o}{(\Delta x)_1}\frac{x^3}{6}\right] + \theta_o x + y_o. \tag{9.19}$$

It is only necessary to know θ and y at the end of the section (that is, at point 1). Substituting $(\Delta x)_1$ for x, and β_1 for $\left(\frac{\Delta x}{EI}\right)_1$ in Eqs. (9.17) and (9.19), gives

$$\theta_1 = \beta_1\left(\frac{M_o}{2} + \frac{M_1}{2}\right) + \theta_o, \tag{9.20}$$

$$y_1 = \beta_1\left(\frac{M_o}{3} + \frac{M_1}{6}\right)(\Delta x)_1 + \theta_o(\Delta x)_1 + y_o, \tag{9.21}$$

$$V_2 = V_1 + m_1 y_1 \omega^2, \tag{9.22}$$

$$M_2 = M_1 + V_2(\Delta x)_2. \tag{9.23}$$

By repeating the above steps across the body, the moment and deflection diagrams can be calculated and drawn.

Generalizing the preceding equations, we find that

$$V_n = V_{n-1} + m_{n-1}\omega^2 y_{n-1}. \tag{9.24}$$

$$M_n = M_{n-1} + V_n(\Delta x)_n. \tag{9.25}$$

$$\theta_n = \beta_n \left[\frac{M_{n-1}}{2} + \frac{M_n}{2}\right] + \theta_{n-1}. \tag{9.26}$$

$$y_n = \beta_n \left[\frac{M_{n-1}}{3} + \frac{M_n}{6}\right](\Delta x)_n + \theta_{n-1}(\Delta x)_n + y_{n-1}, \tag{9.27}$$

where $\beta_n = \left(\dfrac{\Delta x}{EI}\right)_n$.

Let

$$M_n' = \frac{M_{n-1}}{3} + \frac{M_n}{6}, \tag{9.28}$$

$$M_n'' = \frac{M_{n-1}}{6} + \frac{M_n}{3}. \tag{9.29}$$

Then,

$$M_n' + M_n'' = \frac{M_{n-1}}{3} + \frac{M_n}{6} + \frac{M_{n-1}}{6} + \frac{M_n}{3} = \frac{M_{n-1}}{2} + \frac{M_n}{2}.$$

$$\theta_n = \sum_{n=1}^{n} (\beta_n M_n' + \beta_n M_n'') + \theta_o. \tag{9.30}$$

$$y_n = [\beta_n M_n' + \sum_{n=1}^{n-1} (\beta_n M_n' + \beta_n M_n'')](\Delta x)_n + \theta_o(\Delta x)_n + y_{n-1}. \tag{9.31}$$

$$(\Delta V)_n = m_n\omega^2 y_n. \tag{9.32}$$

$$(\Delta M)_n = V_n(\Delta x)_n. \tag{9.33}$$

$$(\Delta y')_n = [\beta_n M_n' + \sum_{n=1}^{n-1} (\beta_n M_n' + \beta_n M_n'')](\Delta x)_n. \tag{9.34}$$

$$(\Delta y'')_n = \theta_o(\Delta x)_n. \tag{9.35}$$

$$(\Delta y)_n = (\Delta y')_n + (\Delta y'')_n = y_n - y_{n-1}. \tag{9.36}$$

It can be demonstrated that V, M, θ, and y at any point in the span are linear functions of the four assumed quantites at the starting end (that is, point O). Hence, y, for example, at point n may be expressed as

$$y_n = A_n V_o + B_n M_o + C_n \theta_o + D_n y_o, \tag{9.37}$$

where A_n, B_n, C_n, and D_n represent numerical coefficients taken from the table. Since two boundary conditions must be known at point O, only two need be evaluated. This is done in two parts on a table.

The boundary conditions for the usual cases are as follows:

Fixed end:

$$y_o = 0, \qquad \theta_o = 0.$$

Simple supported end:

$$y_o = 0, \qquad M_o = 0.$$

Free end:

$$V_o = 0, \qquad M_o = 0.$$

At common point on adjacent spans: θ and y same for both.

Where the values to be placed in the various columns of the table are not obvious from the headings, the following list may be consulted:

Column	
1	$\Delta V = my\omega^2$
2	$V_n = V_{n-1} + \Delta V$
3	Width of step or section
4	$\Delta M = V(\Delta x)$
5	$M_n = M_{n-1} + \Delta M$
8	$M_n' = \dfrac{M_{n-1}}{3} + \dfrac{M_n}{6}; \qquad M_n'' = \dfrac{M_{n-1}}{6} + \dfrac{M_n}{3}$
9	$\beta_n = \left(\dfrac{\Delta x}{EI}\right)_n$
11	$\dfrac{\Delta y'}{\Delta x} = \beta_n M_n' + \sum\limits_{n=1}^{n-1} (\beta_n M_n' + \beta_n M_n'')$
12	Width of step or section
13	$\Delta y' = \Delta x \left(\dfrac{\Delta y'}{\Delta x}\right)$

14 $\Delta y'' = \theta_o(\Delta x)_n$

15 $y_n = (\Delta y')_n + \theta_o(\Delta x)_n + y_{n-1}$

17 $\theta_n = \sum_{n=1}^{n} (\beta_n M_n' + \beta_n M_n'') + \theta_o$

The paper also presents a method of including the gyroscopic effect of large-diameter disks rotating on shafts, which will be omitted here.

The procedure will be illustrated by two simple examples. Actual cases generally have more masses and, hence, are longer, but they follow the method that applies to these simple cases. To obtain satisfactory results, it is generally necessary to use a calculating machine which will carry five or six significant figures.

EXAMPLE 1

This problem is the same as the example of Sec. 9.4, as shown in Fig. 9.4, where the points 0, 1, and 2 listed in Table 9.1 are located on the beam.

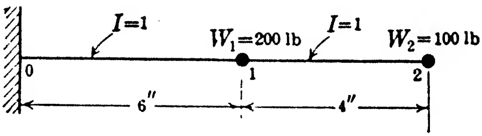

FIG. 9.4

The value of the mass is $\dfrac{W}{g}$ and of β is $\dfrac{\Delta x}{EI}$. Then, $m_1 = \dfrac{200}{386} = 0.518$; $m_2 = \dfrac{100}{386} = 0.259$; $\beta_1 = \dfrac{6}{30(10^6) \times 1} = 0.2(10^{-6})$; $\beta_2 = \dfrac{4}{30(10^6) \times 1} = 0.1333(10^{-6})$.

From the previous example it was found that the lowest natural frequency is 4,775 cpm; hence, a trial value of $\omega^2 = 0.25(10^6)$ will be assumed.

Table 9.1 shows the complete calculation, which is started at the fixed end of the beam for each part. Of the four boundary conditions, the two that are known at the fixed end are $y_o = 0$ and $\theta_o = 0$; whereas the bending moment M_o and the shear V_o there are unknown. For the first part of the calculation one of the

TABLE 9.1

Line	n	(1) ΔV	(2) V	(3) Δx	(4) ΔM	(5) M	(6) $\frac{1}{3}M$	(7) $\frac{1}{6}M$	(8) M' M''	(9) (10^{-6}) β	(10) (10^{-6}) $\beta M'$ $\beta M''$	(11) (10^{-6}) $\frac{\Delta y'}{\Delta x}$	(12) Δx	(13) (10^{-6}) $\Delta y'$	(14) $\Delta y''$	(15) (10^{-6}) y	(16) (10^{6}) $m\omega^2$	(17) θ
1			1															
2	0	0				0	0	0								0	0	0
3			1	6	6				1 2	0.2	0.2 0.4	0.2	6	1.2	0			
4	1	0.1554				6	2	1								1.2	0.1295	
5			1.1554	4	4.6216				3.771 4.542	0.1333	0.5027 0.6053	1.1027	4	4.411	0			
6	2	0.3630				10.6216	3.541	1.771								5.611	0.0647	
7			1.5184															
8			0															
9	0	0				1	0.3333	0.1667								0	0	0
10			0	6	0				0.5 0.5	0.2	0.1 0.1	0.1	6	0.6	0			
11	1	0.0777				1	0.3333	0.1667								0.6	0.1295	
12			0.0777	4	0.3108				0.5517 0.6037	0.1333	0.0735 0.0805	0.2735	4	1.0940	0			
13	2	0.1096				1.3108	0.4369	0.2184								1.694	0.0647	
14			0.1873															

$$V_3 = 1.5184V_o + 0.1873M_o, \qquad V_o = -\frac{0.1873M_o}{1.5184} = -0.1233M_o;$$

$$M_3 = 10.6216V_o + 1.3108M_o, \qquad M_3 = 10.6216(-0.1233M_o) + 1.3108M_o = -0.0012M_o.$$

unknowns V_o is made unity, and the other factors are set equal to zero. For the second part, the other unknown M_o is taken as unity and the other conditions set equal to zero. In this way, equations for the shear V_3 and bending moment M_3 at the free end may be evaluated in terms of the unknowns at the fixed end.

It is known that the shear and bending moment at the free end are both equal to zero, so that one may be expressed in terms of the other to find a remainder. It does not matter whether the shear is expressed in terms of moment to get a shear remainder or the moment expressed in terms of shear to obtain a moment remainder. If this remainder equals zero, a correct assumption of frequency has been made. If it does not, a remainder curve may be plotted as in the Holzer method to aid in selecting a new assumption of frequency.

The table is set up by listing the length of the steps Δx in columns 3 and 12 and the values of β in column 9. The values of $m\omega^2$ are found by multiplying the assumed ω^2 of $0.25(10^6)$ by the masses m and are listed in column 16. The terms in the parentheses in the following explanation refer to the position of the item in Table 9.1.

