The Design of

REINFORCED CONCRETE STRUCTURES

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PREFACE TO SECOND EDITION

The second edition aims to continue the objective of the first edition by presenting a text that will "explain in a consecutive and expanded manner the multitudinous details that make a reinforced concrete design." The illustrative designs and the discussions in the text have been revised to conform, in general, with the 1941 Building Regulations for Reinforced Concrete of the American Concrete Institute or with the 1940 Joint Committee Recommended Practice for Concrete and Reinforced Concrete.

The present theory, based on a straight-line variation of concrete stress and the consideration of stresses caused by loads alone, has fallen into considerable disrepute, and empirical methods of design are increasingly substituted. For this reason a plastic theory of design is presented as a possible alternative. Designs by both theories have been made to give comparison of ease of computation and economy of design.

The material on shrinkage and plastic flow has been rewritten and expanded, as it is essential for discussion of the new articles on prestressed concrete. The chapter on elastic frame analysis has been expanded to include consideration of members of varying depth, curved beams, and three-dimensional space frames. The discussion of space frames leads to consideration of torsional stresses and torsional stiffness. The design of forms has also been added.

This text not only covers the usual content of courses in elementary concrete design but also offers material for advanced courses. It is hoped that it will be a handy reference book for the designer.

D. P., Jr.

Cambridge, Massachusetts
February, 1946

PREFACE TO FIRST EDITION

This book has been written for use in a course in Reinforced Concrete Design of 270 hours of lecture and problem work given at the Massachusetts Institute of Technology. The author's experience of nearly twenty years with the undergraduate and graduate students at the Institute, with the night classes of the Lowell School, and lately with the classes of unemployed engineers during the depression, has convinced him that there is a demand for a textbook which shall explain in a consecutive and expanded manner the multitudinous details that make a reinforced concrete design. During the usual lecture it is not possible for a student to make accurate comprehensive notes and at the same time absorb the general scheme presented by the instructor. It is hoped that this discussion is sufficiently detailed so that the student need take no notes, or may use it to advantage if he must study without instruction.

The general plan is to present the theory, followed by illustrative designs carried through to a sketch sufficiently elaborated to be presented to the field force. Each illustrative problem is a complete design of some unit, while collectively they form the essentials for the design of a complete building. Office practice varies with different firms and localities, and the designs presented are conservative solutions. Emphasis is given to the design fundamentals rather than to the execution of finished drawings and details.

The recommendations of the 1928 Joint Standard Building Code of the American Concrete Institute are used in general for the allowable stresses and methods of commercial design. Where it seemed advisable it has been possible to indicate the changes of procedure recommended by the proposed 1936 revision of the A.C.I. code. The nomenclature is that generally employed in practice.

It is assumed that the student is already equipped with a working knowledge of applied mechanics, particularly with reference to the subjects of statics and the beam theory. Some phases of the work of design lead to highly complex problems, as, for example, the analysis of a reinforced concrete building frame for a variable live load, and for wind loads. Such design problems call for excellent training in the theory of indeterminate structures based upon fundamental theories of applied mechanics and the theory of elasticity. Complex problems are beyond the scope of the present text, which will be confined to the usual problems of design. Since the primary concern of this text lies with design, the matters of construction detail and manipulation, such as building of forms, methods of pouring, etc., are omitted.

Many uses have been found for reinforced concrete. This book is concerned, primarily, with reinforced concrete in the design of buildings. The underlying principles presented are applicable, however, to other classes of structures.

It would be well for the student of reinforced concrete to orient this medium with reference to other classes of building materials before proceeding with the study of its particular properties and problems.

Grateful acknowledgment should be made to Professor Addison F. Holmes for the interest and encouragement that enabled the author to fit the extra load of writing into the teaching schedule. Mr. Alvin Sloane has benefited the text by a critical reading of the manuscript as he drew the illustrations.

D. P., Jr.

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CHAPTER 1

GENERAL PRINCIPLES

1. Reinforced Concrete Construction. Plain concrete, made from natural cements, was employed as a structural material in the time of the Roman Empire. It was used in compression members, such as roads, aqueducts, arches, etc. Some of these structures have survived to this day. Reinforced concrete has been used for about sixty years in members in bending or with tensile stresses. In this short time the study of tests and loaded structures has given rise to the methods of design now in general use. The recent tendency has been to develop more accurate and intricate methods of design which effect savings in materials. Such economies are justified only if the structure adequately supports its load. Variations of the live load must be considered, as well as the effect of temperature changes, and of shrinkage and time-flow, so that the maximum deformations are not excessive. In design it is customary to compute the maximum stress intensities in a member, but the student should always keep in mind that the essential is a reasonable deformation. The stress intensity is a convenient measure of the strain, or unit deformation.

2. Monolithic Construction. The design of reinforced concrete structures involves a study of the elementary parts, consisting of beams, columns, footings, retaining walls, etc. The identification and analysis of these units are more difficult than for steel or timber construction because there is no evident demarcation between slab, beam, or girder. They are poured monolithically, except for "precast" members. In timber construction, such as the "slow-burning" mill building, the floor planks, wooden beams, wooden or cast iron columns, and brick walls are evidently separate units. Riveted steel frames are also designed as units of a single span, both for beams and columns. The modern welded frame, however, gives the more economical rigid connection. The reinforced concrete floor system is poured continuously over large areas, with steel running through all connections and construction joints. The result is a monolith with the advantages pertaining to a rigid frame. When a floor system is poured as a unit for many spans in all directions, the proper division of the floor load between slab, beams, and girders becomes a highly complicated problem. The beams, for instance, are fixed or partially fixed by the columns and by the beams in the adjacent spans. The columns are also continuous from basement to the roof. This type of construction allows readjustment of the stress distribution (deformations) if some member is overloaded. It can be designed to give resistance to wind loads, vibration, and earthquake shocks.

3. Advantage of Reinforced Concrete Construction. Concrete is a material whose tensile strength is much less than its compressive strength. The tensile strength is so low that plain concrete can be used only where the member is in compression. Members in bending, such as slabs, beams, long columns, arches, etc., must be provided with properly placed steel to take the tensile pulls which will otherwise cause cracks and failure in the concrete. Reinforcing steel is supplied as bars of varying sizes which can be rolled cheaply and can be bent to reinforce any part of a member. Steel is best used as tensile reinforcement, because neither bars nor thin plates make satisfactory compression members unless supported laterally.

The usual reinforced section consists of a concrete compression area and a steel tension area. The steel is protected from fire and corrosion by a covering of concrete, which is an excellent fireproofing material. The member has, therefore, great durability as well as adequate strength.

Concrete has a low strength-weight ratio compared with steel. In fact, in many structures the dead weight may be a large percentage of the total load brought to the footings. The members must be correspondingly larger to carry the dead weight, but the effect of movable loads is much reduced. However, with the allowable stresses now used, concrete is a cheaper compression material than steel. For example, the concrete for a column may cost 40 cents per cubic foot, with an allowable compressive stress of 600 lb. per sq. in., or a cost factor of $40 \div 600 = 0.067$. A richer mix may cost 50 cents per cubic foot with an allowable stress of 900 lb. per sq. in., or a cost factor of 0.056. Structural steel may cost 4 cents per pound and have an allowable compressive stress of 13,000 lb. per sq. in. or less. Its cost factor equals $(4 \times 490) \div 13{,}000 = 0.151$. The concrete cost factor is markedly less.

Reinforced concrete is durable and fireproof. It can compete in cost in most localities with steel or wood-brick construction up to heights of 10 to 12 stories. Above such heights the columns are usually of structural steel encased in concrete. Reinforced concrete construction uses unskilled labor for the most part, and the supplies of cement, aggregate, and steel bars are easily obtained. On the other hand, a structural material manufactured at the site can be very variable compared with such a material as structural steel. Some contractors and engineers do not have the experience or the knowledge required to produce even the average concrete assumed in specifications.

The monolithic construction enables machine vibrations to be absorbed, and the rigid structure can resist wind pressures or earthquake shocks. Reinforced concrete is not recommended for structures where alterations are anticipated or for a temporary building which must be razed later.

Concrete has largely superseded masonry in retaining walls, dams, abutments, arches, and conduits. It is frequently used instead of masonry for reservoir walls, roofs, and floors, also for chimneys or towers. Concrete competes with steel, cast iron, or vitrified clay in pipe lines or conduits. In buildings reinforced concrete is generally used for floors and footings, and it competes with steel and masonry for use in columns and walls. It has been employed for water tanks and ships.

4. Physical Properties of Concrete. The designer of reinforced concrete structures must be familiar with the physical properties of concretes in order that the structure may carry its load safely and be durable. The student should consult texts on materials for an extensive discussion of the mass of test data on cements and concretes. A brief summary will be given here of the more important properties of concrete.

Reinforced concrete is often used in comparatively thin sections. It is desirable in such cases to give a minimum covering to the steel. Resistance to frost, imperviousness to water pressure, and protection of the steel from fire or corrosion call for a dense concrete. A dense, impervious concrete is also a strong concrete. Such a concrete can be obtained, with a minimum of the costly cement, by suitably grading the available aggregates and by carefully determining the amount of water. So many variables affect the grade of concrete actually manufactured that it is desirable to determine in advance by tests the strengths realized by the materials chosen, mixed by the methods and in the proportions to be used on the job. It has been the endeavor of recent specifications to encourage such advance tests and also field tests of the concrete made by the contractor.

5. Cement. Portland cements are commonly used and are selected by the specifications of the American Society for Testing Materials. In recent years other cements have been developed for specific properties. Thus, high early strength cements are employed for highways and structures where a reduction of the time of construction justifies some increase in cost. Cements with a low coefficient of shrinkage are in demand for highways, dams, etc. Portland cements frequently have a considerable temperature rise when large masses of concrete are setting. A cement with low heat of setting and low shrinkage coefficient was developed for Boulder Dam.

6. Aggregate. The aggregate is often divided into two sizes. Fine aggregate comprises those particles less than $\frac{1}{4}$ in. in diameter and is obtained from sand or the screened portion of slag or crushed stone. Coarse aggregate is composed of particles larger than $\frac{1}{4}$ in. in diameter and is obtained from gravel, crushed stone, slag, or a more costly vitrified light-weight material. All aggregate should be composed of clean, sound, and strong material.

Sands should not contain more than a trace of loam or other organic material. The grading should not show more than 30 per cent passing the 50 sieve or over 10 per cent passing the 100 sieve. Very fine material has a large surface area for a given volume and requires more cement paste to cover the surfaces. An aggregate which is uniform in size has large percentages of voids which must be filled by the cement paste. If a fixed amount of cement is used, as in a 1 : 2 : 4 mix, the strength will be less for poorly graded aggregates because more water must be used with the cement to produce a workable mix. Excess water will evaporate, and the concrete will be porous and low in strength.

Fine aggregates are selected by a colorimetric test for organic matter, by sieving a sample for particle sizes, and by compression or tension tests for strength.

The coarse aggregate is the material larger than $\frac{1}{4}$ in. For reinforced concrete the maximum size is about $\frac{3}{4}$ in. to 1 in., so that the stone will pass through the narrow clearances around the steel bars. With stones up to 6 in., or even 9 in., in size, large masses of plain concrete can be poured. Gravel concretes are easier to place than broken stone or slag concretes but, if the gravel comprises water-worn material, the smooth surface may give poor adhesion with the cement paste. Gravel concretes usually give the desired workability with less water, but the individual particles are often not as strong as the broken stone particles.

Aggregate should be clean. It is screened to determine particle sizes. The relative gradings of mixtures of fine and coarse aggregate are compared by the "fineness modulus," or summation of the percentages held on certain sieves.

7. Proportioning for Strength. The grading of the aggregate which results in the densest mixture gives a strong concrete. This is also economical because it requires the least amount of cement paste to fill the voids. In practice, somewhat more paste is used than the volume of voids to allow for separation of the aggregate particles by the introduction of the paste. The densest aggregate proportions can be determined by trial or by the use of an ideal curve of gradation, such as Fuller's curve.[1] The strength of concrete increases with the ratio of cement to voids. Aggregate of uniform size, with a high percentage of voids, will require a larger amount of cement to give a desired strength than will the well-graded mixture. The strength of a given grading of aggregate can be increased by using more cement.

The cement paste consists of cement and water. A certain minimum amount of water is needed to complete the chemical process of setting. More water than this amount must be used to give commercial workability. Mr. Duff Abrams proved that there is a relation between strength and the water-cement ratio. The relation is usually expressed

$$S = \frac{A}{B^x} \tag{1}$$

where S = compressive strength
x = water-cement ratio
A and B are constants determined by test.

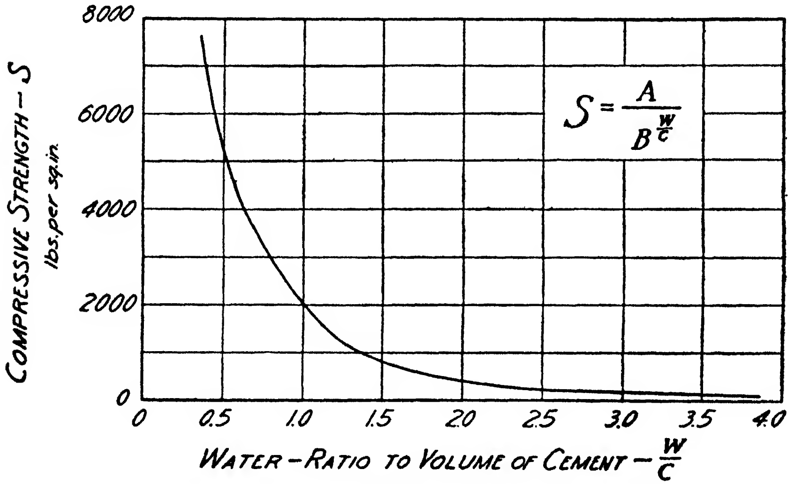


FIG. 1

Figure 1 is a plot of equation 1 for a certain cement and aggregate. From such a plot the maximum water-cement ratio that will give a desired strength can be found.

A given aggregate mixture requires a definite amount of water to wet the aggregate surfaces. A *lean* mix with a small amount of cement will require a higher water-cement ratio to give the same workability as a *rich* mix, and has, therefore, a lower strength.

[1] W. B. Fuller and S. E. Thompson, "Laws of Proportioning Concrete," *Trans. A.S.C.E.*, Vol. LIX, p. 67, 1907.

If the mix is determined from Abrams' curve, it is usual to adopt a water-cement ratio expressed as *gallons per sack* of cement. This water content corresponds to the strength desired and includes the water already present in the aggregate. If the grading of the aggregate varies from day to day the total amount of aggregate per sack of cement is varied, or the ratio of fine to coarse aggregate is adjusted, so that the workability of the mix is satisfactory. The water-cement ratio remains unchanged.

8. Consistency or Workability. There is no completely satisfactory method of determining the consistency of the concrete which can be placed with the minimum of labor. The usual field test is the slump in inches of a standard truncated cone of concrete. The present tendency is to use as dry a mix as possible and to spade or tamp the concrete into place. Mass concrete and that in highways are poured with slumps of 1 to 3 in. Reinforced concrete is placed with slumps of 3 to 6 in. Thin walls of considerable height with several rows of steel reinforcement require wetter consistencies in order to give smooth surfaces and to avoid honeycombing or voids around the steel. Concrete which was formerly considered much too dry can now be placed by the use of vibrators.

Permeability tests show that the mortar sinks as the concrete sets, leaving voids under the coarse aggregate and continuous voids under horizontal or inclined steel. The larger the coarse aggregate the greater continuity of voids is given for flow of water. The presence of horizontal reinforcing steel may also reduce materially the effective depth of concrete to resist water penetration.

9. Concrete Strengths. The compressive strength is the standard by which concretes are rated. The allowable stresses in building codes are based on the compressive strength at the age of 28 days when tested as a cylinder 6 in. in diameter and 12 in. in height. Most concretes are proportioned to give a desired compressive strength. This also insures reasonable tensile and shear strengths because they vary approximately as the compressive strength. The denser, stronger concretes are also the more durable and impervious.

The tensile strength varies from one eighth to one twelfth of the compressive strength. The value is so low that most failures are tensile and occur without warning in plain concrete. With reinforced concrete complete destruction may not occur, but an inadequate steel design will result in unsightly cracks.

The shearing strength of concrete is the resistance to sliding on some plane. Maximum shear stresses on a particle occur on planes at 45° with the planes of maximum compression or tension. Thus, a concrete cylinder tested in compression tends to fail by shear on a 45° plane, and this shear stress will be one half the compression stress on the cross section. The test cylinder has a height twice the diameter, and the plane of failure is actually greater than 45° with the cross section. Shear tests indicate strengths varying from 0.4 to 0.9 the compressive strength.

The push of a column through a slab or footing is a form of shear failure, usually called *punching shear.* The average stress around the column perimeter varies with the shape of column section. The shear strength is great enough so that it seldom determines the failure of a member.

The term *shear failure* is often used colloquially for a tensile failure on some plane inclined to the cross section. Such *diagonal tension* failures occur when the cross section has high shear stresses and the plane of maximum tension for a particle makes a considerable angle with the cross section. The longitudinal steel is not in line with this pull and cannot alone prevent the formation of a tension crack on some inclined plane. Diagonal tension is discussed in Chapter 4.

10. Stress-Strain Relations. The instantaneous unit deformation produced by the application of a load is known as *elastic strain.* Up to a certain limit this deformation is entirely recovered when the load is removed. However, for most loadings on concrete the amount of this recovery decreases as time passes; there has been a permanent set or deformation.