A shear of unity is assumed at the fixed end O and so listed (line 1, column 2). Since the deflection y_o at the fixed end is assumed to be zero (line 2, column 15), the inertia force $m_o y_o \omega^2$ (line 2, column 16) is also zero, and there is no change in the shear at this point (line 2, column 1). Hence, the shear is unity in the first step (line 3, column 2). The change in bending moment over this step equals $\Delta M = V(\Delta x)$, or $1 \times 6 = 6$ (line 3, column 4). The bending moment at point 1 is $M_o + \Delta M = 0 + 6 = 6$ (line 4, column 5).

By Eq. (9.28) the value of $M_{o1}' = \frac{M_o}{3} + \frac{M_1}{6} = 0 + 1 = 1$ (upper part of line 3, column 8); and $M_{o1}'' = \frac{M_o}{6} + \frac{M_1}{3} = 0 + 2 = 2$ (lower part of line 3, column 8). The value of $\Delta y'/\Delta x$ (line 3, column 11) equals $M'_{o1}\beta_1$ by Eq. (9.34), which is $0.2(10^{-6})$. Since the slope at the fixed end is assumed to be zero, that is, $\theta_o = 0$ (line 2, column 17), $\Delta y_{o1}'' = 0$ (line 3, column 14). The deflection at point 1, by the equation for column 15 as given on page 164, equals $1.2(10^{-6}) + 0 + 0 = 1.2(10^{-6})$ (line 4, column 15). The change in the shear at point 1 equals $m_1 y_1 \omega^2$ by Eq. (9.32), which is $1.2(10^{-6}) \times 0.1295(10^6) = 0.1554$ (line 4, column 1).

The remainder of the table is worked out in a similar manner and may be followed by reference to the original equations or the summation given just before this example.

After both parts of the table are completed, the expressions for the shear and bending moment at the free end may be set up in terms of the shear and bending moment at the fixed end by using the bottom values in columns 2 and 5 for each part. In this example the moment at the free end is expressed in terms of the moment at the fixed end, that is, $M_3 = -0.0012M_o$; hence, the remainder is -0.0012. The shear at the free end could also be expressed in terms of the shear at the fixed end if desired.

The moment remainder curve is plotted in Fig. 9.5, and it may be seen that -0.0012 is quite small compared with the values for

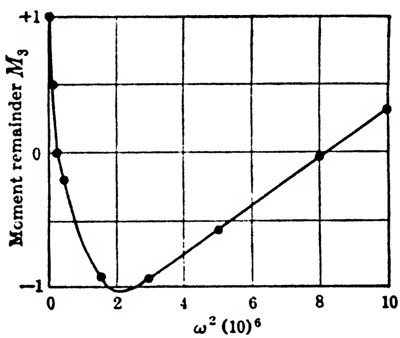

FIG. 9.5

other assumed frequencies, and that 4,775 cpm or a value slightly less than this represents the first natural frequency. The second (and only other, since there are only two masses) natural frequency occurs at 27,350 cpm, or at $\omega^2 = 8.2(10^6)$. It would be instructive for the reader to check this value by a similar calculation.

EXAMPLE 2

A three-bearing shaft with its dimensions and impeller weights is shown in Fig. 9.6. Determine its natural frequency neglecting the shaft weight. The position of the various points along the shaft that are used in Table 9.2 are shown in the figure.

The single weight of 125 lb is divided into two weights of 62.5 lb each located at points 4 and 5 to simplify the calculation. The masses and spring constants are found as in the previous example.

These are $m_1 = m_2 = 0.044$; $m_4 = m_5 = 0.162$; $\beta_1 = \beta_3 = 2.155(10^{-7})$; $\beta_2 = 0.719(10^{-7})$; $\beta_4 = \beta_6 = 1.436(10^{-7})$; $\beta_5 = 0.346(10^{-7})$. Also $(\Delta x)_1 = (\Delta x)_3 = 12$; $(\Delta x)_2 = 4$; $(\Delta x)_4 = (\Delta x)_6 = 8$; $(\Delta x)_5 = 4$.

These values are listed in the appropriate parts of Table 9.2, as in the previous example. A frequency of 11,600 cpm, which corresponds to $\omega^2 = 1.475(10^6)$, will be assumed for the lowest natural frequency or critical speed.

This problem is more complex than the previous one in that the system has two spans. Each span must be worked separately as in the previous example; hence, the calculation is twice as long.

Starting at the left end of the left span, it is known that the moment $M_o = 0$ and the deflection $y_o = 0$; but the shear V_o and the slope θ_o there are unknown. The general procedure is to solve

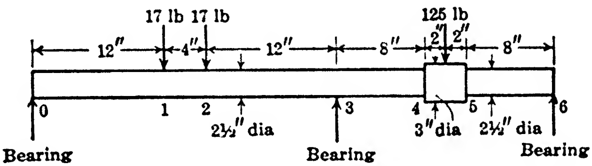

FIG. 9.6

each span in two parts, letting the unknowns in turn equal unity and the known terms equal zero. Thus, for the first part, assume $V_o = 1$, $M_o = \theta_o = y_o = 0$; and for the second part, assume $\theta_o = 1$, $V_o = M_o = y_o = 0$. The method of working the table follows that used in the preceding example.

After completing the two parts of the table for this span, it is possible to write the expressions for the moment M_3, slope θ_3, and deflection y_3 in terms of the unknowns V_o and θ_o. These equations are given on the left-span table.

It is known that the deflection at the middle support is zero, that is, $y_3 = 0$, which gives a relation between V_o and θ_o. ($V_o = -581{,}000\theta_o$.) Substituting this relation in the other two expressions gives θ_3 and M_3 in terms of θ_o ($\theta_3 = -0.996\theta_o$ and $M_3 = +76{,}600\theta_o$).

If we consider the right span, the only known condition at the left end is $y_3 = 0$; but the moment M_3 and the slope θ_3 are the same as at the right end of the left span and are known in terms of the slope at the left end of the beam θ_o. At the natural frequency the beam is in an indifferent state of equilibrium, and θ_o may have any value. It is convenient to consider it unity, in which case $M_3 = +76{,}600$ and $\theta_3 = -0.996$.

For the first part of the tabulation for this span, let $V_3 = 1$ and $M_3 = \theta_3 = y_3 = 0$; and for the second part, let $V_3 = y_3 = 0$ and $M_3 = +76{,}600$ and $\theta_3 = -0.996$.

After completing the two parts of the table for this span, expressions for the moment M_6 and the deflection y_6 in terms of V_3 and

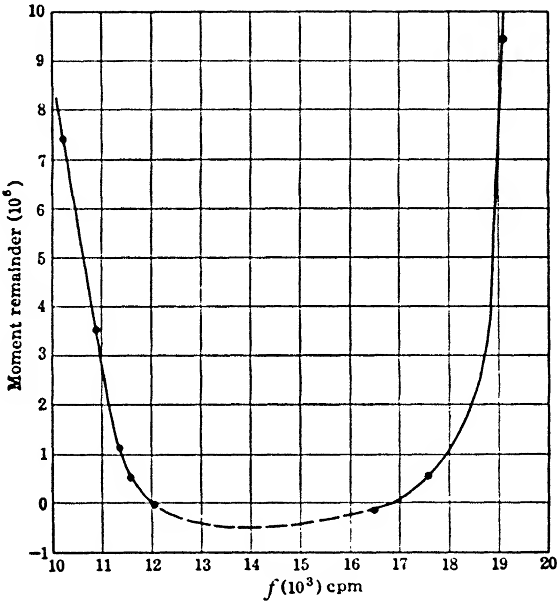

FIG. 9.7

θ_o (which is taken as unity) are obtained and given on the tabulation. It is known that the deflection y_6 equals zero; hence, this expression may be solved to find the shear at the middle bearing. Then, $V_3 = 1{,}398.000$. This value may then be used to find the moment M_6 at the right end of the system; thus, $M_6 = +471{,}000$.

If this moment had equaled zero, the assumed frequency would have been a natural one. Since it does not, a new frequency assumption must be made and the calculation repeated. The moment M_6 may be plotted against assumed frequencies to obtain a remainder curve. The points where this curve (shown in Fig. 9.7) crosses the zero axis represent the natural frequencies. For this

TABLE 9.2. Left Span

Line	n	(1) ΔV	(2) V	(3) Δx	(4) ΔM	(5) M	(6) $\frac{1}{3}M$	(7) $\frac{1}{6}M$	(8) M' M''	(9) (10^{-7}) β	(10) (10^{-7}) $\beta M'$ $\beta M''$	(11) (10^{-7}) $\frac{\Delta y'}{\Delta x}$	(12) Δx	(13) (10^{-7}) $\Delta y'$	(14) $\Delta y''$	(15) (10^{-7}) y	(16) $m\omega^2$	(17) (10^{-7}) θ
1			1.00															
2	0	0				0	0	0								0	0	0
3			1.00	12	12				2 4	2.155	4.31 8.62	4.31	12	51.72	0			
4	1	0.3357				12	4	2								51.72	64,900	
5			1.3357	4	5.343				6.890 7.781	0.719	4.95 5.59	17.88	4	71.52	0			
6	2	0.7998				17.343	5.781	2.890								123.24	64,900	
7			2.1355	12	25.626				12.943 17.213	2.155	27.85 37.10	51.32	12	615.84	0			
8	3	0				42.969	14.323	7.162			88.42 $=-\theta_3$					739.08	0	88.42
9			2.1355															
										(10^{-7})	(10^{9})	(10^{9})		(10^{9})		(10^{9})		(10^{9})
10			0															
11	0	0				0	0	0								0	0	1.0
12			0	12	0				0 0	2.155	0 0	0	12	0	12			
13	1	778,800				0	0	0								12	64,900	
14			778,800	4	3,115,200				519,200 1,038,400	0.719	0.03735 0.07470	0.03735	4	0.1494	4			
15	2	1,048,000				3,115,200	1,038,400	519,200								16.1494	64,900	
16			1,826,800	12	21,921,600				5,211,200 8,864,800	2.155	1.122 1.910	1.234	12	14.808	12			
17	3	0				25,036,800	8,345,600	4,172,800			3.144					42.9574	0	4.144
18			1,826,800															

$$\theta_3 = 0.000008842V_o + 4.144\theta_o, \qquad V_o = -\frac{42.9574\theta_o}{0.000073908} = -581{,}000\theta_o,$$

$$y_3 = 0.000073908V_o + 42.9574\theta_o = 0, \qquad \theta_3 = 0.000008842(-581{,}000\theta_o) + 4.144\theta_o = -0.996\theta_o,$$

$$M_3 = 42.969V_o + 25{,}036{,}800\theta_o, \qquad M_3 = 42.969(-581{,}000\theta_o) + 25{,}036{,}800\theta_o = 76{,}600\theta_o.$$

Table 9.2. Right Span

Line	n	(1) ΔV	(2) V	(3) Δx	(4) ΔM	(5) M	(6) $\frac{1}{3}M$	(7) $\frac{1}{6}M$	(8) M' M''	(9) (10^{-7}) β	(10) (10^{-7}) $\beta M'$ $\beta M''$	(11) (10^{-7}) $\frac{\Delta y'}{\Delta x}$	(12) Δx	(13) (10^{-7}) $\Delta y'$	(14) $\Delta y''$	(15) (10^{-7}) y	(16) $m\omega^2$	(17) (10^{-7}) θ
1			1.00															
2	3	0				0	0	0								0	0	0
3			1 00	8	8				1 333 2 667	1 436	1 913 3 826	1 913	8	15 304	0			
4	4	0 366				8	2 667	1 333								15 304	238,950	
5			1 366	4	5 464				4 911 5 821	0 346	1 700 2 017	7 493	4	29 972	0			
6	5	1 078				13 464	4 488	2 244								45 276	238,950	
7			2 444	8	19 552				9 990 13 249	1 436	14 32 19 01	23 43	8	187 44	0			
8	6	0				33 016	11 005	5 502								232 72	0	
9			2 444															
										(10^{-7})	(10^0)	(10^0)		(10^0)		(10^0)		(10^0)
10			0															
11	3	0				+76,600	25,533	12,766								0	0	−0.996
12			0	8	0				38,300 38,300	1.436	0 0055 0 0055	0 0055	8	0 044	−7 968			
13	4	−1,893,000				+76,600	25,533	12,766								−7 924	238,950	
14			−1,893,000	4	−7,572,000				−1,223,700 −2,485,700	0.346	−0 0435 −0 0861	−0 0325	4	−0 130	−3 984			
15	5	−2,889,000				−7,495,400	−2,498,467	−1,249,233								−12 038	238,950	
16			−4,773,000	8	−38,184,000				−10,111,717 −16,475,733	1 436	−1 451 −2 363	−1 5696	8	−12 557	−7 968			
17	6	0				−45,679,400	−15,226,500	−7,613,250								−32 563	0	
18			−4,773,000															

$$y_6 = 0.000023272V_3 - 32.563\theta_o = 0, \qquad V_3 = \frac{32.563\theta_o}{0.000023272} = 1,398,000\theta_o,$$

$$M_6 = 33.016V_3 - 45,679,400\theta_o, \qquad M_6 = 33.016(1,398,000\theta_o) - 45,679,400\theta_o = +471,000\theta_o.$$

example they are at 12,050 cpm and 16,850 cpm for the two lowest modes of vibration.

PROBLEMS

9.1. For a simple beam made of steel and having the dimensions and loads shown in Fig. 9.8, the deflections at the loads are

$$y_1 = 117.9(10^{-6})F_1 + 179.2(10^{-6})F_2,$$

$$y_2 = 179.2(10^{-6})F_1 + 301.8(10^{-6})F_2.$$

Determine the lowest natural frequency of the system (a) by the Rayleigh method and (b) by the Stodola method. *Ans.* 818 cpm.

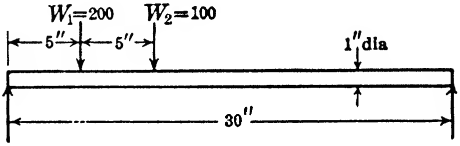

FIG. 9.8

9.2.* Determine the first lateral critical speed of the shaft shown in Fig. 9.9, neglecting the weight of the shaft itself. Use the following scales: space 1 in. = 10 in.; force 1 in. = 100 lb;

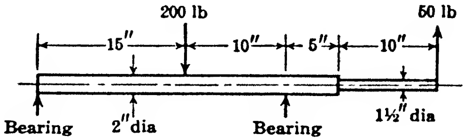

FIG. 9.9

pole distance p = 2 in.; area scale 1 in. = 20,000; pole distance p' = 1.45 in. Divide the M/I diagram into 10-in. lengths and work the problem on a sheet of 8½ by 11-in. paper held vertically. Use $E = 29(10^6)$. *Ans.* 2,000 rpm.

9.3.* A constant-diameter shaft has an impeller located approximately midway between two bearings. What would be the approximate effect in per cent on the lateral critical speed of (a) increasing

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the shaft diameter 20 per cent; (b) increasing the impeller weight 5 per cent; (c) decreasing the bearing span 10 per cent?

Ans. (a) Increase 44 per cent; (b) decrease 2.4 per cent; (c) increase 17 per cent.

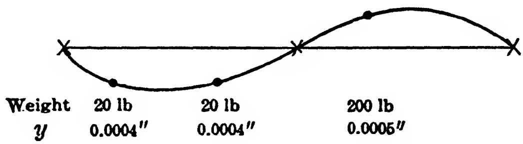

FIG. 9.10

9.4. Solve Prob. 9.2 by the general method to obtain the two natural frequencies. *Ans.* 1,960 cpm and 4,740 cpm.

9.5. The static deflection curve of a three-bearing shaft is shown in Fig. 9.10. If the weights and deflections are as shown there, find the critical speed based on this static deflection and the Rayleigh method. *Ans.* 8,500 cpm.

Chapter 10

BALANCING

10.1 *IMPORTANCE AND DEFINITIONS*

No forces will occur to cause vibration if it is possible to balance a machine completely. If unbalance is present, the machine will be vibrated, which may cause rubbing and excessive wear; moreover, forces are transmitted to the foundation and the ground, with the added result that energy or power is consumed with a consequent loss of efficiency. These effects are obviously undesirable and are to be avoided.

There are two general types of balancing problems that must be considered. One of these is the unbalance that occurs in the design of the machine, as, for example, in a multicylinder engine, a camshaft, and the like. If the member has pure rotation, it is possible to incorporate in the design one or more balance weights properly positioned to eliminate this unbalance, as will be described in the next section. The other type of problem is caused by manufacturing difficulties. The design of the machine may be such as to eliminate any predictable unbalance, but owing to necessary tolerance specifications, nonhomogeneity of the material in the form of gas inclusions, heavy spots, and other such factors, the member may be unbalanced. Relatively small weights not on the axis of rotation may set up large centrifugal forces when the machine runs at operating speed. For example, an unbalance of 1 in.-oz creates a centrifugal force of 23 lb at 3,600 rpm. This condition is remedied with balancing machines, which determine the amount and position of the unbalance that may be corrected by adding or removing weight at the proper points, as will be discussed in Sec. 10.3.

Machines may be divided into two general classes from the balancing standpoint, namely *rotating* and *reciprocating*. Examples of rotating machines are turbines, motors, and centrifugal machines in general, while reciprocating machines include gas and steam engines, reciprocating pumps, and the like. In general, the problem is more complex in the latter class, since the motion of the piston is reciprocating whereas that of the crank is rotating. Therefore, a centrifugal rotational force is set up by the crank, and the piston acceleration produces reciprocating inertia forces. One end of the connecting rod is attached to the piston, and the other end rotates with the crank. As developed in Sec. 8.2, it is customary to consider a portion of the connecting-rod weight to be concentrated at the crankpin and the remainder at the wrist pin or crosshead. It is not possible to balance out all the induced forces in a single-cylinder engine, but for some multicylinder engines, the unbalanced forces may be made to annul each other, and thus complete balance may be attained, as will be outlined in Sec. 10.5.

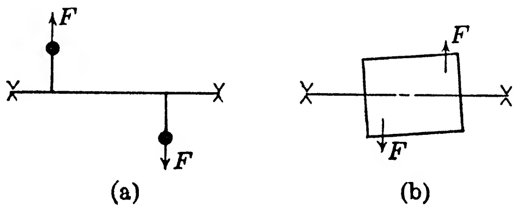

FIG. 10.1

The two types of balance are *static* and *dynamic*. Static balance occurs when the center of gravity of the member coincides with the axis of rotation. If the member does not move when placed on horizontal parallel guides, it is said to be in static balance. It is possible that a member will be in balance when it is static or not rotating, yet will not be in dynamic or rotating balance. Thus, in each case of Fig. 10.1, the center of gravity may lie on the axis of rotation so that the members are in static balance; but when a member is rotating, the equal centrifugal forces F that are developed will create a couple tending to roll the member "end over end," causing dynamic unbalance. In case (a), the magnitude of the unbalance may be known and corrected for by an opposing couple, as will be described in the next section. In case (b), where the axis of the cylinder does not coincide with the axis of rotation, the unbalanced forces may have to be determined with the aid of a balancing machine, as will be described in Sec. 10.3, before the size and position of the balance weights can be predicted.