Concrete is a plastic material, and additional deformations occur immediately after the application of a load. Even though the load remains constant these additional strains increase as time passes. They vary with the amount of load and with the time. The strain increase is called *plastic strain,* or time-flow.

Quite apart from deformations due to the application of a load are deformations due to shrinkage or loss of water from the concrete. *Shrinkage strains* have considerable magnitude if the concrete dries out thoroughly, but will also occur at constant humidity. Subsequent wetting will cause a decrease of the shrinkage, but the recovery is not complete.

Figure 2 from Mr. Glanville's paper [2] shows the general time-strain relation between the three strains at constant temperature, humidity, and load. There can also be deformations caused by temperature changes. In the past, designs have usually taken into consideration the elastic effect of the loading only. The modulus of elasticity for the concrete has been computed, however, from strains which include some of the plastic flow. It is now recognized for members of considerable

[2] W. H. Glanville, Building Research Technical Papers 10, 11, 12.

size, such as floor systems, pavements, dams, and arches, that allowance must be made for plastic flow, shrinkage, and temperature deformations.

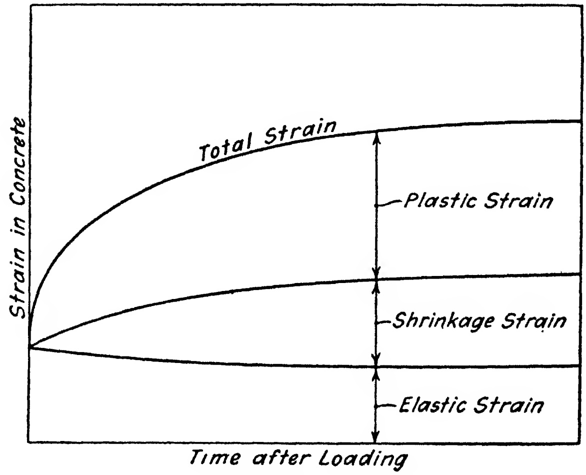
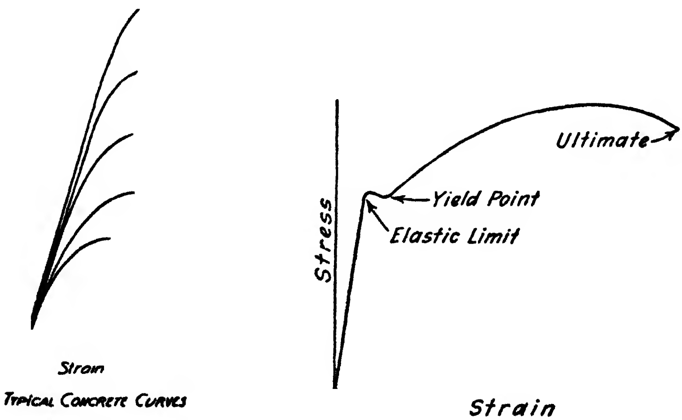


FIG. 2

11. Elastic Stress-Strain Relation. Figure 3*a* shows the stress-strain diagrams for concrete of different strengths as determined by the usual laboratory test. Mr. Glanville determined that some plastic strain is



(*a*) Concrete (*b*) Steel

FIG. 3

included in such a test, even if the strain is recorded within a few seconds of the application of the load. He attributes to plastic flow the flattening of the curve at the higher stresses. The true modulus of elasticity is the slope of the tangent to the curve at the origin, or *initial modulus*. The *secant modulus*, or slope of the line drawn from the origin to some point on the curve, is merely the ratio of stress to total strain.

Professor R. E. Davis of the University of California has suggested that the modulus of elasticity E be defined as the ratio

$$E = \frac{1}{e}$$

where e = elastic strain for a unit stress. He suggests that *modulus of resistance* R be used for the ratio

$$R = \frac{1}{e + c}$$

where c = plastic strain for a unit stress.

The modulus of elasticity E_c of the concrete is assumed in the A.C.I. Code to be equal to $1000f'_c$, where f'_c is the compressive strength at 28 days' age. This criterion is based on tests whose strains include some instantaneous plastic flow. Mr. Glanville's results give the true modulus of elasticity as much greater in value.

If a load is repeatedly applied, the stress-strain relation becomes a straight line up to this load.

12. Shrinkage Strain. Ordinary commercial concrete is poured and sets under atmospheric conditions. As it dries out the concrete shrinks. There may be alternations of expansion and shrinkage as it is subsequently wetted or dried. These deformations due to water content increase with the wetter mixes, with the richer mixes, and with low humidities. Concrete that hardens in shelter will dry out permanently, especially if the structure is heated during the winter. Under such conditions shrinkages from 0.03 per cent in 3 months to 0.08 per cent in 2 years are possible.

13. Methods of Design. In the past reinforced concrete members have been designed to resist safely bending moments and shear forces due to the live and dead loads on the structure. Before the design of slabs, beams, columns, walls, and footings of a structure can be made it must be analyzed for the variation and magnitude of these moments and shear forces in the individual members. In timber-brick construction, where there is little restraint at the supports, each member can be analyzed separately. The same procedure has been used for steel frames with standard riveted connections. The analysis of welded steel and reinforced concrete structures has steadily grown more complex as a result of the realization that end restraint of a member affects the bending moment distribution. In the last decade great progress has been made in the analysis of such monolithic structures on the assumption that they are continuous elastic frames.

It was customary to find for a given member the various possible moment and shear diagrams due to the dead load and various combinations of live loads on nearby spans, also the effect of wind loads on the whole structure; then the individual member was designed for these moments and shears. The computed stresses for reinforced concrete were kept below certain maximum allowable values. These allowable values were set by experience low enough so that the unknown effect of temperature changes, shrinkage, and plastic flow would not result in excessive strains. As time passed methods of computing these additional strains were developed, but the computations are intricate and depend on certain coefficients, determined by test, which are not accurately known for all conditions, so that shrinkage and flow computations are little used in practice.

Recently there has been a tendency to substitute for stress analyses due to loads empirical equations based on tests to failure, which presumably include the effect of shrinkage, of loads, and to a certain extent the effect of plastic flow. The results give the load carried at failure and this load is divided by a factor of safety to give safe working loads. This change in design methods is most marked at present for column design but a similar procedure is proposed and used by certain designers for members in bending. The extent to which these empirical equations will supersede stress analyses due to loads is still in question, so this text will present both methods of design. A method that allows for the effect of all strain causes can naturally use greater allowable stresses than those previously specified for live and dead loads only.

The illustrative problems in this text will use for the most part the allowable stresses given in the 1941 "Building Regulations for Reinforced Concrete" of the American Concrete Institute (see Appendix.)

CHAPTER 2

RECTANGULAR BEAMS

14. Classification of Members. For purposes of analysis structural members are divided into three classes, those in *direct stress,* those in *bending,* and members with *direct stress and bending.* The division is made according to the position of the resultant of the normal stresses acting at a cross section.

A straight member is said to be in direct stress when the resultant force acts along the axis of the member through the center of gravity of each cross section and normal to it. Members in direct stress may be in tension or compression. Reinforced concrete is used for compression members, a typical example being the interior column loaded with an axial load. Such cases are discussed in Chapter 10.

A member is said to be in bending when the resultant of the normal stresses is a couple. Such members are called beams. They may also have shear stresses acting on the cross sections. The beam theory develops the relation between fiber stress, bending couple, and the cross-section dimensions. It is a very important part of reinforced concrete design and will be discussed in this chapter.

A member is in direct stress plus bending when the resultant of the normal stresses is a force which does not act at the center of gravity of the cross section. This is equivalent to a force at the center of gravity producing direct stress and a couple producing bending about the center of gravity. This case is employed for columns and for arches that are subjected to compressive loads and bending. Shear forces may also be present.

15. Homogeneous Beams. Before relations are derived between the bending moment, fiber stress, and section dimensions of reinforced concrete beams the limitations and assumptions underlying the corresponding derivation for beams of homogeneous materials will be surveyed. It is advantageous to know whether the same requirements may be used for the reinforced concrete beam. It is understood that these derivations deal with the immediate elastic strains due to loading, and no account is here taken of shrinkage, plastic flow, or temperature strains.

16. Limitations to Beam Theory. The general theory of flexure for homogeneous materials does not apply the term *beam* to all members in bending. Arches and columns also have bending stresses but are not regarded as examples of simple bending. Five limitations are usually applied to determine what sort of members may be called beams.

Limitation 1. The material of the beam is homogeneous and isotropic. This is not so for reinforced concrete, as the concrete is composed of cement (hydrated or unhydrated), sand and stone particles, and air voids. The addition in certain places of steel reinforcement introduces another material. The same objection can be raised in a certain degree to timber, which has cracks, knots, sap wood, and bark, and is not homogeneous or isotropic.

Limitation 2. The beam is straight and of uniform cross section throughout. This is usually realized as far as external dimensions are concerned. If, however, the concrete and steel areas used in the theoretical derivations are considered, both the compression and tension areas are found to vary as the bending moment increases or decreases, and to become markedly different when the bending moment changes from positive to negative.

Limitation 3. The external forces are in equilibrium and are applied in such a manner that they can be considered equivalent to a system of forces acting in a single plane which will be known as the plane of loading. This limitation can be realized.

Limitation 4. The plane of loading intersects every cross section at an axis of symmetry, and, wherever the term cross section is used, the section at right angles to the central axis is to be understood. This is true for the usual rectangular, tee, and I sections.

Limitation 5. The length of the beam is large in proportion to the greatest dimension of its cross section, and the difference between the depth and greatest width is not excessive. This requirement should be fulfilled.

17. Assumptions. The dimensions of most reinforced concrete beams are determined by the necessity of safely carrying the external bending moment. Therefore, the first derivation will be for the relations existing between the bending moment, fiber stress, and cross section of the beam.

Three assumptions are made in the ordinary beam theory for the behavior of materials in bending. These assumptions are capable of verification by test and they answer the question: *How does a beam act?*

Assumption 1. Plane sections remain plane and normal to the longitudinal fibers after bending. This assumption is justified to 1 per cent accuracy, at least, for reinforced concrete throughout the ordinary range of the working loads, certainly as much so as for timber.

Assumption 2. The material obeys Hooke's law; that is, that stress intensity is proportional to strain throughout the beam. Hooke's law gives the relation between stresses and elastic strains. This law holds for steel up to its elastic limit, which is above the working stresses considered in the beam theory. The stress-strain relation is also a straight line for concretes loaded within the working limits, as can be seen in Figure 3.

Assumption 3. Every longitudinal layer is free to extend, or contract, under stress as if separate from the other layers. This is assumed. The ratio of stress to elastic strain, or modulus of elasticity, is constant for each material (see Art. 11), and the ratio of the two moduli, $\dfrac{E_s}{E_c} = n$, is taken as a constant. In the ordinary beam theory with homogeneous materials, n is assumed to be unity. For the usual concrete mixes n varies from $n = 15$ for concretes whose compressive strength $f'_c = 2000$ lb. per sq. in. to $n = 6$ for 5000-lb. concretes ($f'_c = 5000$ lb. per sq. in.).

STRESSES DUE TO WORKING LOADS

18. Rectangular beams. The beam theory for rectangular sections applies to the design of cross sections whose *compression area is a rectangle,* the tension steel being held in its proper position by sufficient concrete. It is assumed that a crack has appeared at the section of maximum bending moment, and none of the concrete on the tension side is considered in the derivation. Therefore, its shape is immaterial. Thus, in Figure 4, all the sections shown are designed as rectangular

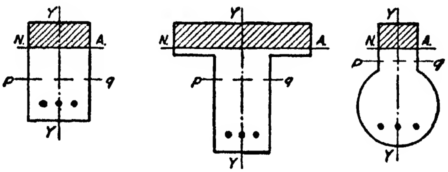


FIG. 4

beams, if the compression areas are the ones shown cross-hatched. Let $N.A.$ be the neutral axis and YY the plane of loading. The compression area and the tension steel must both be symmetrically placed about the axis YY. When the bending moment is much less than the maximum, the concrete on the tension side may not be cracked to the neutral axis and can take some tension. The neglect of this small amount of tension force gives computed stresses in the steel somewhat greater than actually occur.

19. Nomenclature. Let Figure 5 represent the elevation and cross section at a distance x from the support of a rectangular reinforced concrete beam supported at the left end by R_1. Let

b = width of section
h = total depth of section
d = distance from extreme fiber in compression to the center of gravity of the steel in tension
jd = moment arm of internal couple
kd = distance from extreme fiber in compression to neutral axis
E_c = modulus of elasticity of concrete
E_s = modulus of elasticity of steel
n = ratio of moduli, or $\dfrac{E_s}{E_c}$
f_c = maximum intensity of fiber stress in concrete
f_s = average intensity of fiber stress in steel
M = external moment at the section
M_c = moment of resistance expressed in concrete terms
M_s = moment of resistance expressed in steel terms
A_s = area of steel
p = steel ratio $\dfrac{A_s}{bd}$.

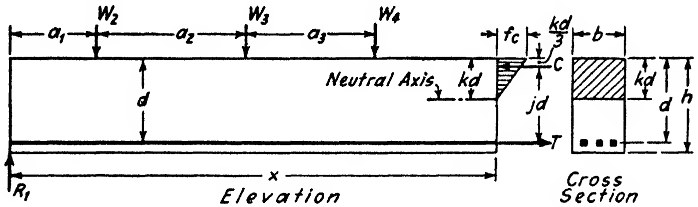


FIG. 5

The amount of tension steel A_s is expressed as the ratio $p = \dfrac{A_s}{bd}$. Since we do not consider the concrete on the tension side it might seem logical to use the ratio of the tension area to the compression area $p = \dfrac{A_s}{bkd}$, but we do not know at first the position of the neutral axis kd.

The steel ratio might also be defined as the ratio $p = \dfrac{A_s}{bh}$, using the total area of the beam. The convenient dimension, however, is the depth to the steel d. The total depth h is greater in order to give adhesion between steel and concrete and to provide fire- or dampproofing. In designing, the steel depth d is first calculated and the total depth h

is made some convenient commercial dimension large enough to give the proper clearance. Therefore, for this derivation, the steel ratio is assumed to be the ratio $p = \dfrac{A_s}{bd}$.

20. Rectangular Beams. The object of the derivation is to deduce relations between the external bending moment M, the fiber stresses f_s and f_c, the area A_s, and the concrete dimensions bd.

Dealing with the portion of the beam span shown in the elevation of Figure 5 and applying limitation 3, the three conditions of equilibrium of statics hold, namely: $\Sigma V = 0$, $\Sigma H = 0$, and $\Sigma M = 0$.

The sum of the vertical forces, $\Sigma V = 0$, gives the vertical shear at the section. For the time being this shear force will not be considered. The resultant of the uniformly varying normal compressive stresses C and the resultant of the pull T in the bars are the only normal forces acting on this portion of the beam. Therefore, since $\Sigma H = 0$, $C = T$ and the force C forms a couple with the force T. The moment arm of this couple will be designated jd, where j is a decimal ratio of d, usually having values between 0.85 and 0.95.

On the assumption that a plane section remains a plane section after bending, the strains of any particle are proportional to the distance of the particle from the neutral axis. The extreme fiber on the compression side has a strain e_c, and the steel has a strain e_s, where

$$e_c = \frac{f_c}{E_c} \qquad \text{and} \qquad e_s = \frac{f_s}{E_s}$$

Therefore

$$\frac{e_c}{e_s} = \frac{\dfrac{f_c}{E_c}}{\dfrac{f_s}{E_s}} = \frac{f_cE_s}{f_sE_c} = \frac{nf_c}{f_s}$$

also

$$\frac{e_c}{e_s} = \frac{kd}{d - kd} = \frac{k}{1 - k}$$

Equating,

$$\frac{nf_c}{f_s} = \frac{k}{1 - k} \tag{2}$$

$$nf_c = nf_ck + f_sk = k(nf_c + f_s)$$

$$k = \frac{nf_c}{nf_c + f_s} = \frac{\dfrac{nf_c}{nf_c}}{\dfrac{nf_c + f_s}{nf_c}} = \frac{1}{1 + \dfrac{f_s}{nf_c}} \tag{3}$$

Equation 3 gives the location of the neutral axis, providing we know the actual stresses and mix, or are assuming allowable values which will be actual values when the design is complete. This equation may well be called the *designer's equation.* The term k is a decimal ratio of d, having values varying from 0.15 to 0.45.

The neutral axis can also be located by using the fact that the compressive force C equals the tensile force T. The resultant C of a uniformly varying stress equals

$$C = ay_0A \tag{4}$$

where a = intensity of stress 1 in. from the neutral axis
y_0 = distance from the neutral axis to the center of gravity of the area stressed
A = area stressed.

The resultant C of the uniformly varying concrete stresses equals

$$C = \left(\frac{f_c}{kd}\right)\left(\frac{kd}{2}\right)(bkd) = \frac{f_cbkd}{2} \tag{5}$$

The resultant of the pulls in the steel bars equals

$$T = f_sA_s = f_spbd \tag{6}$$

Equating

$$C = \frac{f_cbkd}{2} = T = f_spbd$$

$$\frac{f_s}{f_c} = \frac{bkd}{2pbd} = \frac{k}{2p} \tag{7}$$

Revising equation 2

$$\frac{f_s}{f_c} = \frac{n(1-k)}{k} \tag{8}$$

Equating 7 and 8

$$\frac{k}{2p} = \frac{n(1-k)}{k}$$

$$k^2 + 2npk = 2np$$

$$k^2 + 2npk + (np)^2 = 2np + (np)^2$$

$$(k + np)^2 = 2np + (np)^2$$

$$k = \sqrt{2np + (np)^2} - np \tag{9}$$

This expression gives the location of the neutral axis in terms of the mix and steel ratio p. It may well be called the *checker's* or *inspector's* *equation*, for it implies that one is checking a finished design whose mix and steel are known.