More complete information concerning balancing may be found in E. W. Dalby, *Balancing of Engines*, Arnold & Co., London, 1929.

10.2 *ROTATIONAL BALANCE*

If a single weight W has its center of gravity r in. from the axis of rotation, as shown in Fig. 10.2, it may be put in static balance by placing another weight W' at radius r', as shown dotted in the figure,

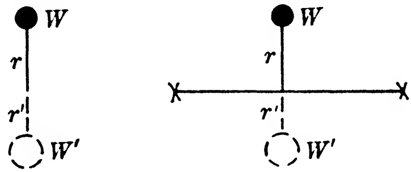

FIG. 10 2

so that $Wr = W'r'$. This will place the center of gravity at the center of rotation.

The centrifugal force developed by the weight W alone is $\left(\frac{W}{g}\right) r\omega^2$, and that of the balance weight W' is $\left(\frac{W'}{g}\right) r'\omega^2$. Equating these two forces, $Wr = W'r'$. Hence, the same correction gives both static and dynamic balance for this case.

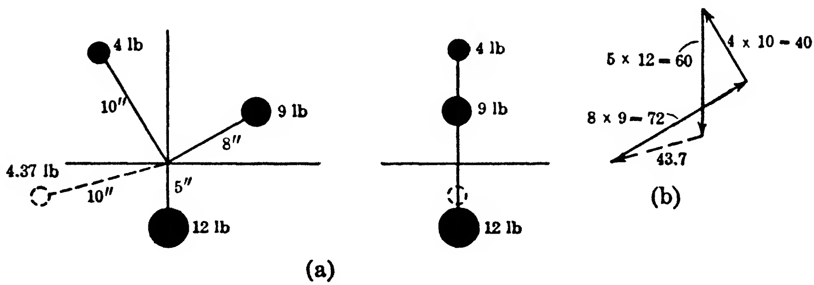

FIG. 10.3

If several masses are attached to a shaft in a single plane of rotation, they may be balanced by a single weight placed in that plane. The vectors representing the Wr values of each weight, acting in directions parallel to the forces, are added. The vector required to close the polygon represents the direction and magnitude of the Wr value of the balance weight that is required to give both static and dynamic equilibrium.

To illustrate, two views of a system of weights are shown in Fig.

10.3a, while part (b) of this figure shows the vector addition of the Wr values. The closing line scales 43.7 lb-in. If the balance weight is to act at a radius of 10 in., the size of the weight should be 4.37 lb. It acts at an angle of 14 deg with the horizontal and is shown dotted in part (a) of the figure.

Two weights acting in the same axial plane, as shown in Fig. 10.4, may have equal Wr values and thus be in static balance. When the shaft rotates, the centrifugal forces developed create a couple equal to Wra, thereby causing dynamic unbalance. To overcome this condition without destroying the static balance, it is necessary to introduce another equal couple in the same plane but acting in the opposite direction. The new couple should have a magnitude of $W'r'b$ equal to Wra. Any values of W', r', or b may be selected as long as the condition just mentioned is satisfied. The balancing couple is shown dotted in the figure.

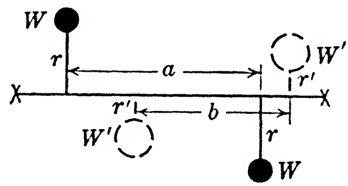

FIG. 10.4

The most usual and general case occurs when there are a number of weights placed at various points and angles along the shaft as shown in Fig. 10.5a. Generally there are two planes perpendicular to the axis of rotation in which it is convenient to place balancing weights. These will be designated as plane O and the reference plane R. For this example they will be located as shown in the figure. The axial distances of the weights from the reference plane R are designated by the letter a with appropriate subscripts.

The solution is based upon the principles developed for the previous cases and consists of first balancing the moments of Wr forces about the reference plane R by drawing a vector polygon of the Wra values. The closing line is the force to be placed in plane O and will equal $W_0r_0a_0$. Knowing a_0, the value and direction of W_0r_0 can be found. Placing this weight in the specified position will give dynamic balance of the member about the reference plane, but the member will still not be in static balance. To obtain static balance, an additional weight is placed in the reference plane;

and since the moment arm of this weight about the reference plane is zero, the dynamic balance already obtained will not be destroyed. The size and position of this weight is found by drawing a vector polygon of Wr forces, including that in plane O. The closing line gives the magnitude and direction of the Wr to be placed in the reference plane.

EXAMPLE

In Fig. 10.5, given the data in the first five columns of the following table, determine the size and position of the balance weights to be placed in planes O and R at a radius of 5 in. to obtain complete balance. Note that plus values of a are measured to the right from the reference plane, and negative values to the left. The angles θ are measured counterclockwise from a reference line OA.

Plane	Weight, W, lb	Radius, r, in.	Angle, θ, deg	Distance, a, in.	Wr, lb-in.	Wra, lb-in.^{2}
1	6	3	0	+15	18	+270
2	3	2	300	+ 5	6	+ 30
3	5	6	135	+10	30	+300
4	4	4	90	− 5	16	− 80
O		5		+20		
R		5		0		

The values of Wr and Wra listed in the last two columns of the table are calculated, and the Wra vector polygon is drawn as shown in Fig. 10.5b. Successive Wra vectors are laid off at the θ angles given above. Note that the vector of the No. 4 moment is negative and is therefore laid off opposite to the usual direction. The closing line representing $W_0r_0a_0$ (shown dashed) scales 129 lb-in.^{2}. Since a_0 is 20 in., $W_0r_0 = \frac{129}{20} = 6.45$ lb-in. As the weight is to be placed at a 5-in. radius, $W_0 = 6.45/5 = 1.29$ lb. The angle θ_0 as scaled from the polygon is 235 deg. The weight thus added gives dynamic balance of the member about a point in the reference plane.

The value for W_0r_0 of 6.45 lb-in. and for θ_0 of 235 deg just found may be added to the above table and the Wr polygon drawn, including W_0r_0, as shown in Fig. 10.5c. The closing line (shown dashed), representing W_Rr_R, scales 26.9 lb-in. As the weight is to be placed at a 5-in. radius, $W_R = 26.9/5 = 5.38$ lb. The angle θ_R as measured on the diagram is 278 deg. As noted previously, this added weight does not destroy the dynamic balance about the reference plane, as the moment arm a_R is zero; but it does give the member static balance. With the two weights added, the member is in complete balance.

The above work may be done mathematically in tabular form by adding the horizontal and vertical components of the Wra and Wr

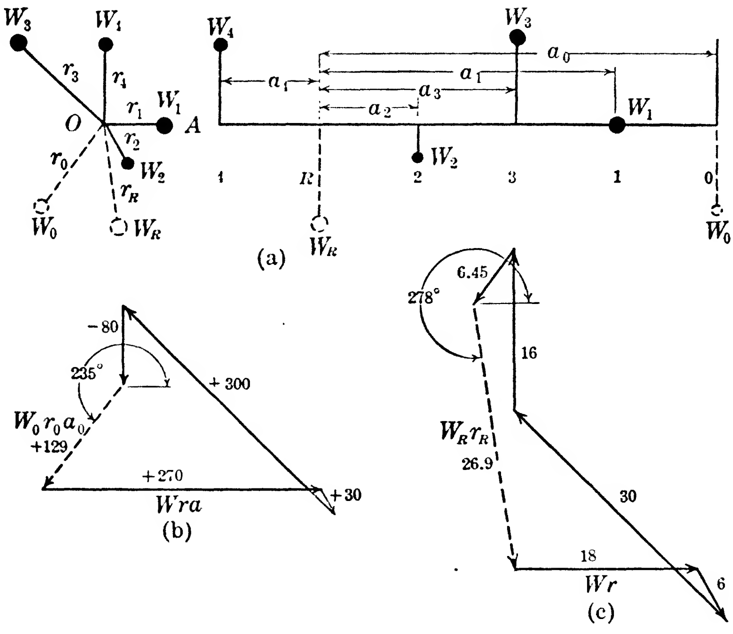

FIG. 10.5

vectors to find the components of $W_0r_0a_0$ and W_Ra_R and the values of θ_0 and θ_R.*

It should be observed that if a rotor is placed in balance for one speed, it will be in balance for any other speed, since the ω^2 term is common to all the masses. An exception to this rule would occur when the speeds are high enough to cause some distortion in the rotor shape. This distortion may be either permanent or within the limit of elastic action.

* C. W. Ham and E. J. Crane, *Mechanics of Machinery*, McGraw-Hill, New York, 1938, pp. 370–373.

Furthermore, if the member is in balance at constant speed, it will be in balance during periods of acceleration and deceleration. During those periods, tangential forces of magnitude $r\alpha$ act on the weights. Since these forces are uniformly perpendicular to the centrifugal forces, they will automatically be in balance, that is, the directions are merely turned through 90 deg.

10.3 *CORRECTIVE BALANCING AND BALANCING MACHINES*

It was shown in the previous section that complete balance can be obtained for any weight distribution by placing weights in two convenient planes of a rotor. In that section, the magnitude and position of the unbalance were known. The problem to be considered here is to determine the size and position of the balance weights to be placed in the two planes that will counteract the effect of unknown unbalance in a rotor, which may be due to such factors as small eccentricities, nonhomogeneity of the material, and the like.

The early procedure, and one that is still used in some cases for balancing rotors in the field, is first to give the rotor a static balance if possible. The rotor is then mounted in bearings and run at a high speed in one direction, and the high spot of the rotor is marked with a scriber. The rotor then runs in the opposite direction at the same speed and the high spot marked again. If the critical speed of the rotor has not been exceeded, the correction weight should be placed opposite the mid-point between the two high spots thus found. This method is not too reliable, and the size of the balance weight must be found by trial.

In this method it is necessary to run the rotor in both directions and take the mid-point of the high spots, since the high spot will not coincide with the position of the unbalance. Reference to Sec. 5.2 and Fig. 5.3 shows that the high spot lags the center of gravity or unbalance by an amount that depends upon the speed of rotation, amount of damping, and the like. At the critical speed the amount of lag is 90 deg, and in that vicinity it changes rapidly as shown by the illustration.

Small rotors may be balanced more rapidly and accurately in a machine. There are a great variety of these on the market,* many

* J. P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill, New York, 1947, pp. 292–309.

of which are designed for production balancing of particular types of rotors. The fundamental principle on which they are based will be outlined here.

The rotor is placed in a light frame that may be pivoted at the balance planes and is supported on relatively weak springs at the bearings. Some means of measuring the amplitude of vibration of the frame is provided: it may be done electrically, by means of mechanics' dial guages, or other instruments. In Fig. 10.6 the rotor

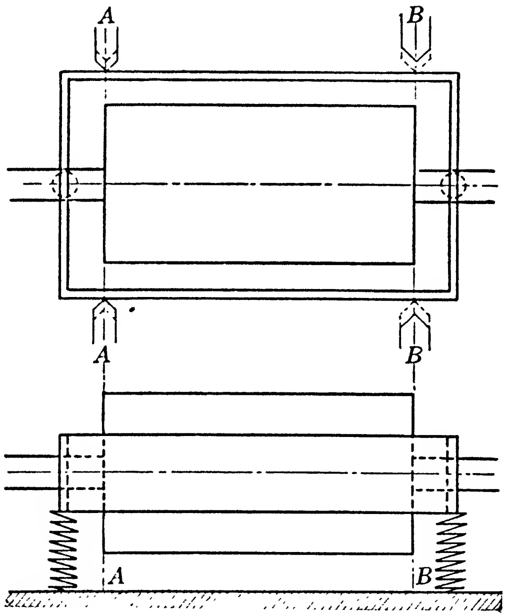

FIG. 10.6

is shown in place, the frame being pivoted at balance plane AA. The rotor is brought up to a speed greater than the natural frequency of the rotor-frame-spring combination, and the speed is then slowly decreased. The maximum amplitude recorded as the rotor passes through this natural frequency is a measure of the unbalance about plane AA. By trial, the magnitude and position of the correct balance weight to be placed in plane BB can be obtained. Then, by releasing the pivot in plane AA and pivoting the frame at BB, the above process may be repeated to find the size and position of the balance weight to be placed in plane AA to give balance about plane *BB*. The rotor will then be in complete balance.

In determining the correct balance weight to be used, a weight of convenient size is moved at a constant radius to various angular positions, and a curve of angle from some reference line on the rotor to the weight position is plotted against the maximum amplitude of vibration, as shown in Fig. 10.7a. The correct position of the balance weight corresponds to the lowest point on this curve (120 deg in the figure). Since the maximum amplitude never reaches zero, it is obvious that the size of the weight is not correct. By plotting a curve of maximum amplitude against the size of the correction weight placed at the 120-deg position, as shown in Fig. 10.7b, and noting where it is zero, the correct magnitude (in this case 6.8 oz) is obtained.

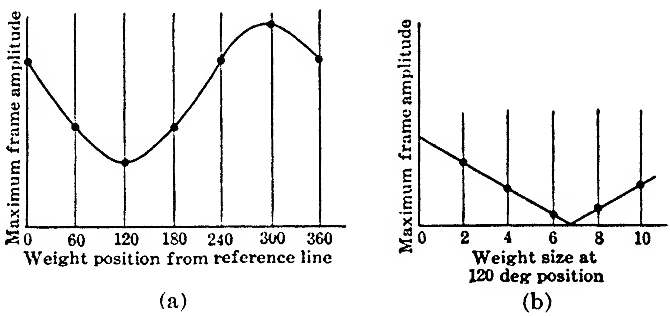

Fig. 10.7

The procedure just outlined is slow and cumbersome. It may be shortened, however, to taking four runs for each balance plane and using the graphical method developed by Den Hartog* and illustrated in Fig. 10.8.

The four runs are made with the following conditions: (a) no balance weight used; (b) a balance weight of known size placed at a convenient radius from the axis of rotation; (c) the same weight placed at the same radius, but diametrically opposite to the position of run (b); (d) the same weight placed at the same radius at any position between the positions of runs (b) and (c). The maximum amplitude of vibration of the frame is recorded for each run. The procedure can best be explained with the aid of an example.

* J. P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill, New York, 1947, pp. 296–297.

EXAMPLE

The maximum amplitudes recorded for the four runs mentioned above are

(a) 0.030 in. for the rotor without a balance weight.
(b) 0.040 in. for a 3-oz weight in the 0-deg position.
(c) 0.025 in. for a 3-oz weight in the 180-deg position.
(d) 0.032 in. for a 3-oz weight in the 150-deg position.

Determine the size and position of the correct balance weight to be used.

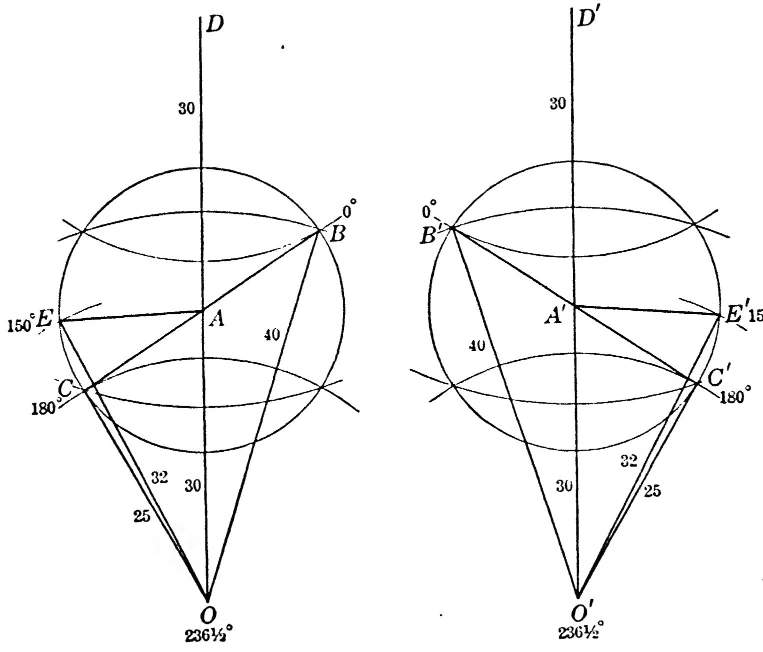

FIG. 10.8

With the data given, two possible diagrams may be drawn, as shown in Fig. 10.8, and each will give the same result. The diagram on the right is labeled with primed letters corresponding to the unprimed letters of the diagram on the left.

The following procedure applies equally well to either diagram:

(1) Lay off line $OA = AD = 30$ units [run (a)].
(2) Draw arcs of radius $DC = OB = 40$ units [run (b)].
(3) Draw arcs of radius $OC = DB = 25$ units [run (c)].
(4) Locate points B and C at intersection of arcs.

(5) Draw a circle with center at A and radius of AC or AB (which are the same).

(6) Draw arc of radius $OE = 32$ units for run (d) to intersect the circle of step 5.

(7) The angle BAE should equal the angular position of the balance weight for run (d) from run (b).

(8) The magnitude of the correct balance weight is given by the line AO drawn to the same scale as that by which AB or AC represents the size of the balance weight used, and it should be placed at the angle BAO with the position of the weight for run (b).

The explanation of the diagram is as follows: The line OA represents the unknown unbalance in the rotor. For run (b), the forces acting on the rotor are the initial unbalance OA and the effect of the added balance weight AB, giving the resultant amplitude OB of 0.040 in. For run (c) the forces acting are the initial unbalance OA and the effect of the added balance weight AC. The force AB acts directly opposite to that of AC, since the two positions of the balance weight are diametrically opposite on the rotor. The fourth run (d) is made to establish the direction in which angles are to be measured, that is, clockwise or counterclockwise. The added unbalance AE equals AB or AC and acts at the known angle of BAE, or 150 deg.

The distance AB, AC, or AE scales as 15 units on the figure; hence, the initial balance or size of the correction weight to be placed at the fixed radius is AO/AB times the size of the trial weight. Thus, $\frac{30}{15} \times 3 = 6$ oz. The correction weight should be placed at $236\frac{1}{2}$ deg with the weight position for run (b).

Rotors that have their mass essentially in one plane, as for example, airplane propellers, require only a static balance, since the moment arms producing dynamic unbalance are negligible. This balancing is accomplished by mounting the member on parallel horizontal guide rails and adding weight until the member remains in any position in which it may be placed.

10.4 *SINGLE-CYLINDER ENGINE*

A. *Piston Displacement, Velocity, and Acceleration*

Before considering the forces acting on the engine, it is desirable to derive an expression for the acceleration of the piston in terms of the crank angle and angular velocity of rotation. In Fig. 10.9, let r be the length of the crank, nr the length of the connecting rod, and x the distance from the center of the crankshaft A to the crosshead B for any position of the mechanism. The stroke of the piston is DE and equals $2r$. When the piston is at midstroke M, the distance x equals nr. Hence, the piston displacement s from the midposition equals $x - nr$.