The desired bending moment relations can now be found. Since the force C is the resultant of uniformly varying stresses, it acts $\frac{2}{3}kd$ from the neutral axis, or $\frac{kd}{3}$ from the extreme fiber.

From Figure 5 it is apparent that

$$d = jd + \frac{kd}{3} \qquad \text{or} \qquad j = 1 - \frac{k}{3} \tag{10}$$

In terms of the concrete stress f_c

$$M_c = C(jd) = \frac{f_c bkd}{2} jd = \frac{f_c}{2} jkbd^2 \tag{11}$$

In terms of the steel stress f_s

$$M_s = T(jd) = f_s A_s jd = f_s pjbd^2 \tag{12}$$

Both expressions for the bending moment must give the same numerical values. Equating 11 and 12,

$$\tfrac{1}{2} f_c jk = f_s pj = K \tag{13}$$

It is convenient at times to write the bending moment equations

$$M_s = M_c = Kbd^2 \tag{14}$$

21. Summary. A reinforced concrete rectangular beam theory should derive relations between the bending moment at a section due to external forces, the internal fiber stresses in the steel or concrete, and the cross-sectional dimensions.

These relations are

$$M_c = \tfrac{1}{2} f_c jkbd^2 = Kbd^2 \tag{11}$$

and

$$M_s = f_s A_s jd = f_s pjbd^2 = Kbd^2 \tag{12}$$

Before these equations can be used, it is necessary to determine the values of the ratios j and k. When one is designing, the position of the neutral axis kd can be found by use of

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \tag{3}$$

When one is checking a completed design

$$k = \sqrt{2np + (np)^2} - np \tag{9}$$

The moment arm jd of the internal couple can be found by

$$j = 1 - \frac{k}{3} \tag{10}$$

The most economical rectangular beam is one whose dimensions and ratio of steel are such that at the section of maximum bending moment the maximum allowable concrete stress f_c and the maximum allowable steel stress f_s are given simultaneously.

22. Transformed Section. It is possible to substitute for the reinforced concrete section a hypothetical section of a single material, concrete or steel, which has the same deformations as the reinforced section. This substitute is called the *transformed section*, and the usual equations for a homogeneous beam may be employed. Theoretically, a beam of plain concrete using the transformed section will deform as does the reinforced beam; practically, it is not possible to construct or load this section. It is customary to substitute a transformed section of concrete.

The assumption that the tension area has cracked is again made. The reinforced section consists of a rectangular concrete compression area and tension steel. The steel has a tensile strain e_t. The substituted concrete must be in the same position and have the same strain.

$$\frac{f_s}{E_s} = e_t = \frac{f_t}{E_c}$$

where f_t = tensile stress in the substitute concrete.

$$f_t = \frac{E_c}{E_s} f_s = \frac{f_s}{n}$$

The substituted concrete must have an area A_t sufficient to give a resultant tensile force T.

$$T = f_s A_s = f_t A_t$$

The transformed area equals

$$A_t = \frac{f_s A_s}{f_t} = nA_s \tag{15}$$

The transformed section consists of a rectangular compression area, wide and kd deep, plus an unconnected tension area nA_s, which has the height of the steel bars and the width to give the necessary area.

The center of this narrow rectangle is at the depth d from the compression particles with maximum stress (Fig. 6*b*). The neutral axis is located at the center of gravity of the two rectangles. The moment of inertia of the transformed section about the neutral axis is found and the fiber stresses can be computed, using

$$f = \frac{My}{I}$$

where f = fiber stress intensity
M = bending moment
y = distance from neutral axis to particle considered
I = moment of inertia about neutral axis.

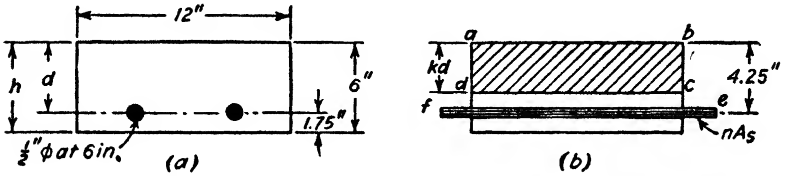


FIG. 6

MAXIMUM MOMENT AT FAILURE OF BEAM [1]

23. Plastic Theory. It has been stated in Chapter 1 that the elastic strains due to application of a load give only a portion of the actual strains at any time in the concrete and steel. Before the load application strains due to shrinkage have been produced. A long-continuing load, such as the dead load, will give increasing plastic strains as time passes. For members in bending it is impossible to compute the final strain condition of a particular particle, especially if the member is part of a continuous frame. In any case the relation between steel and concrete strains will be very different from that given by the equations derived above for stresses due to loads only. Therefore, certain designers have advocated designing for failure, the method of analysis being based upon tests of beams to failure. By this approach the ultimate strains will include shrinkage, load, and some plastic flow strains.

In the past similar analyses for stresses in a beam up to the ultimate load assumed that the compressive stress-strain curve was a parabola like those of Figure 3*a*. The modern discussion modifies this assump-

[1] This discussion is based on "Plastic Theory of Reinforced Concrete Design," by Charles S. Whitney, in the *Proc. A.S.C.E.*, Dec., 1940, p. 1749. See also "The Plasticity Ratio of Concrete and Its Effect on the Ultimate Strength of Beams," by V. P. Jensen, in *Jour. A.C.I.*, June 1943, p. 565.

tion. Tests of beams seem to indicate that the maximum stress occurs for strains of 0.0015 to 0.0020 (Fig. 7). If the beam is strained beyond these values, failure starts and the stress decreases.

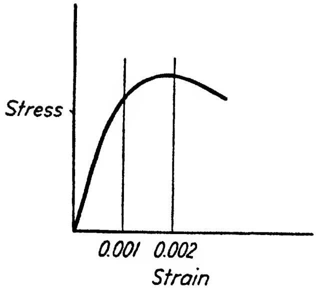


Fig. 7

In a beam approaching failure, if a plane section remains plane, the strains will be uniformly varying at any section but the stresses will vary as in Figure 7. Since many sections are lightly strained the beam will not fail until the maximum strain on the most severely strained section exceeds 0.0020. At this section the maximum stress will not occur at the extreme fiber. If the proper per cent of steel is used, the steel will have reached its yield point stress at failure. If less steel than this amount is used, failure will be initiated by the increasing excessive steel strains, while the steel stress remains constant at the yield point value. Therefore, for balanced design, the steel strain at the most severely strained section will be equal to $e_s = \dfrac{\text{yield point stress}}{E_s}$ and the concrete strain at the extreme fiber will be greater than $e_c = 0.0020$. Figure 8 shows such a section where the maximum concrete strain equals 0.0030 and the steel strain equals $e_s = \dfrac{50{,}000}{30{,}000{,}000} = 0.00167$, the yield point being at 50,000 lb. per sq. in. The compression stress distribution will be as shown, the maximum value being the compressive strength of the concrete f'_c which occurs in the particles some distance below the top of the beam.

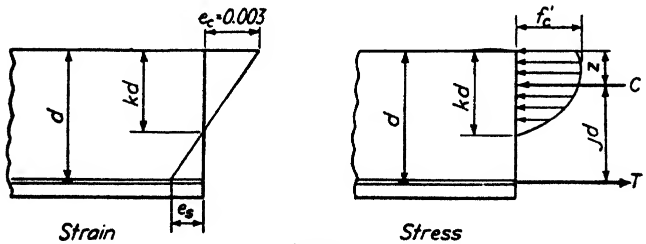


Fig. 8

In this case

$$kd = \frac{0.00300}{0.00467} d = 0.643d$$

The compressive force $C = b \int f\,dy$ where f equals fiber stress on a particle at a distance y from the neutral axis. This integral is the area of the stress diagram and the force C acts at the center of gravity of this area.

The tensile force $T = f_y A_s$ where f_y equals yield point stress of steel.

The stress-strain diagram for the concrete will vary with the mix and the water-cement ratio. For practical designs it is sufficiently accurate to substitute for the concrete stress diagram a constant stress of $0.85f'_c$ (Fig. 9) with a depth of a, instead of kd, whose resultant force C will

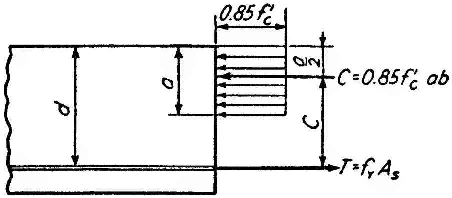


FIG. 9

act at the center of gravity of the stress diagram of Figure 8. The depth a is less than the distance to the neutral axis kd.

Since $T = C$

$$a = \frac{f_y A_s}{0.85f'_c b} \tag{16}$$

This corresponds to the checker's equation (equation 9) of the other derivation. In the past the area of tensile steel has usually been so small that the steel reaches its yield point before the concrete fails. In such a case, if the load is increased, the steel stress remains unchanged but its strain increases until the concrete, in turn, is overstrained and failure occurs with the concrete stress distribution of Figure 8.

24. Under-Reinforced Beams. The expressions for bending moment at failure in terms of section dimensions and stresses follow.

$$M = T \cdot c = f_y A_s \left(d - \frac{a}{2}\right) = f_y pbd \left(d - \frac{f_y pd}{2 \times 0.85f'_c}\right)$$

where $f_y pbd = 0.85f'_c ab$.

$$M = f_y p \left(1 - \frac{mp}{2}\right) bd^2 = K'bd^2 \tag{17}$$

where $m = \dfrac{fy}{0.85f'_c}$.

Also

$$M = C \cdot c = 0.85f'_c ab\left(d - \frac{a}{2}\right) \tag{18}$$

$$0.85f'_c b(a^2 - 2ad) = -2M$$

$$a^2 - 2ad + d^2 = -\frac{2M}{0.85f'_c b} + d^2$$

$$a - d = \pm d\sqrt{1 - \frac{2M}{0.85f'_c bd^2}}$$

Since a must be less than d, use negative sign with the square root term.

$$\frac{a}{d} = \left[1 - \sqrt{1 - \frac{2.35M}{f'_c bd^2}}\right] \tag{19}$$

The ratio $\frac{a}{d}$ corresponds to the equation 3 for k in the straight-line derivation and equation 18 is in a form to be used by the designer, providing he already has chosen the section dimensions. This is the case if the beam is under reinforced.

25. Balanced Design. For economy of design the designer should endeavor to have both materials fail simultaneously. A study of beam tests led Mr. Whitney to assume that a equals $0.537d$ for balanced design. Then

$$c = d - \frac{a}{2} = 0.732d$$

Substituting these values in equation 18,

$$M = C \cdot c = 0.85f'_c(0.537d)b(0.732d) = \frac{f'_c}{3} bd^2 \tag{20}$$

This is a simple equation by which to determine the section dimensions.

$$M = T \cdot c = f_y A_s(0.732d)$$

or

$$A_s = \frac{M}{0.732f_y d} = \frac{f'_c bd^2}{3 \times 0.732f_y d} = 0.456\frac{f'_c}{f_y} bd \tag{21}$$

This balanced steel area is considerably more than has been used generally. The following tabulation gives the percentage of steel p for balanced design by the two methods. The plastic theory tends to give a smaller beam section with more steel than the working-stress straight-line theory.

STEEL RATIO FOR BALANCED DESIGN

Compressive strength f'_c	2000	2500	3000
A.C.I. working-stress straight-line theory	0.009	0.011	0.014
Plastic theory	0.018	0.023	0.027

There is no advantage in using more steel than called for by balanced design, as the load at failure is not increased. This load is now determined by failure of the concrete. There is an advantage in using a steel area somewhat less than that called for by balanced design, as overloading will not cause a sudden compressive failure in the concrete, but incipient failure will be evidenced by a gradual yielding of the steel at the stress f_y with marked increase of deflection and cracks in the beam. Such phenomena would be easily noticed in time to reduce the load or strengthen the beam. Also, it often happens that the under-reinforced beam is more economical.

CHAPTER 3

SLABS WITH ONE-WAY STEEL

26. One-Way Slabs. The first application of the rectangular beam theory will be made to the design of slabs whose steel runs perpendicularly to the supporting beams. The term *slab* is used to denote the floor in contrast with the supporting beams and girders. The top surface may be finished to form the actual floor surface, or additional flooring, such as wood, cork, etc., may be laid over it. Sometimes a granolithic finish is applied as a wearing surface. This is counted as part of the slab if it is applied soon enough to bond with the concrete. The slab is generally poured for many spans, giving a continuous monolithic floor. The supporting beams are poured with the slab and all form one mass, so that the slabs are restrained at their supports against a change of slope by these supporting beams and the slab of the adjacent span. The resistance of a beam to the turning tendency, which is torsion for the beam, is not as great as the resistance of a column or wall. Also the restraint of the end span, where there is no slab beyond, is not so great as for the interior spans. Therefore, the moment distribution is different for end spans than for interior spans of the same length.

27. Continuous Members. A reinforced concrete floor system is poured simultaneously for a considerable distance. The slab steel runs through the beams, the beam steel runs through the girders, and both the beam and girder steel run through the columns. The result is that the whole floor system is tied together. The column steel runs into the column above, and the column stack also has continuity at its connections.

If we consider a single unit, such as one span of the beam, or a single story height for the column stack, we must include in the loading diagram the effect of the continuity of the member. This is done by computing the member with a fixed or partially fixed support. The span for such a member is taken as the clear span, as is customary for fixed-end beams.

A structural member whose ends are fixed may have a reversal of curvature of its neutral layer. Thus a fixed-end beam with a load in the span will *sag* in the center, giving compressive stresses near the top of a section and tensile stresses on the particles near the bottom. This is called positive bending. At or near the support the beam will *hog* with tension at the top and compression at the bottom. This is called negative bending.

The effect of continuity upon the bending moment is somewhat uncertain in concrete construction. It is well known that, if a beam is continuous over a number of supports, two kinds of moment are developed: positive at or near the middle of the span and negative over and adjacent to the supports. These moments cannot be determined by the

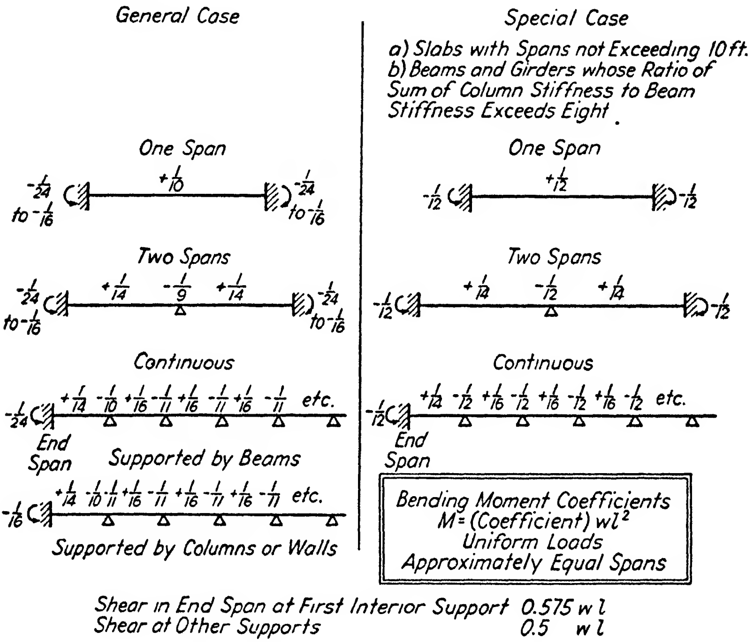


FIG. 10

equations of statics, and recourse is made to solutions involving the elasticity of the material. When the material is reinforced concrete, so many uncertainties arise that in general the bending moment values are matters of judgment which may be to some extent determined by suitably modifying the application of the theory of elasticity for homogeneous structures under similar conditions.

In Chapter 14 the effect of restraint of the supports is discussed in detail.

The A.C.I. Code specifies maximum bending moments for continuous beams of approximately equal spans. "Approximately equal" is a rather elastic term as the limit is said to be reached if the larger of two adjacent spans is not more than 20 per cent longer than the shorter. The values are given in A.C.I. Article 701 in the Appendix. These recommendations are listed in Figure 10 with the addition of suggested negative moments at the exterior support. Slabs of short span and beams supported by massive columns are regarded as fixed at the ends. The ratio of moment of inertia I of the section to the unsupported length l or h is taken as the relative stiffness $\frac{I}{l}$ of a beam or column. The greater the restraint at the support the greater will be the maximum negative bending moment. For a uniformly distributed load w the moment $\frac{wl^2}{12}$ corresponds to a fully fixed connection.

28. Reduction of Live Load. The live load on a structure is the removable load of people, vehicles, furniture, machinery, supplies, snow, wind, etc. It is the variation of this live load from day to day that causes variations in the shear forces and bending moments in beams and columns. If the live load is given as a maximum load per square foot, or per foot of length, it is probable that the full loading will seldom occur in many types of structures. If the member in question supports a large area, this probability is increased, and building officials permit a decrease in the maximum load used in design. This decrease should never be applied for warehouses, garages, etc., which may be fully loaded.

The dead load consists of the weights of the structures plus other permanent installations. Obviously the dead load should not be reduced as it is always present in full amount.

The tentative (1932) Building Code of the New England Building Officials Conference states that "a reduction of the total live load used in the design of two-way slabs and flat slabs of a certain area and in beams, girders, and columns based on a certain tributary floor area shall be permitted as noted in the following schedules." These live-load reductions are given in the Appendix.