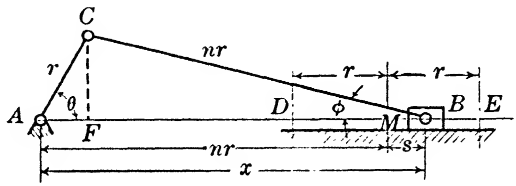

FIG. 10.9

If the angle between the crank and piston motion is θ and that between the connecting rod and piston motion is ϕ, the displacement s is given by

$$s = x - nr = r \cos \theta + nr \cos \phi - nr.$$

Now, $\sin \phi = \dfrac{CF}{nr}$ and $\sin \theta = \dfrac{CF}{r}$; hence, $\sin \phi = \dfrac{\sin \theta}{n}$. Since $\cos^2 \phi = 1 - \sin^2 \phi$, then $\cos \phi = \sqrt{1 - \dfrac{\sin^2 \theta}{n^2}}$; and

$$\begin{aligned} s &= r \cos \theta + nr \sqrt{\frac{n^2 - \sin^2 \theta}{n^2}} - nr \\ &= r(\cos \theta + \sqrt{n^2 - \sin^2 \theta} - n). \end{aligned} \tag{10.1}$$

If the crank rotates with a constant angular velocity ω, then $\theta = \omega t$ and $d\theta/dt = \omega$. Differentiating Eq. (10.1) with respect to time t to find the velocity of the piston v,

$$\begin{aligned} v = \frac{ds}{dt} &= r \left[- \sin \theta \frac{d\theta}{dt} + \tfrac{1}{2}(n^2 - \sin^2 \theta)^{-\frac{1}{2}}(-2 \sin \theta \cos \theta) \frac{d\theta}{dt} \right] \\ &= -r\omega \left[\sin \theta + \frac{\sin 2\theta}{2 \sqrt{n^2 - \sin^2 \theta}} \right]. \end{aligned} \tag{10.2}$$

The minus sign specifies the direction of the velocity and may be neglected. If the $\sin^2 \theta$ term in the radical is neglected, the error is small, since its maximum value is 1. Equation (10.2) may then be written in the form

$$v = r\omega\left(\sin\theta + \frac{\sin 2\theta}{2n}\right). \tag{10.3}$$

If Eq. (10.3) is differentiated with respect to time t, an expression for the acceleration of the piston a results; thus,

$$a = \frac{dv}{dt} = r\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right). \tag{10.4}$$

When n equals 4, the maximum error in the acceleration resulting from the use of the approximate equation is only about 0.6 per cent.

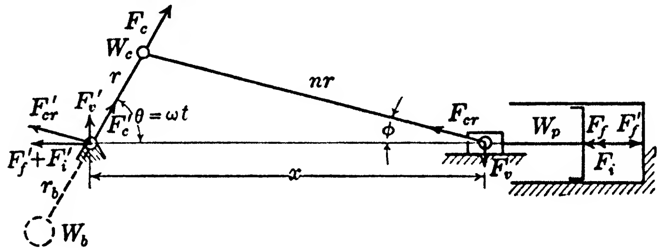

FIG. 10.10

B. *Force Analysis*

If the connecting-rod mass is divided between the crankpin and piston as outlined in Sec. 8.2, the forces acting on the moving parts of the engine are

(1) Centrifugal force F_c due to the crank weight.

(2) Fluid pressure F_f due to the expanding gas or steam in the cylinder.

(3) Inertia force F_i causing the acceleration or deceleration of the piston.

These forces as they act on the moving parts of the engine are shown in Fig. 10.10. Their magnitude and the manner in which they act on the frame of the engine will now be considered.

(1) *Centrifugal Force F_c.* The centrifugal action of the weight of the crank, crankpin, and portion of the connecting rod acts radially

away from the center of rotation. Its magnitude equals $\left(\frac{W_c}{g}\right) r\omega^2$, where W_c is the weight of the rotating parts considered to be concentrated at the crank radius r. The action of this force on the frame is a pull in the same direction at the crankshaft bearings as shown on Fig. 10.10 by the force F_c'.

It may be equalized by a balance weight placed opposite to the crank. The product of its weight and the radius of its center of gravity should equal that of the unbalance; that is, $W_b r_b = W_c r$. The balance weight is shown in dotted lines on Fig. 10.10.

(2) *Fluid Force F_f.* The force due to the expanding steam or gas acts on the piston, as shown in the figure. Its magnitude equals $\left(\frac{\pi}{4}\right) d_p^2 p_f$, where d_p is the diameter of the piston and p_f the pressure exerted by the fluid at any moment. This force is transmitted to the crosshead or wrist pin, where it is broken up into two components F_{cr} and F_v. The force transmitted along the connecting rod then is $F_{cr} = \frac{F_f}{\cos \phi}$ and hence is greater than F_f. The other component F_v represents the push of the crosshead on its guide; thus, $F_v = F_{cr} \sin \phi = F_f \tan \phi$.

The connecting-rod component F_{cr} is transmitted along the connecting rod and down the crank. It acts on the crankshaft bearings in an amount and direction equal and parallel to the component at the wrist pin or crosshead and is shown on the sketch as force F_{cr}'. The horizontal component of this action F_f' equals F_f and acts in the same direction, while the vertical component F_v' equals F_v, but acts upward. Hence, the forces F_v and F_v' acting at a distance x apart form a couple that is equal to the torque developed by the engine. As long as the engine delivers power or torque, this couple must necessarily be present. The horizontal component F_f' is balanced in the engine frame by an equal and opposite force exerted by the expanding fluid acting on the cylinder head.

As the magnitude of the fluid force continually varies with the pressure in the cylinder, the torque on the engine produced by the F_v components will be vibratory. This vibration cannot be eliminated in a single-cylinder engine.

(3) *Inertia Force F_i.* The force F_i is induced by the acceleration or deceleration of the piston as given by the fundamental equation

$F_i = \left(\frac{W_p}{g}\right) a_p$, where a_p is the acceleration of the piston as given by Eq. (10.4) and W_p is the weight of the reciprocating parts including a portion of the connecting rod weight. Hence,

$$F_i = \frac{W_p}{g} r\omega^2 \left(\cos\theta + \frac{\cos 2\theta}{n}\right). \tag{10.5}$$

It may be noted that if the connecting rod is infinitely long, this equation reduces to $F_i = \left(\frac{W_p}{g}\right) r\omega^2 \cos\theta$, and the force is harmonic.

Equation (10.5) may be written

$$F_i = \frac{W_p}{g} r\omega^2 \cos\theta + \frac{W_p}{g}\frac{r}{4n}(2\omega)^2 \cos 2\theta. \tag{10.6}$$

In this form it may be observed that the inertia force is made up of two terms; the first is known as the primary inertia force, and the other as the secondary inertia force. The secondary inertia force acts at twice the frequency of the primary, is smaller in magnitude, and is due to the angularity of the connecting rod, or the variation in the angle ϕ which prevents the piston from moving with simple harmonic motion.

When the piston is accelerating, the force acts on it in the direction of motion; and when the piston is decelerating, the force acts opposite to the direction of motion. The action of this force on the frame of the machine is similar to that of the fluid force F_f, except that the horizontal component at the crankshaft bearings is not balanced out by an equal force on the cylinder head; hence, its effect on the frame is relatively large.

The vertical component of the inertia force acts to increase or decrease the turning couple exerted by the fluid force and is usually neglected in balancing.

The horizontal inertia force at the bearings F_i' can be balanced to some extent. By increasing the magnitude of the balance weight by an amount W_b', some or all of the primary inertia force may be balanced out, since the horizontal component of the balance-weight force and the primary inertia force are both functions of $\cos\theta$. The difficulty with this procedure is that a new vertical unbalance of $\left(\frac{W_b'}{g}\right) r\omega^2 \sin\theta$ is introduced, which creates an additional torque on the engine. This matter will be discussed at greater length in the following example. Since the frequency of the secondary inertia force is twice that of the engine, it is difficult to reduce its unbalance.

To summarize, it is impossible to balance completely a single-cylinder engine.

Example

To illustrate the preceding discussion, consider a single-cylinder steam engine, given the following information:

Engine speed, 225 rpm.
Crank length, 10 in.
Connecting-rod length, 40 in.
Connecting-rod weight, 245 lb.
Weight of reciprocating parts (piston, rod, crosshead), 230 lb.
Equivalent crank weight at 10 in. radius, 151 lb.
Distance of center of gravity of connecting rod from crankpin center, 13.3 in.

The fluid force will be neglected in this example and also the vertical components of the inertia forces. Then:

$$\text{Angular velocity of crank} = \omega = \frac{225}{60}\,2\pi = 23.55 \text{ rad per sec.}$$

$$\text{Ratio of connecting-rod length to crank length} = n = \frac{40}{10} = 4.$$

$$\text{Weight of connecting rod acting at piston} = \left(\frac{13.3}{40}\right) \times 245 = 81.5 \text{ lb.}$$

$$\text{Weight of connecting rod acting at crankpin} = 245 - 81.5 = 163.5 \text{ lb.}$$

$$\text{Total reciprocating weight} = W_p = 230 + 81.5 = 311.5 \text{ lb.}$$

$$\text{Total rotating weight} = W_c = 151 + 163.5 = 314.5 \text{ lb.}$$

$$\text{Centrifugal force at crank} = \left(\frac{W_c}{g}\right) r\omega^2 = \left(\frac{314.5}{386}\right) 10(23.55)^2 = 4{,}520 \text{ lb.}$$

$$\text{Primary inertia force} = \left(\frac{W_p}{g}\right) r\omega^2 \cos\theta$$

$$= \left(\frac{311.5}{386}\right) 10(23.55)^2 \cos\theta$$

$$= 4{,}470 \cos\theta \text{ lb.}$$

$$\text{Secondary inertia force} = \left(\frac{W_p}{g}\right)\left(\frac{r}{4n}\right)(2\omega)^2 \cos 2\theta$$

$$= \left(\frac{311.5}{386}\right)\left(\frac{10}{4\times 4}\right)(2\times 23.5)^2 \cos 2\theta$$

$$= 1{,}120 \cos 2\theta \text{ lb.}$$

$$\text{Total inertia force} = 4{,}470 \cos\theta + 1{,}120 \cos 2\theta.$$

To summarize the forces acting on the crankshaft bearings as calculated above, it is convenient to plot them on a polar diagram,

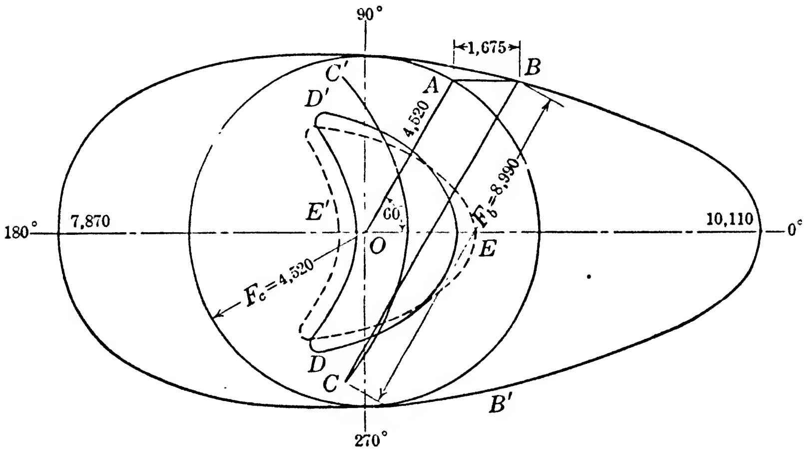

FIG. 10.11

as shown in Fig. 10.11. The inertia forces for various values of θ may be calculated in table form. Positive and negative values resulting from the sign of the cosine function indicate the direction of action (right and left).

A circle with a radius of 4,520 units representing the centrifugal force of the crank F_c is first drawn about pole O. Then, for various values of θ the inertia force is laid off horizontally from the corresponding point on this circle. For example, when $\theta = 60$ deg,

$F_i = 2{,}235 - 560 = 1{,}675$ lb. Where the 60-deg radial line intersects the F_c circle (at A) a horizontal line 1,675 units long is laid off to the right to locate point B. The resultant load on the bearings for this position is then OB, or approximately 5,550 lb. By a similar procedure, other points are obtained to draw the curve through B and B', which represent the unbalanced centrifugal and inertia forces. The maximum horizontal unbalance force $= 4{,}520 + 4{,}470 + 1{,}120 = 10{,}110$ lb.

Assume now that a balance weight is placed opposite the crank (shown dotted on Fig. 10.10). By varying the size and/or the radius to its center of gravity, any desired balance force can be obtained. For any engine the balance force should balance the centrifugal force completely and part of the primary inertia force so as to make the resultant load on the bearings, and hence foundation, a minimum. The effect of the balance force on the diagram of Fig. 10.11 is to produce a force acting at the angle θ with the horizontal, starting from the previously found point (B in the sample construction).

If the balance weight is to counteract the centrifugal force and all the maximum primary inertia force, its magnitude F_b must be $4{,}520 + 4{,}470 = 8{,}990$ lb. On a line drawn at a 60-deg angle with the horizontal from B, the distance of 8,990 units is laid off to obtain point C. Through similar points the curve through C and C' may be drawn. It retraces itself and, hence, appears as a line rather than as a closed curve, giving a smaller resultant force than when there is with no balance weight. As mentioned previously, the force in a horizontal direction is very small, but it is fairly large in the vertical direction.

The optimum condition will occur when the maximum horizontal and vertical forces are approximately equal and a minimum. For the conditions of this example, the optimum condition is found to occur when the balance weight overcomes the centrifugal force and from 0.6 to 0.7 of the maximum primary inertia force. The solid curve through D and D' is the curve of the resultant force when the balance weight annuls F_c and 0.7 of the maximum primary inertia force; whereas the dotted curve through E and E' is for a balance weight annulling F_c and 0.6 of a maximum primary inertia force. Both have approximately the same maximum and nearly equal horizontal and vertical maximums. Generally, the balance-weight size is taken so as to care for the centrifugal force and two thirds of the maximum primary inertia force for steam engines.

10.5 *MULTICYLINDER ENGINES*

In some cases it is possible to cancel some or all of the unbalanced forces in multicylinder engines by the proper positioning of the cranks of the various cylinders. In addition to balancing the forces, the couples created by these forces along the crankshaft must be considered and, if possible, made equal to zero.

For a given crankshaft the angles between the various cranks is fixed and constant, although their angle θ with a fixed radial reference line continually varies as the shaft rotates. Thus, in Fig. 10.12, which represents a four-cylinder engine, the angles ψ, which

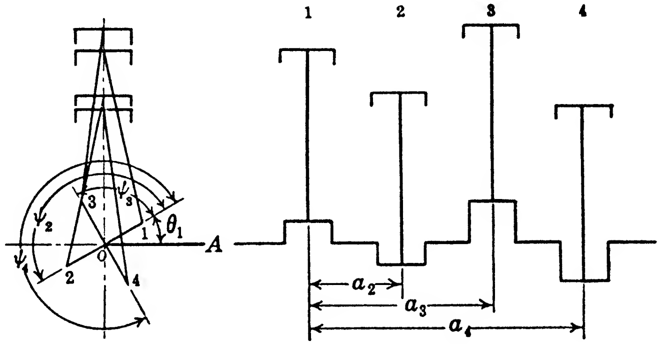

FIG. 10.12

the various cranks make with the No. 1 crank, is constant, but the angles θ, which they make with the radial reference line OA, varies as the crank rotates. The position of any crank θ is determined by the addition of the angles ψ and θ_1.

The distances from the center line of the No. 1 cylinder to the center line of the other cylinders is dimension a with the appropriate subscript, as shown in the figure.

The primary inertia force, as given by the first term of Eq. (10.6), is $\dfrac{W_p}{g} r\omega^2 \cos\theta$. Considering all the cranks of the engine,

$$F_p = \frac{W_{p_1}}{g} r_1\omega^2 \cos\theta_1 + \frac{W_{p_2}}{g} r_2\omega^2 \cos\theta_2 + \cdots .$$

Since $\theta_1 = \theta_1 + \psi_1$ (note that $\psi_1 = 0$), $\theta_2 = \theta_1 + \psi_2, \cdots,$

$$F_p = \frac{W_{p_1}}{g} r_1\omega^2 \cos(\theta_1 + \psi_1) + \left(\frac{W_{p_2}}{g}\right) r_2\omega^2 \cos(\theta_1 + \psi_2) + \cdots .$$

But

$$\cos(\theta_1 + \psi) = \cos\theta_1 \cos\psi - \sin\theta_1 \sin\psi;$$

hence,

$$F_p = \frac{\omega^2}{g}\cos\theta_1(W_{p_1}r_1\cos\psi_1 + W_{p_2}r_2\cos\psi_2 + \cdots) -$$

$$\frac{\omega^2}{g}\sin\theta_1(W_{p_1}r_1\sin\psi_1 + W_{p_2}r_2\sin\psi_2 + \cdots).$$

It may be observed that the cosine and sine terms of this equation represent, respectively, the horizontal and vertical components of the primary force; so that the equation may be broken into the two equations that follow:

$$F_{pH} = \frac{\omega^2}{g}\cos\theta_1 \Sigma W_p r\cos\psi, \tag{10.7}$$

$$F_{pV} = \frac{\omega^2}{g}\sin\theta_1 \Sigma W_p r\sin\psi. \tag{10.8}$$

If the moments of these primary forces are taken about the No. 1 cylinder, the primary couple is

$$C_p = \frac{W_{p_1}}{g}r_1\omega^2 a_1\cos\theta_1 + \frac{W_{p_2}}{g}r_2\omega^2 a_2\cos\theta_2 + \cdots.$$

Following through steps similar to those above, the horizontal and vertical components of this couple are

$$C_{pH} = \frac{\omega^2}{g}\cos\theta_1 \Sigma W_p ra\cos\psi, \tag{10.9}$$

$$C_{pV} = \frac{\omega^2}{g}\sin\theta_1 \Sigma W_p ra\sin\psi. \tag{10.10}$$

The secondary inertia force, as given by the second term of Eq. (10.6), may be written as $\frac{W_p}{g}\frac{r}{n}\omega^2\cos 2\theta$. Considering all the cranks,

$$F_s = \frac{W_{p_1}}{g}\frac{r_1}{n_1}\omega^2\cos 2\theta_1 + \frac{W_{p_2}}{g}\frac{r_2}{n_2}\omega^2\cos 2\theta_2 + \cdots.$$

Since $\theta_1 = \theta_1 + \psi_1$ (again $\psi_1 = 0$), $\theta_2 = \theta_1 + \psi_2$, and so on. Then,

$$F_s = \frac{W_{p_1}}{g}\frac{r_1}{n_1}\omega^2\cos(2\theta_1 + 2\psi_1) + \frac{W_{p_2}}{g}\frac{r_2}{n_2}\omega^2\cos(2\theta_1 + 2\psi_2) + \cdots,$$

and

$$F_s = \frac{\omega^2}{g}\cos 2\theta_1\left(W_{p_1}\frac{r_1}{n_1}\cos 2\psi_1 + W_{p_2}\frac{r_2}{n_2}\cos 2\psi_2 + \cdots\right) -$$

$$\frac{\omega^2}{g}\sin 2\theta_1\left(W_{p_1}\frac{r_1}{n_1}\sin 2\psi_1 + W_{p_2}\frac{r_2}{n_2}\sin 2\psi_2 + \cdots\right).$$