29. Allowable Stresses. The ultimate compressive strength of the concrete f'_c is defined as the compressive strength of a concrete cylinder, 6 in. in diameter by 12 in. high, at the age of 28 days. The allowable stresses for design, whether they be fiber, shear, or bond stresses, are expressed as a percentage of the ultimate compressive strength f'_c.

The term mix will be used to denote the quality of concrete under consideration, which is based upon the ultimate compressive strength f'_c. Thus, a 2000-lb. mix refers to the ultimate strength $f'_c = 2000$ lb. per sq. in. at the age of 28 days.

The allowable stresses used in this text are those recommended in the 1941 Building Regulations for Reinforced Concrete of the American Concrete Institute. The allowable values are given in A.C.I. Articles 305 and 306 in the Appendix. These stresses should be regarded as maxima self-imposed by the designer. They are a mutual agreement between the inspector, the engineer, and the designer. The designer should not exceed a maximum stress. He should regard it much as he does his bank balance, as a limit not to be overdrawn. It is true that concrete materials vary, that design loadings may not always be completely present, and that a factor of safety has been applied. All these facts have been considered in the determination of the allowable stress. The honest designer should not rely on them a second time.

30. Commercial Sizes of Steel. Round bars, plain or deformed, can be obtained from $\frac{1}{4}$ in. to 1 in. inclusive, varying by $\frac{1}{8}$-in. sizes. The only square bars kept in stock are the $\frac{1}{2}$-in., 1-in., $1\frac{1}{8}$-in., and $1\frac{1}{4}$-in. See Table 1 in the Appendix. In this text deformed bars are used as reinforcement.

31. Weight of Concrete. The average weight of reinforced concrete will be taken as 150 lb. per cu. ft.

WORKING-STRESS STRAIGHT-LINE THEORY

DESIGN OF SLAB—ILLUSTRATIVE PROBLEM 1

32. Design of a Floor Slab. Given the interior span of a continuous slab (Fig. 11) loaded with a live load of 130 lb. per sq. ft., the span being 14 ft. 6 in. Compute the slab thickness and necessary steel, and include a detailed sketch of the steel.

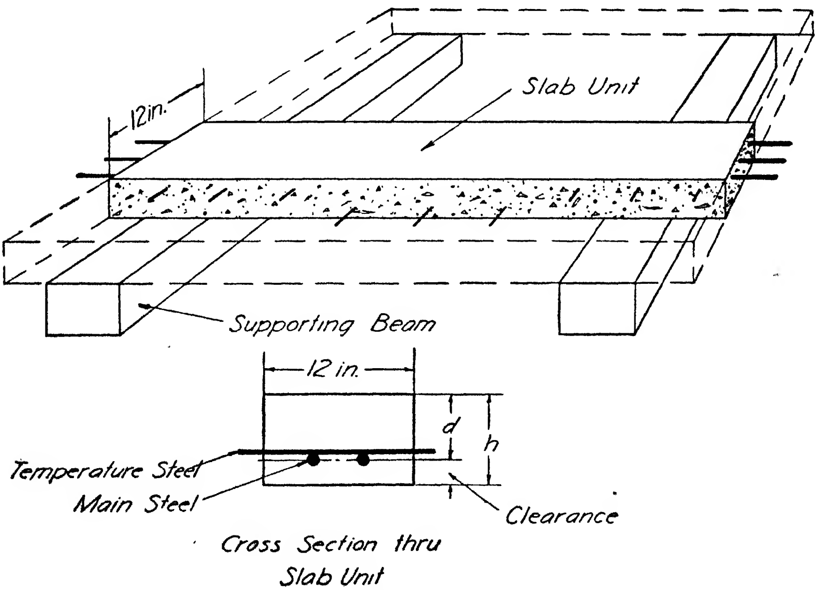


FIG. 11

Let us adopt a concrete testing f'_c = 2000 lb. per sq. in.,[1] and make the preliminary assumption that the slab is 6 in. thick and the beam stem 12 in. wide. It is necessary to make initial estimates of the slabs and beam sizes and correct them, if required, as these dimensions are not known. The experienced designer seldom has to make corrections.

33. Algebraic Solution. Rather than design simultaneously a large expanse of the slab we shall take a strip 1 ft. wide as our beam. The adjacent strips, each 1 ft. wide, are loaded with the same load and all deflect alike. There is no tendency for one strip to slide by another—in other words, no shear on the sides of the strips, and each can be regarded as a separate beam, just as a series of wooden planks are separate beams. If this strip is safely designed, the adjacent strips 1 ft. wide, loaded in exactly the same manner and deflecting alike, will also be safe.

The 6-in. slab, 1 ft. wide, will weigh 75 lb. per ft. of length. The live load is 130 lb. per sq. ft.; and the total load w is 205 lb. per ft. of length.

We have the case of a continuous beam, interior span; the maximum positive bending moment M_p by A.C.I. Article 701 (in the Appendix) is $M_p = \dfrac{wl^2}{16}$ and the maximum negative bending moment $M_n = -\dfrac{wl^2}{11}$. We will design the slab depth for the greater negative moment.

By A.C.I. Article 700 we use the average of the two adjacent clear spans for the negative moment. In this case, since the spans are equal, the clear span is 13 ft. 6 in.

$$M_n = \frac{wl^2}{11} = \frac{205(13.5)^2}{11} = 3400 \text{ ft.-lb.} = 40{,}800 \text{ in.-lb.}$$

With $f_c = 0.45 \times 2000 = 900$ lb. per sq. in. and $f_s = 20{,}000$ lb. per sq. in. as allowable stresses (A.C.I. Articles 305 and 306)

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} = \frac{1}{1 + \dfrac{20{,}000}{15 \times 900}} = 0.40$$

$$j = 1 - \frac{k}{3} = 0.87$$

$$M_n = \frac{f_c}{2} jkbd^2 = \frac{900}{2} \times 0.87 \times 0.40 \times 12 \times d^2 = 157 \times 12d^2 = 40{,}800 \text{ n.-lb.}$$

Solving,

$$\text{Minimum } d = \sqrt{\frac{40{,}800}{157 \times 12}} = 4.65 \text{ in.}$$

34. Clearance between *d* and *h*. The A.C.I. Code provides clearances for bond in Article 507 and the 1928 A.C.I. Code gives fireproofing clearances in Article 506 (see Appendix).

We shall assume that our aggregates are from glacial gravel. The protective covering for the top steel will be $\frac{3}{4}$ in. and the fireproofing for the positive steel will be $1\frac{1}{2}$ in. The size of steel is not known, but its estimated diameter will be taken some-

[1] The modern floor concrete usually has strength of f'_c = 2500 to 3000 lb. per sq. in. The rather low strength f'_c = 2000 lb. per sq. in. is used in these illustrative problems since the low allowable stresses require greater attention to design details.

what large, so that we shall not be obliged to refigure the minimum clearance. Esti mating the steel as not larger than $\frac{1}{2}$-in. bars (Fig. 11),

$$\text{Minimum } h = d + \text{clearance} = 4.65 + 0.75 + 0.25 = 5.65 \text{ in.}$$

Practically it does not pay to vary the slab thickness by amounts less than half an inch. We shall adopt a slab 6 in. thick with an actual $d = 6 - 1.00 = 5$ in.

It may seem excessively fussy to compute the depth d to hundredths of an inch when one has observed the haste and inaccuracies of placing steel. Nevertheless, the actual value of d frequently figures an odd fraction of an inch. The values of A_s, j, k, etc., should not be computed to great precision as the actual d may vary by an eighth of an inch.

If the original estimate of the slab thickness is not correct, the computations are refigured for a corrected weight until the commercial h agrees with the estimated thickness. No steel computations are made until the concrete stresses and dimensions are satisfactory.

35. Minimum Depth for Positive Moment. The maximum positive moment

$$M_p = \frac{wl}{16} = 28{,}000 \text{ in.-lb.}$$

$$\text{Minimum } d = \sqrt{\frac{28{,}000}{157 \times 12}} = 3.86 \text{ in.}$$

$$\text{Minimum } h = d + 1.50 + 0.25 = 5.61 \text{ in.}$$

We shall adopt a 6-in. slab and use an actual depth d for positive bending of $d = 6 - 1.75 = 4.25$ in.

36. Steel. The positive tension steel A_p can be computed.

$$A_p = \frac{M_p}{f_s jd} = \frac{28{,}000}{20{,}000 \times 0.87 \times 4.25} = 0.38 \text{ sq. in. per ft. width}$$

The A.C.I. Code provides for the maximum spacing of the main steel in A.C.I. Article 710 (in the Appendix). The tension steel must not be too closely spaced because it is then difficult to place and the coarse aggregate will not pass through readily. It must not be too widely spaced, or there will be portions of the slab which are not served by the reinforcement.

The tension steel is best spaced at intervals approximately the depth of the slab. In this problem the required area can be given by $\frac{1}{2}$-in. round bars spaced 6 in. on centers, or by $\frac{1}{2}$-in. square bars spaced 8 in. on centers. Use the $\frac{1}{2}$-in. round bars. If it were desirable to use $\frac{3}{8}$-in. round bars, a new depth $d = 6 - 1.5 - 0.19 = 4.31$ in. would be used to compute the area A_p. If $\frac{5}{8}$-in. round bars were considered, the actual $d = 6 - 1.5 - 0.31 = 4.19$ in. In no case can the actual d be less than the minimum d of 3.86 in.

The maximum negative moment M_n at the support is 40,800 in.-lb. and the value of $d = 5$ in. The negative tension steel A_n is computed:

$$A_n = \frac{M_n}{f_s jd} = \frac{40{,}800}{20{,}000 \times 0.87 \times 5} = 0.47 \text{ sq. in. per ft. width}$$

This steel will be supplied by bending up the positive steel at an angle approximately 30°. The ratio of these areas, $\frac{0.47}{0.38} = 1.24$, calls for about 25 per cent more steel at the support than at the center of the slab. If bent bars shaped like Type A (Fig. 12) were used, there would be the same area of steel over the support as there is in the bottom near the center. If the bars of Type B (Fig. 13) were employed,

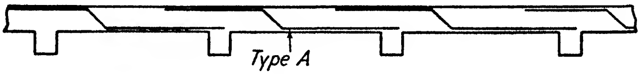


FIG. 12

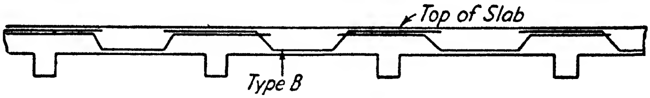


FIG. 13

there would be twice as much negative steel as positive. To give the desired ratio between negative and positive steel use $\frac{1}{2}$-in. round bars spaced at 6 in. in the center of the span, using Type B bars for every fourth bar. We shall then have three Type A for every one Type B bar and the negative steel will average 25 per cent more than the positive steel (Fig. 14).

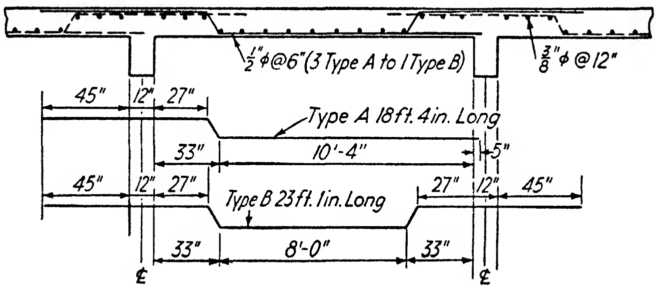


FIG. 14

Another possible steel arrangement is to use Type B bars with Type C bars, which are straight bars at the bottom of the slab. If every third bar is straight, there will be four bars at each support for every three bars at the bottom in the center, or one third more steel at the support (Fig. 15).

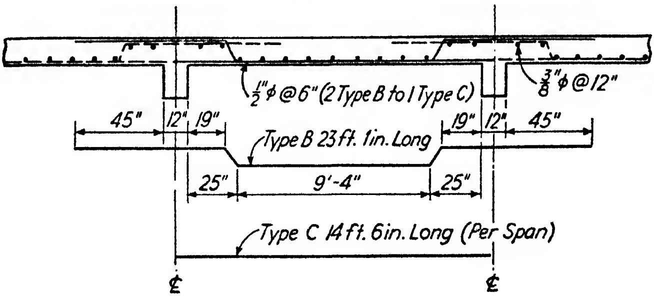


FIG. 15

37. Bending Steel. *First Arrangement.* Let us discuss the placing of the steel arrangement of Figure 14. Diagram 5 (in the Appendix) gives the parabolic bending moment curve that can be used for uniformly distributed loads (see explanation in Art. 340). The base line, $M = 0$, for the positive bending moment $M_p = \frac{wl^2}{16}$ is halfway up on the $\frac{wl^2}{8}$ curve. The positive moment equals zero where this base line cuts the curve at 15 per cent of the span (25 in. from edge of support). The center of this steel is $4\frac{1}{4}$ in. from the top. When bent up, its center is 1 in. from the top, so the rise of this bar is $3\frac{1}{4}$ in. If bent at approximately 30° it will reach its upper position in 6 in. If all the positive steel is bent up at 25 in., it will reach the top at 19 in. from the edge of the support.

Again using Diagram 5, the maximum negative moment of $M_n = \frac{wl^2}{11}$ has a base line ($M = 0$) at 73 per cent of $\frac{wl^2}{8}$. This occurs at 24 per cent of the span (39 in. from the edge of the support). Some of the negative steel must remain at the top until it is 39 in. out. If the Type B bars from the adjacent span are extended this far before they end and the Types A and B in this span are not at the top after 19 in. out, only one fifth of the negative steel runs at the top beyond 19 in. (12 per cent of span). By Diagram 5 the negative moment at 12 per cent of the span is $\frac{73 - 41}{73} = 0.44$ the maximum. In other words 20 per cent of the steel cannot care for 44 per cent of the moment. To satisfy the conflicting positive and negative requirements would require different bending points for the Type A bars and the reversal of some of them. This is not desirable. Some designers detail an intermediate bending position and expect the steel foreman and the inspector to encourage some staggering of the steel as it is placed. In Figure 14 a bending point of 33 in. is assumed (20 per cent of the span). The steel that ends on the bottom is run 10 diameters (5 in.) into the supporting beam. The steel that ends in the top is run to the point of inflection (39 in.) and anchored 12 diameters (6 in.) farther out. The bar details are shown in Figure 14.

Second Arrangement (Fig. 15). In this arrangement for every three positive bars there are four negative bars at each support; two of the four come from the adjacent span. In this case, if the two Type B bars in this span are bent down at 19 in. out, there are two bars available from the adjacent span to run out to the point of inflection at 39 in. In other words 50 per cent of the negative steel remains at 19 in. to carry 44 per cent of the maximum moment. This is satisfactory. The bar details are shown in Figure 15. The Type C bars will run continuously for 2 spans (29 ft. long) or 3 spans (43 ft. 6 in. long).

38. Selection of Arrangement. From the details of Figures 14 and 15 the weight of steel per foot of slab width figures 26 lb. for Figure 14 and 27 lb. for Figure 15. The second arrangement (Fig. 15) is more flexible and will be adopted.

39. Temperature Steel. The steel has been figured for its full allowable stress to carry the bending moment. It is also true that the shrinkage of the concrete as it sets and the temperature and moisture changes after it sets will tend to crack the concrete and hence stress the steel. This is cared for by putting in additional steel at right angles to the main steel which is known as shrinkage and temperature steel. A.C.I. Article 707 (in the Appendix) requires $p = 0.002$ for floor slabs with deformed bars.

$$A_s = pbd = 0.002 \times 12 \times 4.25 = 0.11 \text{ sq. in.}$$

Use $\frac{3}{8}$-in. rounds at 12-in. spacings.

This temperature steel can be supplied also as wire mesh. In any case it is wired to the main steel on the side nearest the center of the slab in order to preserve the fireproofing clearance (Figs. 14 and 15). This steel helps likewise to tie together the main steel into a continuous mat which will tend to resist as a unit. Thus, if a concentrated load is applied to the floor, one bar will not be obliged to resist the load alone.

40. End Span. The maximum bending moments in the end span are $-\dfrac{wl^2}{10}$ and $+\dfrac{wl^2}{14}$. A 6-in. slab is still satisfactory for concrete fiber stress. The steel areas and steel arrangement will differ from the interior spans and can be satisfied by possibly three different types of bars. Only enough steel will be bent up at the exterior support to care for $M_n = -\dfrac{wl^2}{24}$.

ILLUSTRATIVE PROBLEM 2

41. Solution by Plots. The previous solution was made by means of the equations derived in Chapter 2. The practical designer needs short-cut methods. The use of these same equations plotted gives a more rapid solution.

It will be noticed that both equations 11 and 12 (Art. 20) for the bending moment can be written in the form

$$M = Kbd^2$$

where $K = \frac{1}{2}f_c jk$ or $K = f_s pj$.

The plot for rectangular beams is given in Diagram 2 in the Appendix, where the steel ratios p are abscissae and values of K are ordinates. Thus, if $f_s = 20{,}000$ lb. per sq. in. and $f_c = 900$ lb. per sq. in., $K = 157$, which is the value given by the intersection of the $f_s = 20{,}000$ and $f_c = 900$ curves. The steel ratio $p = 0.009$. If more than this amount of steel is used, the steel stress will be less than 20,000 lb. per sq. in. when the concrete stress reaches 900 lb. per sq. in.; if less steel is used, the stress will be higher than 20,000. Using $K = 157$ in equation 20 the same depths d will be found as in the solution of Problem 1 and the design continues as in that problem.

ILLUSTRATIVE PROBLEM 3

42. Solution by Transformed Section. The algebraic equations for rectangular beams have graphical counterparts. Such a solution is made by the use of the transformed section (Art. 22). In other words, for the reinforced section is substituted a hypothetical section which is either all concrete or all steel and which has the same effective areas and effective moment of inertia as the actual section. In the problem given (Fig. 11), the unit beam was 1 ft. wide. Figure 6*a* shows the dimensions of the section. According to the beam theory we consider in our computations the concrete only on the compression side of the section. In Figure 6*b* the shaded rectangle *abcd* represents the compression area above the neutral axis.

For the same strain 1 sq. in. of steel can carry a stress n times that of 1 sq. in. of concrete, or it requires n square inches of concrete to carry the same force that 1 sq. in. of steel takes. On the tension side we count no actual concrete area but we substitute for the steel n times as much concrete area.

Revising equation 2 (Art. 20), we have

$$\frac{f_c}{\frac{f_s}{n}} = \frac{k}{1 - k}$$

which is the algebraic statement that the equivalent concrete stresses vary with the distance from the neutral axis. If the maximum allowable concrete stress f_c equals 900 lb. per sq. in. and the maximum stress f_s equals 20,000 lb. per sq. in.

$$k = 0.40 \qquad \text{and} \qquad j = 0.87 \text{ (Art. 33)}$$

By equation 13 (Art. 20),

$$p = \frac{f_c}{2f_s}k = \frac{900 \times 0.40}{2 \times 20{,}000} = 0.009$$

The moment of inertia of the transformed section (Fig. 6*b*) about the neutral axis is:

Tension Area. The area

$$ef = nA_s = npbd = 15 \times 0.009 \times 12 \times d = 1.62d$$

Assuming $\frac{1}{2}$-in. steel the area ef will be $\frac{1}{2}$ in. high, and have a length $\frac{1.62d}{\frac{1}{2}}$ or $3.24d$.

The moment of inertia about the neutral axis $= I_{N.A.} = I_{C.G.} + Aa^2$, where $I_{N.A.}$ and $I_{C.G.}$ are the moments of inertia about the neutral axis and center of gravity of the area A, respectively. The term a is the distance between the neutral axis and the center of gravity of the area. The moment of inertia of the tension area equals

$$\frac{bh}{12} + Aa^2 = \frac{3.24d}{12}\left(\frac{1}{2}\right)^{\prime} + 1.62d(d - kd)^{\prime}$$

$$= 0.0338d + 1.62d^3(1 - k)^2$$

It is customary to neglect the term $\frac{bh^3}{12}$, as it is small compared with the product Aa^2. The moment of inertia of the tension area equals

$$1.62d^3(1 - 0.40)^2 = 0.584d^3$$

Compression Area. The compression area is a rectangle. The moment of inertia about its base is

$$\frac{bh^3}{3} = \frac{12(kd)^3}{3} = \frac{12 \times (0.40)^3d^3}{3} = 0.256d^3$$

Total moment of inertia $I = 0.584d^3 + 0.256d^3 = 0.840d^3$.

From the general beam theory

$$f = \frac{My}{I}$$

For the extreme particles in compression with positive bending

$$900 = \frac{28{,}000(kd)}{0.840d^3} = \frac{28{,}000 \times 0.40}{0.840d^2}$$

$$d^2 = 14.8 \qquad \text{or} \qquad \text{minimum } d = 3.85 \text{ in.}$$

Commercial Size. When we adopt the actual $d = 4.25$ in. for a 6-in. slab, the fiber stress f_c in the concrete will be reduced, because the concrete area is increased. The actual steel area should be such that its fiber stress $f_s = 20{,}000$ lb. per sq. in. Taking moments about the neutral axis,

$$(12kd)\frac{kd}{2} = (npbd)(d - kd)$$

$$6(4.25)^2k^2 = 15 \times 12(4.25)^2(1 - k)p$$

or

$$k^2 + 30kp - 30p = 0 \tag{22}$$

On the tension side the equivalent concrete stress equals

$$\frac{f_s}{n} = \frac{20{,}000}{15} = 1330 \text{ lb. per sq. in.}$$

and

$$I = (npbd)(d - kd)^2 + \frac{b(kd)^3}{3} = 15 \times 12(4.25)^3(1 - k)^2p + \frac{12(4.25)\ k}{3}$$

$$I = 13{,}820(1 - k)^2p + 307k^3 = 307(k^3 + 45k^2p - 90kp + 45p)$$

From

$$= \frac{My}{I}$$

$$1330 = \frac{28{,}000(d - kd)}{I} = \frac{28{,}000 \times 4.25(1 - k)}{307(k^3 + 45k^2p - 90kp + 45p)}$$

$$k^3 + 45k^2p - 90kp + 0.292k + 45p - 0.292 = 0 \tag{23}$$

The solution of equations 22 and 23 simultaneously gives

$$k = 0.37 \qquad \text{and} \qquad p = 0.0074$$

The solution is difficult, however, and the method of transformed sections is recommended only for the determination of minimum depth d. This method gives a simpler solution when a completed design is checked (Art. 45). Even then the algebraic solution is quicker.

The steel area equals

$$A_s = pbd = 0.0074 \times 12 \times 4.25 = 0.38 \text{ sq. in.}$$

This is the same value computed by the algebraic solution (Art. 36). Solutions by the transformed area are best made wholly graphically. Plotted at large scale they give accurate results.[2]

[2] See textbooks on graphic statics. Also "Stresses in a Composite Member Subjected to Bending and Direct Stress," by B. A. Rich and W. W. Bigelow, *Jour. Boston Soc. of C. E.*, Feb., 1926, p. 62.

CHECK OF DESIGN

ILLUSTRATIVE PROBLEM 4

43. Checking Designs. *Algebraic.* If a drafting-room checker or a building inspector were to check the design given above, he could not assume that $f_s = 20{,}000$ lb. per sq. in., or that $f_c = 900$ lb. per sq. in., as the design may not be made for these values. He would note that

(*a*) The mix is 2000 lb. per sq. in. concrete.
(*b*) The slab is 6 in. deep.
(*c*) The positive steel is $\frac{1}{2}$-in. round bars at 6 in. on centers.

Checking the section of maximum positive bending,

$$n = 15 \quad p = \frac{0.39}{12 \times 4.25} = 0.00765 \quad np = 0.115$$

From equation 9 (Art. 20),

$$k = \sqrt{2np + (np)^2} - np = 0.379$$

$$= 1 - \frac{k}{3} = 0.874$$

$$f_c = \frac{2M}{jkbd^2} = \frac{2 \times 28{,}000}{0.874 \times 0.379 \times 12 \times (4.25)} = 780 \text{ lb. per sq. in.}$$

Allowable $f_c = 900$ lb. per sq. in.

$$f_s = \frac{M}{pjbd^2} = \frac{28{,}000}{0.00765 \times 0.874 \times 12 \times (4.25)^2} = 19{,}300 \text{ lb. per sq. in.}$$

Allowable $f_s = 20{,}000$ lb. per sq. in.

Similarly, checking the section of maximum negative bending (Fig. 15),

$$p = \frac{0.39 \times \frac{4}{3}}{12 \times 5} = 0.0087 \quad k = 0.397 \quad j = 0.868$$

$$f_c = \frac{2 \times 40{,}800}{0.868 \times 0.397 \times 12 \times (5)^2} = 790 \text{ lb. per sq. in.} \quad \text{Safe.}$$

$$f_s = \frac{40{,}800}{0.0087 \times 0.868 \times 12 \times (5)^2} = 18{,}000 \text{ lb. per sq. in.} \quad \text{Safe.}$$

ILLUSTRATIVE PROBLEM 5

44. Checking by Plots. Diagram 2 in the Appendix is also useful for checking a finished design. Let us check the design of Problem 1 by the plots. For the maximum positive moment

$$K = \frac{M}{bd^2} = \frac{28{,}000}{12 \times (4.25)^2} = 129 \qquad \text{and} \qquad p = 0.00765$$

The intersection of this ordinate and abscissa gives

$$f_c = 780 \text{ lb. per sq. in.} \qquad \text{and} \qquad f_s = 19{,}300 \text{ lb. per sq. in.}$$

ILLUSTRATIVE PROBLEM 6

45. Checking by Transformed Section. The transformed section affords an alternative method of checking a finished design. A design by transformed section has already been given in Article 42. The solution is much easier if the amount of steel is known.

Once again checking Problem 1, the unit beam is 1 ft. wide and the positive steel equals 0.39 sq. in. per ft. width. In Figure 6, $h = 6$ in. and $d = 4.25$ in. The transformed tension area equals

$$nA_s = 15 \times 0.39 = 5.85 \text{ sq. in. of equivalent concrete}$$

The area *ef* represents this. It is $\frac{1}{2}$ in. high and 11.7 in. long, the center line being 4.25 in. from the top. If we wish to check the design by this method, there is the difficulty that the position of the actual neutral axis kd is not known. The neutral axis, however, must be at the center of gravity of the shaded areas in Figure 6*b*. Taking moments about the neutral axis,

$$(12kd)\frac{kd}{2} = 5.85(4.25 - kd)$$

$$6(kd)^2 + 5.85kd - 24.86 = 0 \qquad \text{and} \qquad kd = 1.60 \text{ in.}$$

$$k = \frac{1.60}{4.25} = 0.377$$

The moment of inertia of the transformed area about the neutral axis is

Compression Area.

$$\frac{b(kd)^3}{3} = \frac{12 \times (1.60)^3}{3} = 16.4$$

Tension Area.

$$\frac{11.7 \times (\frac{1}{2})^3}{12} + 5.85(2.65)^2 = 0.1 + 41.1$$

Total $I = 57.6$ (in.)^4.

Fiber Stress.

$$f = \frac{M_y}{I} = \frac{28{,}000}{57.6}y = 486y$$

Extreme fiber stress in the concrete

$$f_c = 486 \times 1.60 = 780 \text{ lb. per sq. in.}$$

The average fiber stress in the steel equals

$$f = 486 \times 2.65 = 1288 \text{ lb. per sq. in.}$$

$$f_s = nf = 15 \times 1288 = 19{,}300 \text{ lb. per sq. in.}$$

The section of maximum negative moment can be checked by similar computations.

PLASTIC THEORY—DESIGN OF SLAB

ILLUSTRATIVE PROBLEM 7

46. Plastic Theory. Let us solve Problem 1 by the plastic theory, assuming balanced design. As before, we take a slab strip 1 ft. wide and assume it to be 6 in. deep. The uniformly distributed load $w = 205$ lb. per ft. Mr. Whitney recommends [3] that the allowable dead plus live load be taken as four tenths of the ultimate load; in other words, that the allowable stresses be taken as four tenths the compressive strength of the concrete and four tenths the yield point of the steel.

Negative Bending.

$$M_n = \frac{wl^2}{11} = \frac{205 \times (13.5)^2 \times 12}{11} = 40{,}800 \text{ in.-lb.}$$

By equation 20 (Art. 25),

$$d^2 = \frac{3M}{f'_c b} = \frac{3 \times 40{,}800}{800 \times 12} = 12.7 \text{ in.} \qquad \text{and} \qquad d = 3.56 \text{ in.}$$

$$h = 3.56 + 0.75 + 0.25 = 4.56 \text{ in.}$$

Positive Bending.

$$M_p = \frac{wl^2}{16} = 28{,}000 \text{ in.-lb.}$$

$$d = \frac{3 \times 28{,}000}{800 \times 12} = 2.96 \text{ in.}$$

$$h = 2.96 + 1.5 + 0.25 = 4.71 \text{ in.}$$

Revising the slab thickness to 5-in. depth, the load $w = 193$ lb. per ft., $M_n = 38{,}400$ in.-lb., and $M_p = 26{,}400$ in.-lb. The minimum depth $h = 4.46$ in. for negative bending and $h = 4.62$ in. for positive bending. Use a slab 5 in. thick.

Steel. Equation 21 gives the steel area for balanced design.

Positive Bending.

$$A_s = \frac{M}{0.732 f_y d} = \frac{26{,}400}{0.732 \times 20{,}000 \times 3.25} = 0.56 \text{ sq. in.}$$

Use $\frac{1}{2}$-in. square bars spaced at 5 in. (area = 0.60 sq. in.).

Negative Bending.

$$A_s = \frac{38{,}400}{0.732 \times 20{,}000 \times 4} = 0.66 \text{ sq. in.}$$

This area is 10 per cent more than the actual positive steel. Adopting the steel arrangement of Figure 14, we get one fifth increase by using one Type B bar for every four Type A bars.

Economy. A comparison can be made of the above results with those of Problem 1 if costs are available. Let us assume that the forms cost the same in both designs, that 2000-lb. concrete costs 35 cents per cubic foot and steel costs 5 cents a pound in

[3] *Proc. A.S.C.E.*, Vol. 66, No. 10, Dec., 1940, p. 1778.

place. Since the commercial spacings of the steel are often in excess of the theoretical, this comparison is made using the same bar types but spacing them at the minimum possible of 6.19 in. for the working-stress design and 5.35 in. for the plastic theory. A strip of slab 1 ft. wide and $14\frac{1}{2}$ ft. long will cost:

	Working-Stress Theory	Balanced	Plastic Theory Under-Reinforced
Concrete slab, 6 in.	\$2.54		\$2.54
5 in.		\$2.12	
Steel, 26.2-lb., $\frac{1}{2}$-in. round	1.31		
36.8-lb., $\frac{1}{2}$-in. square		1.84	
24.3-lb., $\frac{1}{2}$-in. round			1.22
Total cost	\$3.85	\$3.96	\$3.76

47. Under-Reinforced Slab. Let us assume a 6-in. slab which requires less than balanced-design steel. Using the plastic theory and the negative moment, by equation 19,

$$\frac{a}{d} = \left[1 - \sqrt{1 - \frac{2.35 \times 40{,}800}{800 \times 12 \times (5)^2}}\right] = 0.225$$

$$\frac{\ }{d} = 1 - \frac{a}{2d} = 0.887 \qquad \text{and} \qquad c = 0.887 \times 5 = 4.43 \text{ in.}$$

$$\text{Negative } A_s = \frac{M}{f_y c} = \frac{40{,}800}{20{,}000 \times 4.43} = 0.46 \text{ sq. in.}$$

For the positive moment of 28,000 in.-lb., and $d = 4.25$ in.,

$$\frac{a}{d} = 0.212 \qquad \frac{c}{d} = 0.894 \qquad \text{and } c = 3.80 \text{ n}$$

$$\text{Positive } A_s = \frac{28{,}000}{20{,}000 \times 3.80} = 0.37 \text{ sq. in.}$$

Use $\frac{1}{2}$-in. round bars spaced at 6 in. (Area = 0.39 sq. in.)

The required negative steel area is 18 per cent more than the actual positive steel. Use the arrangement of Figure 14 with four Type A bars for one Type B bar. The relative cost is included in the tabulation above, assuming the $\frac{1}{2}$-in. round bars to be spaced 6.35 in. on centers. It will be noticed that the balanced design of the plastic theory is somewhat greater in cost than the working-stress design, and it uses less concrete and more steel. The cost of the under-reinforced design is much cheaper than the balanced, and slightly less than the working-stress design.

48. Summary. A designer or inspector working in a given city soon memorizes the values of K called for by the city building code and works without reference to any text or plot. However, the student is advised to try all the methods indicated so that he may use any one with facility and also determine which one suits his own personal taste. The experienced designer uses the plots and tables unless he deals with a concrete strength for which there are no plots, whereupon he designs on the basis of the fundamental algebraic equations.

One-way slabs are designed as rectangular beams to carry the applied bending moment. Unusually heavy loads for short spans require a check for shear stresses (see Chapter 5). Slabs with very long spans should be checked for deflection (Art. 165).

With the exception of the plastic theory the designs illustrated in this chapter are computed due to the loading only. The allowable stresses have been adjusted by experience low enough so that the usual shrinkage, flow, and temperature changes do not give excessive deformations.

CHAPTER 4

SHEAR, BOND, ANCHORAGE, AND DIAGONAL TENSION

The fiber stress equations for a rectangular reinforced concrete beam can be used to design or check rectangular beams when the fiber stress at the section of maximum bending moment is the critical stress. This is the case for floor slabs of ordinary dimensions, and the illustrative problems have covered such slab designs. The safe design of the supporting beams with their greater depths and loads requires that the maximum shear stress be also investigated. The shear stress equation is derived in this chapter.

Whenever there are large shear stresses it is possible that tensile cracks may appear in the concrete on other planes than the cross sections. Tension on these inclined planes, called *diagonal tension,* will also be discussed and the computation of the diagonal tension steel known as *web steel.*

We have previously stated that tension steel is used in a reinforced concrete beam to take the tensile pulls which otherwise cause cracks in the concrete. This statement is incomplete because the steel must also transfer its load back to the concrete before the end of the bar is reached. The load, or pull T, in the bar is determined by the relation $T = \frac{M}{jd}$ as long as the bar is in tension. The decrease of the pull T requires that the adhesion, or friction, between the steel and concrete shall be sufficient to keep the steel from slipping. This frictional stress, or *bond stress,* is also discussed herewith. After the bar passes the point of inflection it ceases to be in tension, as it is now in a compression region. Any further length that may be given it may be regarded as *anchorage.* The usual requirements for anchorage are explained in subsequent articles of this chapter.

49. Fiber Stress and Shear Stress Variation. The fiber stresses in cross sections of beams of timber or steel are figured by the usual relation

$$f = \frac{My}{I}$$

The fiber stress is uniformly varying, and the maximum tension or compression occurs on the particles farthest from the neutral axis (Fig. 16, section AA).