This equation may be separated into horizontal and vertical components as for the primary forces; thus,

$$F_{sH} = \frac{\omega^2}{g}\cos 2\theta_1 \Sigma W_p \frac{r}{n}\cos 2\psi$$

$$= \frac{\omega^2}{g}\cos 2\theta_1 \Sigma W_p \frac{r}{n}(\cos^2\psi - \sin^2\psi), \tag{10.11}$$

$$F_{sV} = \frac{\omega^2}{g}\sin 2\theta_1 \Sigma W_p \frac{r}{n}\sin 2\psi$$

$$= \frac{\omega^2}{g} 2\sin 2\theta_1 \Sigma W_p \frac{r}{n}\sin\psi\cos\psi. \tag{10.12}$$

Taking moments of the secondary forces about the No. 1 cylinder as before, we obtain

$$C_{sH} = \frac{\omega^2}{g}\cos 2\theta_1 \Sigma W_p \frac{r}{n} a\cos 2\psi$$

$$= \frac{\omega^2}{g}\cos 2\theta_1 \Sigma W_p \frac{r}{n} a(\cos^2\psi - \sin^2\psi), \tag{10.13}$$

$$C_{sV} = \frac{\omega^2}{g}\cos 2\theta_1 \Sigma W_p \frac{r}{n} a\sin 2\psi$$

$$= \frac{\omega^2}{g} 2\sin 2\theta_1 \Sigma W_p \frac{r}{n} a\sin\psi\cos\psi. \tag{10.14}$$

If the eight equations (10.7) to (10.14) equal zero, the engine is in balance. If they do not, the amount of unbalance in terms of θ_1 may be found from them.

Example

Determine the balance conditions in a four-cylinder engine for which the following data are available. The angular position of the cranks is 0 deg, 180 deg, 180 deg, 0 deg. The weight of the reciprocating parts (including portion of connecting-rod weight) is 2 lb. The engine speed is 2,000 rpm. The connecting-rod length is 11 in. The crank radius is $2\frac{1}{4}$ in. The spacing of the cylinder center lines is $3\frac{1}{2}$ in., 4 in., $3\frac{1}{2}$ in.

It may be noted that for this example the values of W_p, r, and n are the same for all the cranks, and may be neglected except when the amount of unbalance is being calculated. The balance may be investigated by a tabulation as shown below:

	Crank 1	Crank 2	Crank 3	Crank 4	Summation	Reference equation
ψ.	0	180	180	0		
$\cos\psi$	+1	−1	−1	+ 1	0	(10.7)
$\sin\psi$	0	0	0	0	0	(10.8)
a..	0	$3\frac{1}{2}$	$7\frac{1}{2}$	11		
$a\cos\psi$	0	$-3\frac{1}{2}$	$-7\frac{1}{2}$	+11	0	(10.9)
$a\sin\psi$	0	0	0	0	0	(10.10)
$\cos^2\psi$	1	1	1	1		
$\sin^2\psi$	0	0	0	0		
$\cos^2\psi - \sin^2\psi$	1	1	1	1	4	(10.11)
$\sin\psi\cos\psi$.	0	0	0	0	0	(10.12)
$a(\cos^2\psi - \sin^2\psi)$	0	$3\frac{1}{2}$	$7\frac{1}{2}$	11	22	(10.13)
$a\sin\psi\cos\psi$	0	0	0	0	0	(10.14)

From this tabulation it is apparent that the engine is in balance for all forces and couples except the secondary horizontal force and the secondary horizontal couple.

Applying Eq. (10.11), the magnitude of the secondary horizontal inertia force in terms of θ_1 is

$$F_{sH} = \frac{\omega^2}{g}\cos 2\theta_1 W_p \frac{r}{n}\,\Sigma(\cos^2\psi - \sin^2\psi)$$

$$= \left(\frac{2{,}000}{60}\,2\pi\right)^2 \frac{2}{386}\,\frac{2\frac{1}{4}}{11/2\frac{1}{4}}\,(\cos 2\theta_1)(4)$$

$$= 417.5\cos 2\theta_1.$$

Applying Eq. (10.13), the magnitude of the secondary horizontal inertia couple in terms of θ_1 is

$$C_{sH} = \frac{\omega^2}{g}\,W_p\,\frac{r}{n}\cos 2\theta_1\Sigma a(\cos^2\psi - \sin^2\psi),$$

$$C_{sH} = \left(\frac{2{,}000}{60}\,2\pi\right)^2 \frac{2}{386}\,\frac{2\frac{1}{4}}{11/2\frac{1}{4}}\,(\cos 2\theta_1)(22)$$
$$= 2{,}300 \cos 2\theta_1.$$

PROBLEMS

10.1. Given the following data for two weights on a shaft, find the balance weights to be placed in the O and R planes. All distances are measured from the reference plane R.

Plane	W	θ	r	a
1	3	0	1	+5
2	4	90	$\frac{1}{2}$	−10
O			2	+15
R			2	0

Ans. $W_O = 0.83$ lb, $\theta_O = 127$ deg; $W_R = 1.95$ lb, $\theta_R = 239$ deg.

10.2. The conditions are the same as in Prob. 10.1, except that the data are as follows:

Plane	W	θ	r	a
1	32.2	45	2	+10
2	64.4	180	1	−5
O			3	+15
R			3	0

Ans. $W_O = 20$ lb, $\theta_O = 210$ deg; $W_R = 24.2$ lb, $\theta_R = 348$ deg.

10.3. The conditions are the same as in Prob. 10.1, except that the data are as follows:

Plane	W	θ	r	a
1	50	0	2	+10
2	25	90	4	−10
O			3	−20
R			3	0

Ans. $W_O = -23.57$ lb, $\theta_O = 135$ deg; $W_R = 53$ lb, $\theta_R = 198$ deg.

10.4. The conditions are the same as in Prob. 10.1, except that three weights are on the shaft and the data are as follows:

Plane	W	θ	r	a
1	10	30	$\frac{1}{2}$	+10
2	20	270	2	+20
3	30	150	1	− 5
O			3	+30
R			3	0

Ans. $W_O = 9.65$ lb, $\theta_O = 101.5$ deg; $W_R = 9.35$ lb, $\theta_R = 348$ deg.

10.5. A rotor being balanced about a pivot in a machine has the following maximum amplitudes of vibration at the natural frequency. Find the size and position of the correcting balance weight to be used.

(a) 0.020 in. for a rotor without additional weight
(b) 0.020 in. for a 4-oz weight in the 0-deg position
(c) 0.025 in. for a 4-oz weight in the 180-deg position
(d) 0.029 in. for a 4-oz weight in the 135-deg position

Ans. 7.62 oz at 285 deg.

10.6. Given the following data on a single-cylinder steam engine, determine (a) the unbalanced centrifugal force at the crankpin; (b) the piston inertia force in terms of θ; (c) the maximum horizontal unbalanced force at the bearings; (d) the size of the balance force to overcome all the crank centrifugal force and two thirds of the maximum horizontal primary inertia force: Engine speed, 250 rpm; stroke, 18 in.; connecting-rod length, 45 in.; connecting-rod weight, 222 lb.; weight of piston and crosshead, 200 lb; equivalent cranks weight at 9 in. radius, 143 lb; distance of center of gravity of connecting rod from crankpin, in.

Ans. (a) 4,825 lb; (b) 4,190 cos θ + 835 cos 2θ; (c) 9,825 lb; (d) 7,610 lb.

10.7. A single-cylinder horizontal steam engine rotates at 180 rpm with a stroke of 9 in. The equivalent weight of the rotating parts concentrated at the crankpin is 100 lb, and that of the reciprocating parts concentrated at the crosshead is 150 lb. The connecting-rod length is 27 in. Determine (a) the maximum value of the unbalanced inertia and centrifugal forces; (b) the horizontal and vertical forces acting on the crankshaft bearings when the crank angle θ is 90 deg and no counterbalancing is used.

Ans. (a) 1,142 lb; (b) $F_v = 432$ lb, $F_n = 104$ lb.

10.8. Check the condition of balance in an eight-cylinder in-line engine having ψ angles of 0–180–180–0–90–270–270–90 deg. All cylinders are equally spaced on 3-in. centers. The weight of the reciprocating parts (including a portion of the connecting rod) is $1\frac{1}{2}$ lb. The engine speed is 2,100 rpm. The connecting-rod lengths are 10 in., and the crank radii are 2 in. If any unbalance is present, state its magnitude in terms of θ_1.

Ans. $C_{s_H} = 3{,}600 \cos 2\theta_1$; rest in balance.

10.9. The conditions are the same as Prob. 10.8, except that the ψ angles are 90–270–180–0–0–180–270–90 deg. *Ans.* In balance.

10.10. The conditions are the same as in Prob. 10.8, except that it is a six-cylinder engine having ψ angles of 0–120–240–240–120–0 deg. *Ans.* In balance.

10.11. The conditions are the same as in Prob. 10.8, except that it is a four-cylinder engine having ψ angles of 0–180–90–270 deg.

Ans. $C_{p_H} = 1{,}126 \cos \theta_1$; $C_{p_V} = 1{,}126 \sin \theta_1$; $C_{s_H} = 751 \cos 2\theta_1$; rest in balance.

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