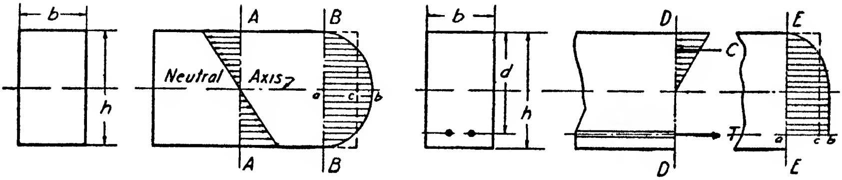


FIG. 16

The particles in a cross section of a steel or timber beam also have vertical shearing stresses due to the shear force at the section. The magnitude of this shearing stress is given by the equation

$$v = \frac{VQ}{bI}$$

where v = the intensity of vertical shearing stress
V = the total shear force at the section
Q = the statical moment about the neutral axis of the portion of the cross section beyond the layer containing the particle
b = the width of the layer containing the particle
I = the moment of inertia of the whole cross section about the neutral axis.

For a rectangular beam the stress varies in a parabolic relation, as shown in section BB of Figure 16. These stresses are vertical stresses, even though they are plotted horizontally for comparison. The maximum value $v_{\text{max.}}$ occurs at the neutral axis and is

$$v_{\text{max.}} = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A}$$

where A = the area of section = bh.

50. Fiber Stress Variation—Reinforced Concrete. In Chapter 2 the variations of the concrete fiber stresses and the average value of the steel fiber stress due to working loads were derived for reinforced concrete beams. The concrete compression stresses are uniformly varying (section DD, Fig. 16) while the tension in the concrete is not considered. The steel stresses are averaged at the center of gravity of the steel area.

SHEAR

51. Shear Stress—Reinforced Concrete. It is now necessary to derive expressions for the variation of vertical shear stress on the cross section of a reinforced concrete beam.

Let us take a rectangular reinforced concrete beam which has a width b, height h, and depth d from the extreme fiber in compression to the center of gravity of the tension steel. This section is shown in Figure 17a with the compression area above the neutral axis cross-hatched. We

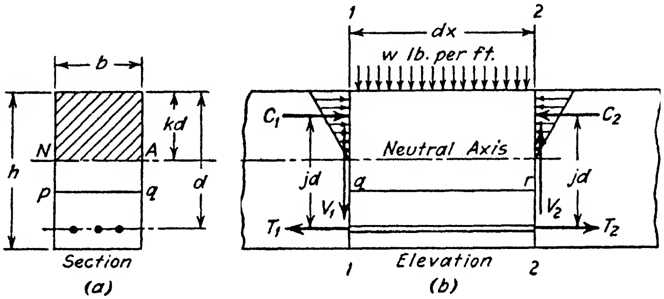


FIG. 17

will take as a rigid body the portion of the beam lying between the cross sections dx apart (Fig. 17b). The external forces acting on this body are

On sections *1–1* and *2–2*	C_1 and C_2	= resultant of the uniformly varying compressive stress
	T_1 and T_2	= resultant of the tensile steel stresses
	V_1 and V_2	= resultant of the vertical shear stresses
On top surface	w	= average intensity of the distributed loads on the beam between the two sections

Since these two cross sections are very near the neutral axes may be assumed to be the same distance kd down from the top on both sections and the two forces C_1 and C_2 will act along the same line, dx being small enough to insure 1 per cent accuracy for this assumption. The two forces T_1 and T_2 also act in line and the moment arm jd of the couples at each section will be the same. If there is a load wdx acting on the body, V_1 does not equal V_2. By the condition of equilibrium of statics, $\Sigma V = 0$.

$$V_2 - V_1 - wdx = 0$$

If $V_1 = 10{,}000$ lb. and dx is so small that $wdx = 10$ lb., then $V_2 = 10{,}010$ lb. In such a case the statement that $V_1 = V_2 = V$ is well within 1 per cent accuracy.

If we take moments about a point on the neutral axis at its center $\frac{dx}{2}$ from section *1–1*, we have, by the condition of equilibrium $\Sigma M = 0$,

$$T_1(jd) - T_2(jd) - V_1\frac{dx}{2} - \frac{V_2dx}{2} + (wdx) \times 0 = 0$$

$$(T_1 - T_2)jd - Vdx = 0$$

This assumes that the couple at section *1–1* is greater than that at section *2–2*. Then

$$T_1 - T_2 = \frac{Vdx}{jd} \tag{24}$$

$T_1 - T_2$ is the difference in pull at the ends of the steel bars; in this case it tends to pull the lower part of the body to the left. The push $C_1 - C_2$ will be equal to $T_1 - T_2$ and will tend to move the top part of the body to the right. There will be, therefore, shear forces on longitudinal planes through the body.

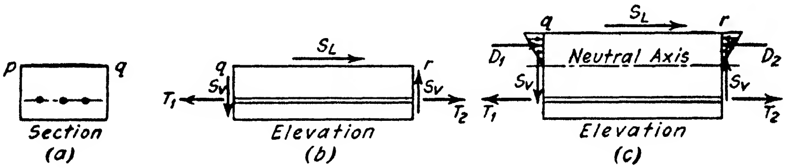


FIG. 18

Taking as a rigid body that portion of the rigid body of Figure 17 which lies below the longitudinal plane *pqr* (Fig. 18*b*) we have the following external forces acting.

On sections *1–1* and *2–2*: T_1 and T_2 = tensile force in steel; S_v = a portion of the vertical shear force V_1 or V_2

On longitudinal plane *pqr*: S_l = resultant of longitudinal shear stresses on plane *pqr*

Since we have assumed $V_1 = V_2 = V$ and are considering identical parts of sections *1–1* and *2–2* for the rigid body, the portions S_v of the vertical shears V_1 and V_2 are equal. Applying the condition of equilibrium, $\Sigma H = 0$, to the body gives

$$T_2 + S_l - T_1 = 0 \qquad \text{or} \qquad T_1 - T_2 = S_l$$

If the section *pqr* is taken above the neutral axis (Fig. 18*c*), there acts on the body a portion of the compression stresses whose resultants are D_1 and D_2.

In that case, $\Sigma H = 0$ gives

$$(T_1 - T_2) - (D_1 - D_2) = S_l$$

Between the steel and the neutral axis the maximum longitudinal shear force S_l amounts to $T_1 - T_2$. Above the neutral axis the longitudinal shear S_l will be gradually reduced as the compressive forces equalize it. The top particles have no longitudinal shear, as the sum of the longitudinal forces

$$(T_1 - T_2) - (C_1 - C_2) = 0$$

This force S_l is equal to the average stress s_l times the area, or

$$S_l = s_l b dx$$

$$S_l = s_l b dx = (T_1 - T_2) = \frac{V dx}{jd}$$

and

$$s_l = \frac{V}{bjd}$$

By a theorem of the theory of elasticity, the shear stress intensities on perpendicular planes through a particle are equal. The intensity of vertical shear stress v is equal to the longitudinal shear stress intensity s_l, or

$$v_{\text{max.}} = \frac{V}{bjd} \tag{25}$$

The variation of the vertical shear stress v for the given rectangular beam is shown at section *EE*, Figure 16. Equation 25 gives the maximum stress only, which is a constant between the neutral axis and the tension steel. There is a variety of possible cross sections whose compression areas are rectangular, and which are, therefore, rectangular reinforced concrete beams. For the sections of Figure 4 the shear force $S_l = T_1 - T_2$ will be unchanged on any longitudinal plane *pqr* (Fig. 17) between the tension steel and the neutral axis. However, the maximum intensity of shear stress will occur on the layer *pqr* between the tension steel and the neutral axis which has the least width.

52. Allowable Shear Stresses. A.C.I. Article 305 (see Appendix) provides four values of the allowable vertical shear stresses. If the cross section is unreinforced, the maximum intensity of vertical shear at section *AB* cannot exceed $0.02f'_c$.

If there is steel passing through the section, the resistance to vertical shear is much increased, since the steel must also be sliced off to produce failure. Therefore a maximum shear intensity of $0{,}06f'_c$ is allowed for ordinary anchorage.

If there is steel at both top and bottom, there is still greater resistance to shear, and even higher allowable values may be used for the combined action of steel and concrete. This is also true if greater attention is given to steel placement and anchorage. The maximum allowable value of shear intensity is $0.12f'_c$ for special anchorage. For continuous beams there is steel in the top and bottom of the section of maximum shear. For these beams the higher allowable stresses of special anchorage require a more rigid beam-column connection produced by greater anchorage lengths. This excess weight of steel is usually justified because of the increased values of shear and bond stresses that may be used for computations. The requirements for ordinary and special anchorage are discussed under A.C.I. Articles 902 and 903 (see Appendix). An illustration of the use of the shear stress formula is given in Problem 8 in Chapter 5.

BOND

53. Adhesion. It has been previously stated (Art. 51) that the unbalanced pull $T_1 - T_2$ in the tension steel tends to produce longitudinal shear in the concrete. This cannot occur, of course, unless there is adhesion, or bond, in the form of friction between the steel and concrete. The deformed bar has projections, also, which must plow their way through the concrete if the bar slips. There is a direct pressure on these projections in addition to frictional resistance. On this account the deformed bar has an advantage, even though harder to handle, and deformed bars will be used in the illustrative problems of this book.

54. Bond Stress Equations. The bond stress u can be computed by use of the rigid body of Figure 17 and the relations derived in computing the shear stress v. From equation 24

$$T_1 - T_2 = \frac{Vdx}{jd}$$

This unbalanced pull in the steel will cause slipping of the bar unless balanced by the adhesion between the concrete and steel in the length dx. This bond force is equal to the average bond stress u in the length dx times the area of contact between the steel and concrete. The area of contact is equal to the sum of the perimeters of the bars times the length and is written as (Σo) (dx).

$$\text{Bond force} = u(\Sigma o)dx = T_1 - T_2 = \frac{Vdx}{jd}$$

$$\text{Bond stress} = u = \frac{V}{(\Sigma o)jd} \tag{26}$$

The bond stress can also be obtained in terms of the bending moment.

$$M_1 = T_1jd \qquad \text{or} \qquad T_1 = \frac{M_1}{jd} \qquad \text{and} \qquad T_2 = \frac{M_2}{jd}$$

$$T_1 - T_2 = \frac{M_1 - M_2}{jd}$$

But the distance between these sections is only dx, and $M_1 - M_2$ can be written as dM, a small difference in bending moment. The bond force in the length dx equals $T_1 - T_2$, or $\frac{dM}{jd}$.

$$\text{Bond force} = u(\Sigma o)dx = T_1 - T_2 = \frac{dM}{jd}$$

$$u = \frac{T_1 - T_2}{\Sigma odx} = \frac{dM}{dx}\frac{1}{\Sigma ojd} \tag{27}$$

This is another form of equation 26 since, by applied mechanics, $V = \frac{dM}{dx}$, or the vertical shear equals the rate of change of bending moment. The bond stress u varies with the difference in pulls $T_1 - T_2$ of the bars, and, therefore, with the rate of change of the bending moment.

55. Position of Maximum Bond Stresses. If a beam is fixed or partially fixed at the ends and is loaded with a uniformly distributed load of w pounds per foot, the bending moment diagram will be of the shape shown in Figure 19. Let us assume that partial fixity gives $M_n = M_p = 500{,}000$ in.-lb.

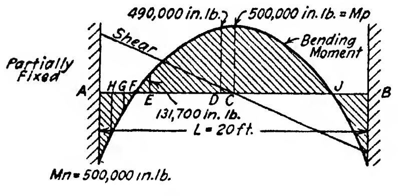


FIG. 19

The positive tension steel in the bottom of the beam has the maximum allowable stress $f_s = 20{,}000$ lb. per sq. in. at section C of maximum positive bending moment $M_p = 500{,}000$ in.-lb. The curve does not vary rapidly here, and a foot away at D the bending moment is also large, say 490,000 in.-lb. The change of bending moment $\frac{dM}{dx}$ in a foot is 10,000 in.-lb. The bending moment at F, the point of inflection, is zero, and a foot away at E is 131,700 in.-lb. Here the change of bending moment per foot is 131,700 in.-lb. and is a much more severe case for bond. This is also indicated, of course, by the fact that the vertical shear V is greater at F than at C.

On the other hand, for negative bending near the support the tension steel will be at the top of the beam. At A, the section of maximum negative bending moment, the moment $M_n = -500{,}000$ in.-lb. A foot away at H the moment equals $-310{,}000$ in.-lb., giving a change of moment per foot of 190,000 in.-lb. At the point of inflection F the moment is zero; a foot away at G the moment is 15,000 in.-lb. The most severe bond conditions for the negative tension steel are at the support where the shear force V is a maximum.

In general it can be stated that the greatest bond stresses occur where the bending moment curve is the steepest, or in other words at the section of maximum shear force *while the bar is in the tension side of the section.*

56. Difference between Bond and Anchorage. The designer does not have control of the variation of the bond stresses, for they vary with the bending moment diagram. It does not help to use a longer bar. He can obtain lower bond values by using smaller bars or by fulfilling the requirements for special anchorage (A.C.I., Art. 903). A 1-in. square bar has an area of 1 sq. in. and a perimeter of 4 in. Four $\frac{1}{2}$-in. square bars have an area of 1 sq. in. and total perimeter $\Sigma o = 8$ in. Using four $\frac{1}{2}$-in. square bars instead of one 1-in. square gives the same area and weight of steel but reduces the bond stresses by half, since the perimeter is doubled.

It should be emphasized that tension steel is checked for bond only while in active tension. Thus in Figure 19 the positive tension steel is checked only between sections F and J, whereas the negative tension steel at the supports is checked only between sections A and F, or B and J.

After the positive tension steel passes beyond the point of inflection F, it is no longer in tension and the distance it then runs is regarded as *anchorage.* Theoretically the stress in the bar is zero at F and its use as tension steel is completed, but it is counted for bond at this point and it is customary to run the bars farther to permit such use. In fact, some of this steel is continued on the bottom into the support to give a more rigid column-beam connection and to support stirrups.

Similarly the negative tension steel is checked for bond only between sections A and F. Any of this steel which continues on the top toward the center is being anchored. It is customary to run some of this steel to the center line to afford support for stirrups.

ANCHORAGE

57. Anchorage. Anchorage does not begin for tension steel until it has passed out of the region of tension. The length of anchorage, and hence the stress variation, are entirely at the control of the designer.

Thus, in Figure 20*a*, if a bar is embedded in concrete and is pulled by a force T, the necessary length of embedment can be computed. Let u equal the average bond stress between the concrete and steel. If

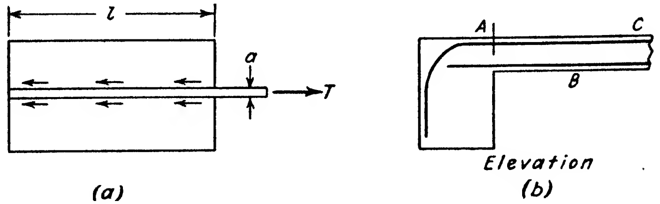


FIG. 20

the bar does not slip, the pull T must be balanced by the bond force between the concrete and steel. Let a be the diameter of a round bar or the side of a square bar.

Round Bar.

$$T = f_s \frac{\pi a^2}{4}$$

$$\text{Bond force} = u\pi a l$$

Then

$$f_s \frac{\pi a^2}{4} = u\pi a l$$

and

$$l = \frac{f_s a}{4u} \tag{28}$$

Square Bar.

$$T = f_s a^2$$

$$\text{Bond force} = u4al$$

$$f_s a^2 = u4al$$

$$l = \frac{f_s a}{4u} \tag{28}$$

Figure 20*b* represents a wall beam supporting the exterior end of the exterior span of a slab. At the face of the support A the continuous slab is subjected to a maximum negative bending moment $M_n = -\frac{wl^2}{24}$. The negative tension steel at the top of the slab has a stress of 20,000 lb. per sq. in. At A the bar must run far enough in both directions to anchor this pull. If the concrete tests $f'_c = 2000$ lb. per sq. in., the bond stress $u = 0.05f'_c = 100$ lb. per sq. in. (see A.C.I. Art. 305 in the Appendix). By equation 28

$$l = \frac{20{,}000 \times a}{4 \times 100} = 50a$$

To the right of A while the bar is in active tension, the bar runs more than $50a$. To the left of A it must also run $50a$ into the beam *for anchorage*. At this same support some of the positive steel has not been bent up. When it passes the positive point of inflection B it has zero stress theoretically, but it is customary to anchor zero stresses by $10a$ for ordinary anchorage or by a hook for special anchorage. Hence, carrying the positive bar $10a$ to the left of the point of inflection will anchor it, but this steel will be run into the support with an anchorage of $10a$ to give a more rigid beam-slab connection.

58. Allowable Bond Stresses. The bond strength has usually been determined by tests which pull a bar from a concrete block. In the pull-out tests the block is usually set on the top of the machine and the steel bar grasped and pulled. The concrete is in compression and the steel in tension. In the actual beam both are in tension. A few tests have been made with the concrete in tension. They do not indicate a great variation from the usual results. The allowable bond stresses u given in A.C.I. Article 305 are based on the compressive strength f'_c of the concrete. The factor of safety has been so adjusted that these allowable values compare favorably with the bond computation data from reinforced concrete beam tests. If a moderately good design is contemplated (ordinary anchorage), bond stresses of $u = 0.05f'_c$ are allowed for deformed bars. If greater attention is paid to anchoring the bars, as in special anchorage, allowable bond stresses of $u = 0.075f'_c$ are permitted for deformed bars in beams, but u must not exceed 250 lb. per sq. in.

Anchorage lengths are always computed for the lower frictional stress of $u = 0.05f'_c$.

The application of the bond stress formula and of the anchorage requirements to a design is given in Problem 8 (Chapter 5) and the subsequent beam problems.

Steel which is in compression seldom requires a check for bond stresses. As will be shown in Article 83, the compression force C'_s in the steel does not change rapidly in magnitude, since it forms only a part of the total compression force C at any section. The necessary bond stresses are small and are aided by the bearing of the end of the bar against the concrete.

DIAGONAL TENSION

59. Diagonal Tension. A plain concrete beam will scarcely support its own weight because the tensile strength of concrete is so low. Such a beam will fail suddenly without warning at the section of maximum bending moment (Fig. 21*a*) by a crack which starts on the tension side. The collapse is immediate, and the beam falls into two pieces.

This sudden failure can be prevented and the applied load much increased by introducing steel to take the pull that would otherwise cause the cracking of the concrete. This steel is computed by the bending moment equation 12. In order to take the pull, this steel must transfer the stress back to the concrete, before it ends, by suitable anchorage in

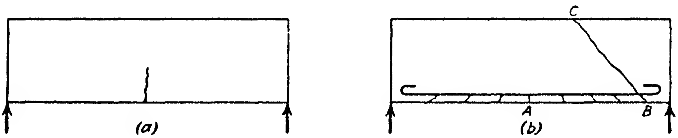


FIG. 21

some compression region. With the introduction of bending moment steel the concrete will crack slightly at the section A of maximum bending moment (Fig. 21*b*). As the load increases, sections of smaller bending moment will be overstressed in tension and the concrete will crack as far up as the bending moment steel. Eventually the beam will fail suddenly, cracking along a line such as BC. This is a tension failure, as evidenced by the opening up of this crack, but the presence of the longitudinal bending moment steel does not prevent this tension failure. However, the steel does prevent the beam from collapsing in two separate parts.

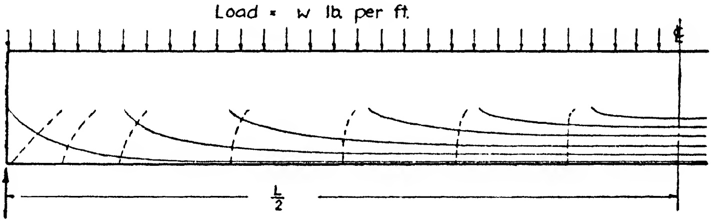
The appearance of these cracks and the failure by tension on the diagonal plane BC can be predicted by the stress analysis of the theory of elasticity. If we consider a plain concrete beam, simply supported and loaded with a uniformly distributed load, the fiber stress f and the shear stress v can be figured for any particle in any cross section. The shear stress s_l on a longitudinal plane through the same particle are known to be equal to the vertical shear stress v. The principal tensile stress n_1 can be computed by the relation

$$n_1 = \frac{f}{2} + \frac{1}{2}\sqrt{f^2 + 4v^2} \tag{29}$$

The principal tensile plane, or plane with the maximum tensile stress, will make an angle with the cross section equal to α, where

$$\tan 2\alpha = \frac{2v}{f}$$

At the section of maximum bending moment the shear equals zero and $n_1 = f$. The cross section is the principal tensile plane, and a crack will tend to form along this cross section. At other sections shear forces will be acting, and the principal tensile stresses will be inclined at different angles to the cross section, depending on the relative magnitudes of the fiber stress f and the shear stress v. If we consider an unreinforced



BEAM WITH SUPPORTED ENDS - HALF SPAN

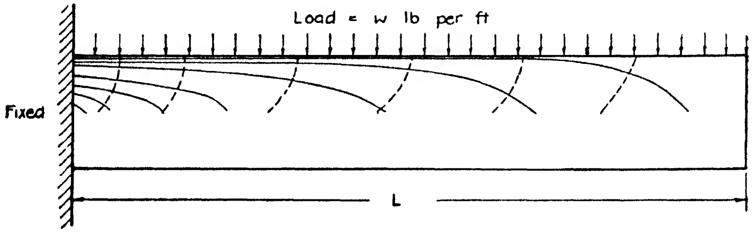
FIG. 22

rectangular concrete beam of span l and loading w pounds per foot, the direction of these principal tensile stresses can be sketched by a series of lines such as the unbroken lines in Figure 22. All these lines cross the neutral axis at 45°, since there $f = 0$ and $\tan 2\alpha = \infty$, giving $2\alpha = 90°$ and $\alpha = 45°$.

The dotted lines in Figure 22 drawn perpendicular to the direction of maximum tension mark the direction of possible cracks in different parts of the beam. It will be noted that in the center the cracks nearly coincide with the cross section, and the longitudinal steel perpendicular to these possible cracks prevents their appearance. Near the end the cracks are about 45° with the longitudinal and the bending moment steel cannot alone take this pull.

Figure 23 shows the same full lines of maximum tension and the dotted lines of possible cracks for a cantilever beam of span l and loaded uniformly with w pounds per foot length. The cracks are all inclined, and longitudinal steel alone will not prevent their appearance.

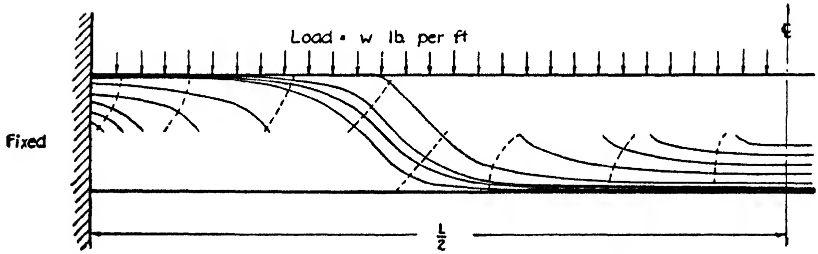
In Figure 24 a beam with fixed supports is represented. Its span is l and the load is w pounds per foot uniformly distributed. Such a beam has a maximum negative bending moment of $M_n = -\frac{wl^2}{12}$ at the ends and a maximum positive bending moment of $M_p = +\frac{wl^2}{24}$. Here, again, for a large part of the span the possible cracks are inclined and longitudinal steel alone is not sufficient.



CANTILEVER BEAM - FULL SPAN

Direction of Maximum Tension

FIG. 23



BEAM WITH FIXED ENDS - HALF SPAN

Direction of Maximum Tension

FIG. 24

The minimum weight of steel is required, if the longitudinal steel is bent into curves coinciding with the lines of maximum tension, so that the steel is always in the direction of the pull. A more practical possibility is to approximate these lines as in Figure 25. For commercial

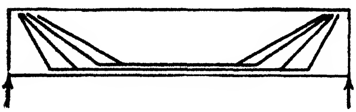


FIG. 25

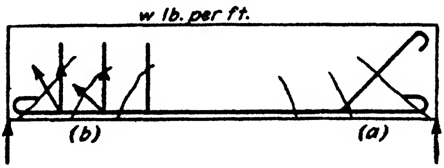


FIG. 26

work, however, it is customary to make bends of 45°, or thereabouts (Fig. 26a). Such a bend is satisfactory near the support of the beam of Figure 22. It is reasonably satisfactory throughout the span for a cantilever beam (Fig. 23) and from the support to near the quarter span for the fixed-end beam (Fig. 24). Most concrete beams in a building are partially fixed, and 45° bends are satisfactory between the support and the quarter or fifth span.

Certain designers prefer to use straight bars for all bending moment steel. They provide for diagonal tension cracks by adding vertical steel

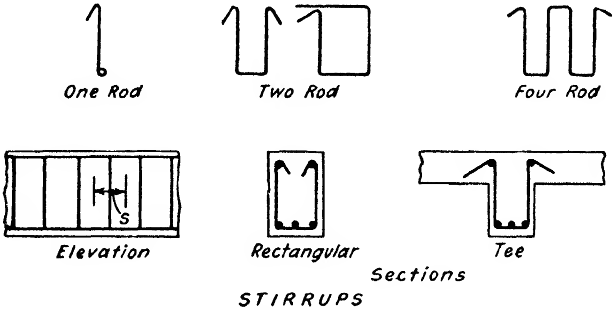


FIG. 27

in the form of stirrups (Fig. 27) which act with the longitudinal steel to give a resultant pull in the proper direction (Fig. 26*b*).

60. Steel Systems. There are many systems of steel placement for continuous concrete beams, but three types are in general use.

Type 1 (Fig. 28) consists of straight bars for longitudinal bending moment steel, and stirrups for diagonal tension reinforcement. The positive steel on the bottom can be cut off as the bending moment reduces,

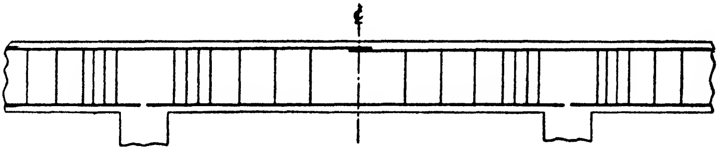


FIG. 28

but the anchorage specifications require that some of the bars run into the support to assist in forming a rigid connection. The negative tension steel at the top is needed in full amount at the support. Some of these bars run to the center line and lap with the bars from the other support. The stirrups are wired to the top and bottom steel so that they cannot slip as the concrete is poured. This system is easy to design, as separate bars are supplied for positive and negative steel and for diagonal tension. It gives a greater weight of steel than the other methods. Its advocates claim, however, that the total cost is not more, as it can be placed rapidly and requires no bends except for the stirrups. An illustrative design using this system is given in Problem 8 (Chapter 5).

Type 2. The second system of steel placement consists of straight bars in the bottom for positive steel as before, and the inverted U bars for the negative steel at the top over the support also act as diagonal tension steel (Fig. 29). These bars are bent down as near as possible

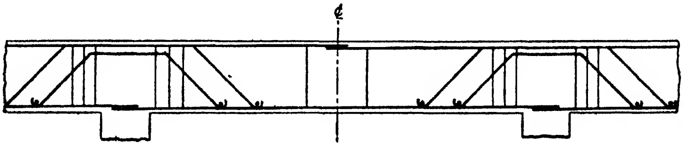


Fig. 29

to the support. They are hooked at the bottom about spacer bars which are wired to the bottom steel. Two negative bars run to the center line to carry the few stirrups which may be needed to complete the diagonal tension reinforcement. This system is nearly as flexible as type 1, though the negative steel must perform two functions, acting as negative bending moment steel and later as diagonal tension steel. The weight of steel is less; but the negative bars, which are usually medium size, have four bends each. An illustrative design using this system is given in Problem 17 (Chapter 7).

Type 3. A third system of steel placement provides the positive steel in two types of bar shapes (Fig. 30). The bottom row consists of straight

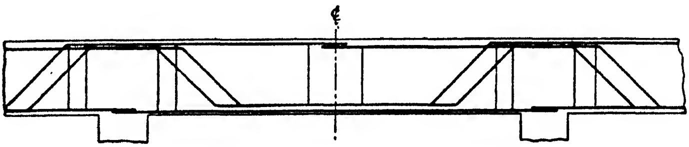


Fig. 30

bars as in the previous cases. Some of these straight bars run into the support for anchorage. The top row are shaped as in Figure 31 and are known as truss bars. These truss bars act as positive steel and diagonal tension steel in this span, and as negative tension steel at each support. The use of one bar for three functions necessitates great care in the design. The system is not flexible, and it is difficult to meet the varying requirements for moment for end spans and unusual loadings. The weight of steel is often more than for type 2, while for long

Fig. 31

spans the truss bars have excessive lengths. The truss bars have four bends. The labor cost is probably greater than for type 2. An illustrative design using this system is given for the girder design of Problem 17 (Chapter 7).

61. Diagonal Tension Reinforcement. For any particle in the beam the tensile stress on the principal plane is given by equation 29 (Art. 59).

$$t = \frac{f}{2} + \frac{1}{2}\sqrt{f^2 + 4v^2} \tag{30}$$

When the fiber stress f is large in comparison with the shear stress v the maximum tension t is nearly longitudinal in direction, and the longitudinal, or bending moment, steel prevents the spread of tension cracks in the concrete. The maximum tension t makes a considerable angle with the longitudinal direction when the shear stress v is large in comparison with the fiber stress f. The use of equation 30 involves much labor, and, therefore, for commercial designs it is customary to assume that t is proportional to v in diagonal tension computations. Wherever the fiber stress f = zero, $t = v$. Therefore, at the neutral axis of all sections $t = v$. Wherever the maximum tension t is considerably inclined, t may equal $1.5v$, or $2v$, or $t = kv$, depending on how small the fiber stress f may be at the particle considered. In commercial work it is customary to design the diagonal tension reinforcement in terms of the maximum shear stress v in the cross section, assuming that $t = kv$. The constant k is taken large enough and the allowable value of v is adjusted so that the portion assigned to the concrete of the total tension on a particle does not exceed the allowable tension of concrete. Therefore, in the analysis of diagonal tension the concrete is ssumed to take some tension. This can be safely done as the sections in question have small bending moments and the concrete has not yet cracked in tension, while fine hair cracks may have already appeared at the sections of large bending moment.

The allowable shear stresses v, or *equivalent tension* stresses, which will not exceed the allowable tension in the concrete are given in the A.C.I. Code (see Appendix), a distinction being there made between ordinary and special anchorage of the steel.

The tension on a particle is divided between the diagonal tension steel and the concrete, so that

$$v = v_c + v_s$$

where v = maximum shear stress at a given cross section

v_c = allowable shear stress, or *tension equivalent*, taken by concrete

v_s = remaining shear stress, or *tension equivalent*, to be taken by diagonal tension reinforcement.

The amount of diagonal tension steel can be computed by relations derived as follows.

62. Diagonal Tension Equation for Stirrups and Bent Bars. Let Figure 32a represent one end of a beam [1] with bars bent up at an angle α with the longitudinal steel and crossing a plane AB of a possible diagonal tension crack which makes an angle θ with the longitudinal steel. The bent bars are spaced a distance c perpendicularly, or s longitudinally. The beam is b inches wide.

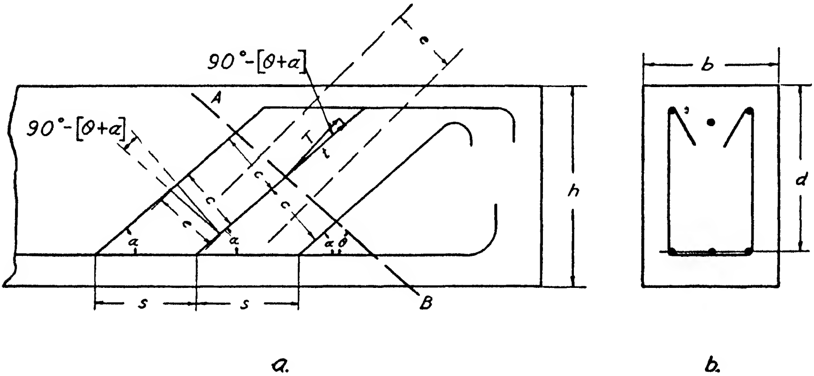


FIG. 32

Assumption 1. For values of α between 45° and 90°. Let us assume that the bent bars are nearly perpendicular to the crack or, in other words, nearly in line with the pull. The tensile stress T on plane AB is perpendicular to the crack AB and makes an angle of $90° - (\theta + \alpha)$ with the bent bars. The stress T can be resolved into a component t along the bar and one perpendicular to the bar.

The component perpendicular to the steel equals

$$T \sin [90° - (\theta + \alpha)]$$

The angle is small, by the assumption, and the concrete can withstand this stress, which can be resolved into tension and shear on the plane of the crack.

The component along the bar equals

$$T \cos [90° - (\theta + \alpha)]$$

This component is withstood in part by the concrete and in part by the bent bar. Let us designate the portion taken by the steel as $v_s \cos [90° - (\theta + \alpha)]$. The middle bar of the three shown in Figure 32a must

[1] See *Journal, Boston Soc. C. E.*, February 1925, pp. 75–76, by Professor Hale Sutherland.

withstand the pull on an area of the crack e long and b wide, where e is measured half way to the adjacent bars in the *plane* of the crack. The steel takes a total pull,

$$T_s = v_s be \cos [90° - (\theta + \alpha)]$$

By trigonometry

$$\frac{c}{e} = \cos [90° - (\theta + \alpha)]$$

and

$$\frac{c}{s} = \sin \alpha$$

therefore

$$e \cos [90° - (\theta + \alpha)] = c = s \sin \alpha$$

also (Art. 51)

$$v_s = \frac{V_s}{bjd}$$

then

$$T_s = v_s bs \sin \alpha$$

But

$$T_s = a_s f_s$$

where a_s = area of diagonal tension steel
f_s = tension stress in this steel.

Then

$$a_s f_s = v_s bs \sin \alpha$$

or

$$v_s = \frac{a_s f_s}{bs \sin \alpha} \tag{31}$$

and

$$v = v_c + v_s = v_c + \frac{a_s f_s}{bs \sin \alpha} \tag{32}$$

Equation 32 is given by the 1924 Joint Committee Report.

Assumption 2. For values of α less than 45°. In this case it is assumed that the steel makes a considerable angle with the crack and a resolution of stresses as in assumption 1 would give too great a component perpendicular to the bar to be taken by the concrete as tension and shear on the plane of the crack. The total tensile stress T on the plane is taken partly by the concrete tensile stress v_c and the rest by the steel stress v_s. In assumption 2 the tension stress T, which is perpendicular to the plane of the possible crack, is resolved into a component t_l parallel to the longitudinal steel and a component t parallel to the bent bars (Fig. 33). This assumption is justified because the crack will tend to start at the bottom where the longitudinal steel and bent bars are both present and able to take tensile pulls.

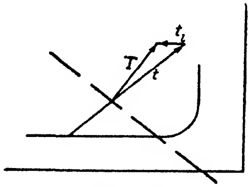


FIG. 33

The tensile force F acting on the plane of the crack for a length e equals

$$F = Tbe$$

Its vertical component is

$$F_v = Tbe \sin (90^\circ - \theta) = Tbe \cos \theta$$

Since the component stress t_l is horizontal, the vertical component of t equals the vertical component of T (Fig. 33). The vertical component B_v of the force parallel to the bent bar is

$$B_v = Tbe \cos \theta$$

The force B can now be computed by dividing by *sin* α. The force B is supplied partly by tension in the concrete, partly by the tension in the steel bar. Let v_s be the portion of the stress T taken by the steel. Then the force T_s in the bent bar equals

$$T_s = \frac{v_s be \cos \theta}{\sin \alpha}$$

As before

$$e \cos [90^\circ - (\theta + \alpha)] = s \sin \alpha$$

or

$$e \sin (\theta + \alpha) = s \sin \alpha$$

and

$$e = \frac{s \sin \alpha}{\sin (\theta + \alpha)}$$

substituting,

$$T_s = \frac{v_s b \cos \theta}{\sin \alpha} \; \frac{s \sin \alpha}{\sin (\theta + \alpha)} = \frac{v_s bs \cos \theta}{\sin (\theta + \alpha)}$$

$$T_s = v_s bs \frac{\cos \theta}{\sin \theta \cos \alpha + \cos \theta \sin \alpha}$$

as before

$$T_s = a_s f_s$$

$$v_s = \frac{a_s f_s}{bs} \; \frac{\sin \theta \cos \alpha + \cos \theta \sin \alpha}{\cos \theta}$$

If the crack angle $\theta = 45°$, $\sin\theta = \cos\theta$ and

$$v_s = \frac{a_s f_s}{bs}(\sin\alpha + \cos\alpha) \tag{33}$$

This is the equation given for this case in the 1924 Joint Committee Report.

For the special case, when $\alpha = 45°$, equations 32 and 33 give identical values of

$$v_s = \frac{1.41 a_s f_s}{bs} \tag{34}$$

63. Application to Design. Since bent bars in beams are usually at an angle of 45° and stirrups are vertical, equation 32 can be used for all diagonal tension computations. Illustrations of the application to stirrups and to bent bars are given in Articles 75, 135, and 146.

The A.C.I. Code recommends a formula for the area of diagonal tension steel in the form of vertical stirrups:

$$a_s = \frac{V_s s}{f_s jd} \tag{35}$$

This is obtained from equation 31 by substituting:

$$v_s = \frac{V_s}{bjd}$$

where V_s = portion of total shear force on the section taken by the steel. Then

$$\frac{V_s}{bjd} = \frac{a_s f_s}{bs \sin\alpha}$$

$$a_s = \frac{V_s s}{f_s jd}$$

Equation 35 is extensively employed commercially. However, the author will use in his illustrative problems the Joint Committee formulae which he regards as more convenient of application.

64. Maximum Spacing. The formulae given above for the computation of spacing of diagonal tension steel are not foolproof. If one uses two $1\frac{1}{4}$ square bent bars, it is possible that spacings of 60 in. or more may result. It is, of course, absurd to expect that a bar 5 or 6 ft. away will prevent the spread of a crack. It is the object of diagonal tension reinforcement to prevent the spread of cracks beyond the neutral axis into the compression area. The concrete on the tension side is assumed to be cracked in all fiber stress computations. To prevent ill-advised spacings, empirical equations are given to govern the maximum spacing longitudinally. A.C.I. Article 806 states that web reinforcement shall be spaced closely enough so that every potential 45° crack is crossed by at least one line of reinforcement before the crack reaches mid-depth of the beam. If special anchorage be used, wherever the shear stress exceeds $v = 0.06f'_c$ two lines of web reinforcement must cross the potential crack-line. Mid-depth is a convenient dimension and it insures that the crack does not extend farther up to the neutral axis.

65. Bent Bars. In Figure 34 are shown stirrups with a spacing of s such that each stirrup stops the first possible crack (dashed) before it

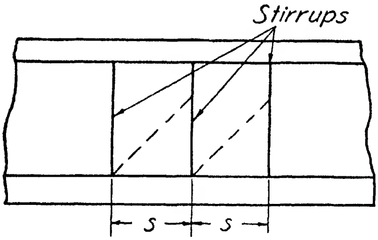


FIG. 34

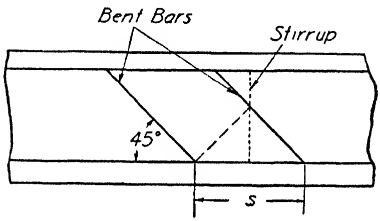


FIG. 35

reaches mid-depth. It is evident that bent bars may be spaced longitudinally much farther apart than stirrups and yet intercept this first possible crack at the same point (Fig. 35). Assuming both crack and bent bar to be at angles of 45° with the longitudinal, the spacing of the bent bars can be twice the stirrup spacing. Using A.C.I. Article 806 for sections where

(*a*) Maximum shear stress v is less than $0.06f'_c$

$$\begin{aligned} \text{Maximum } s &= 0.5d \text{ for stirrups} \\ \text{Maximum } s &= d \text{ for bent bars} \end{aligned} \tag{36}$$

(*b*) Maximum shear stress v is greater than $0.06f'_c$

$$\begin{aligned} \text{Maximum } s &= 0.25d \text{ for stirrups} \\ \text{Maximum } s &= 0.50d \text{ for bent bars} \end{aligned} \tag{37}$$

CHAPTER 5

RECTANGULAR BEAM DESIGN

The preceding chapters have discussed the theoretical and empirical information which forms the basis of practical beam designs. In this chapter a design will be made of a typical beam span. It is suggested that the reader note particularly the order of procedure and that the student check the various steps in order to understand the reasons for each operation.

66. Economy. *Stresses.* The most economical rectangular concrete beam is the one stressing steel and concrete to their maximum allowable values. It is proper, then, to design rectangular beams by use of the theoretical equations for fiber stress derived in Chapter 2.

Shape. A rectangular beam is poured into forms which have bottom and sides but no top. If the space to be filled is too narrow, the concrete cannot be tamped into place easily about the bottom steel and the removal of the forms will show "honeycombing" which can only be laboriously and unsatisfactorily patched. A beam too wide, and hence shallow, will have too much steel. Experience has shown that beams having ratios of depth to width not exceeding 2 to 1 give economical sections.

Mix. The choice of the proper strength of concrete to give the greatest economy involves many variables. Professor Inge Lyse [1] analyzes the results of tests on concrete to arrive at the following conclusions:

"Above a minimum number of cement particles necessary to give workability and binding strength to concrete, the strength of concrete increases in direct proportion to the increase in number of cement particles per unit of water." The relation can be written

$$S = A + B\frac{c}{w}$$

where S = strength of concrete, in pounds per square inch
A and B are constants
c = cement content
w = water content.

[1] "Relation between Quality and Economy of Concrete," *Jour. A.C.I.*, March–April 1933, p. 325.

The statement assumes that, in "the range of practical concrete mixes," the straight-line relation serves as well as Abrams' curve. Figure 1 shows the original Abrams curve which gives the relation between strength S and the water-cement ratio $\frac{w}{c}$.

The study of other tests causes Professor Lyse to conclude that "the factors which give high strength will also give high impermeability, greater resistance to freezing and thawing, and high fire resistance. On the other hand, the volume changes due to soaking and drying of the concrete will increase with the increase in cement content.

"In the practical application of the above study the cost of mixing, placing, curing and possible finishing of the concrete and also the cost of formwork and labor contribute to the total cost of the final structure. However, the cost of preparing, handling and curing the concrete is nearly the same for rich and lean mixes. The cost of formwork, labor, and finish may be somewhat different for the different concretes, since the cross-sectional area of the member changes with the strength of the concrete and is greater for lean than for rich mixes. However, in most cases the differences in the cost of these items are so small that for studies of the nature presented in this paper they may be neglected. . . . Generally the decrease in cost of the concrete will more than offset the increase of cost of the steel and the protective cover," if rich mixes are used. Professor Lyse comes to the conclusion that the concrete beam cost is little affected by the variation in the strength of the concrete.

We should consider also that, if concrete strengths of 4000 to 5000 lb. per sq. in. are used instead of 2000 to 2500 lb. per sq. in., slabs would be so thin that extra labor and inspection would be necessary to insure satisfactory results in mixing and placing the concrete and correctly locating the steel.

67. Order of Procedure in Design. In all reinforced concrete beam designs the concrete dimensions should be definitely adopted before the steel is computed. A logical order of design is:

1. Determine concrete size by bending moment requirements.
2. Check size for shear.
3. Figure steel area for bending moment.
4. Check steel for bond and anchorage requirements.
5. Figure diagonal tension steel.
6. Detail beam for the steel foreman.

Before studying the following problem the student should first read the discussion of partially restrained beams in Chapter 14 (especially Art. 340).

WORKING-STRESS STRAIGHT-LINE THEORY

ILLUSTRATIVE PROBLEM 8

Given continuous rectangular concrete beams supporting a removable wooden floor consisting of planks weighing 20 lb. per sq. ft. (Fig. 36). The interior spans are supported by concrete columns 30 in. square and 18 ft. high in the clear. The beams are spaced 14 ft. 6 in. apart, and have a span of 29 ft. center to center of columns.

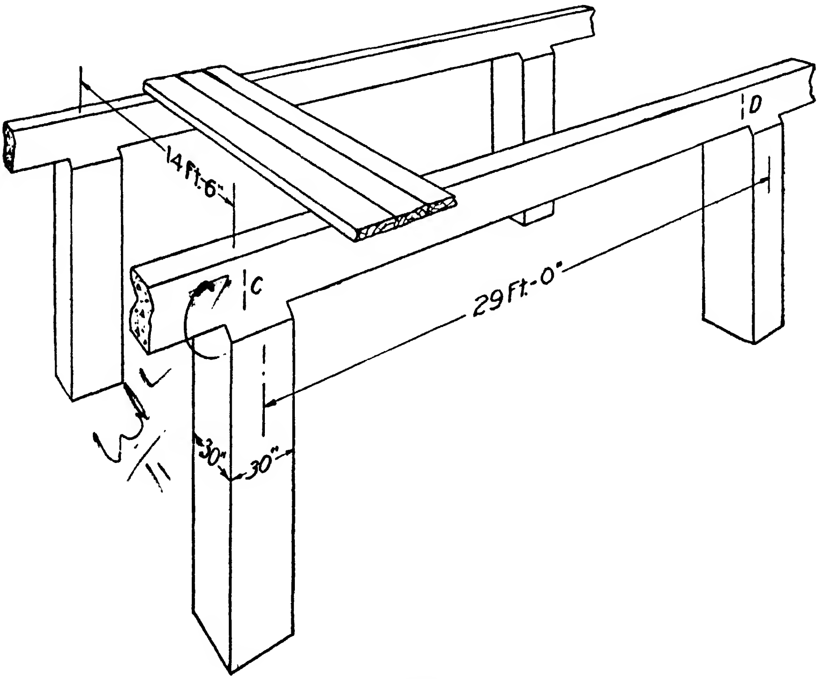


FIG. 36

The live load is 130 lb. per sq. ft. Using 2000-lb. concrete,[2] design the interior span *CD*. Employ type 1 steel placement (Fig. 28).

68. Beam Size by Bending Moment The beam carries 14.5 sq. ft. of floor surface for each foot of length of beam. It is also loaded with its own weight. The designer makes a guess of the size and corrects his estimate later if the guess is not accurate. After trials of a 14-in. width, the beam is assumed to have a cross section 16 in. by 32 in., which is the deepest 16-in. beam.[3]

[2] See footnote for Article 32.

[3] If wood planks are used as beam-bottom forms, the width should be the actual plank width, $13\frac{1}{2}$ in. or $15\frac{1}{2}$ in. in this case.

$$
\begin{array}{llll}
\text{Live load} & = 130 \text{ lb. per sq. ft.} & & \\
\text{Planks} & = \underline{20} \text{ lb. per sq. ft.} & & \\
\text{Total floor load} & = 150 \times 14.5 & = 2175 \text{ lb per ft.} & \\
\text{Weight of beam} & = \dfrac{16 \times 32}{144} \times 150 & = \underline{530} \text{ lb. per ft.} & \\
\quad \text{Total beam load } w & & = 2705 \text{ lb. per ft.} &
\end{array}
$$

Since span CD is an interior span of several equal spans, the maximum bending moment coefficients can be selected from A.C.I. Article 701 (in the Appendix). Assuming the beam section to be 16 in. wide by 32 in. deep, and using the clear span,

$$\text{Beam stiffness} = \frac{I}{l} = \frac{bh^3}{12l} = \frac{16(32)^3}{12(26.5 \times 12)} = 137 \text{ in.}^3$$

$$\text{Column stiffness} = \frac{I}{h} = \frac{bt^3}{12h} = \frac{30(30)^3}{12(18 \times 12)} = 312 \text{ in.}^3$$

$$\text{Ratio of stiffnesses} = \frac{312}{137} = 2.3$$

This ratio is less than 8, so the maximum negative moment equals $\dfrac{wl^2}{11}$ and the maximum positive $\dfrac{wl^2}{16}$.

The maximum numerical moment

$$M_n = \frac{wl^2}{11} = \frac{2705(26.5)^2 \times 12}{11} = 2{,}070{,}000 \text{ in.-lb.}$$

By A.C.I. Articles 305 and 306, $f_c = 0.45f'_c = 900$ lb. per sq. in., $f_s = 20{,}000$ lb. per sq. in., and the constant for rectangular beams $K = 157$.

$$\text{Minimum } d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{2{,}070{,}000}{157 \times 16}} = 28.7 \text{ in.}$$

By A.C.I. Article 507 the protection for this top steel is 1.5 in. Assuming ordinary anchorage and tension steel 1 in. square, or less, in one row with $\frac{1}{2}$-in. stirrups, the total depth h includes (Fig. 37)

$$
\begin{array}{ll}
\text{Minimum } d & = 28.7 \text{ in.} \\
\text{Minimum clearance to stirrup} & = 1.5 \\
\text{Stirrup} & = 0.5 \\
\text{Distance to center of tension steel} & = \underline{0.5} \\
\quad \text{Minimum } h & = 31.2 \text{ in.}
\end{array}
$$

Adopt tentatively a 16-in. by 32-in. beam. The neutral axis ratio k, corresponding to the design stresses of $f_c = 900$ lb. per sq. in. and $f_s = 20{,}000$ lb. per sq. in., equals 0.40 by equation 3, and $j = 0.87$ by equation 10.

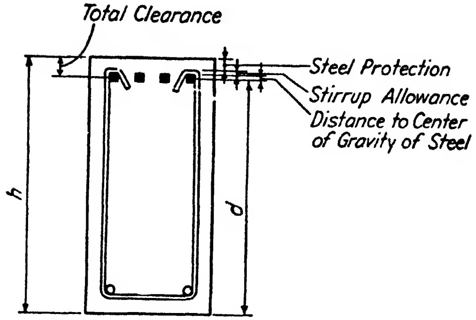


FIG. 37

69. Size Checked for Maximum Positive Bending Moment.

$$\text{Maximum positive moment } M_p = \frac{wl^2}{16} = 1{,}420{,}000 \text{ in.-lb.}$$

This moment is smaller than the maximum negative moment, but the fireproofing clearance is 2 in. for glacial gravel (1928 A.C.I. Code, Art. 506). Therefore, we should check the fiber stress f_c. Assuming 1-in. bars in one row the value of $(h - d)$ equals

Fireproofing clearance	= 2.0 in.
Stirrup	= 0.5
Distance to center of tension steel	= 0.5
Total	= 3.0 in.

The value of d equals 29 in. (Fig. 38).

$$K = \frac{1{,}420{,}000}{16(29)^2} = 105$$

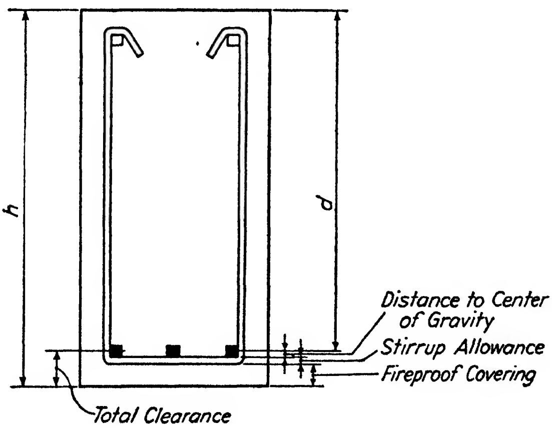


FIG. 38

On Diagram 2 (Appendix) follow to the right on the $K = 105$ line until the curve of $f_s = 20{,}000$ is intercepted.

$$p = 0.006 \qquad \text{and} \qquad f_c = 700 \text{ lb. per sq. in.}$$

Allowable $f_c = 900$ lb. per sq. in. By Diagram 1 (Appendix), using $p = 0.006$ and $n = 15$,

$$j = 0.885$$

The beam section of 16 in. by 32 in. is satisfactory for bending moment requirements.

70. Size Checked for Shear. The shear diagram for interior spans of continuous beams will be assumed to be symmetrical about the center line (Art. 349). At either

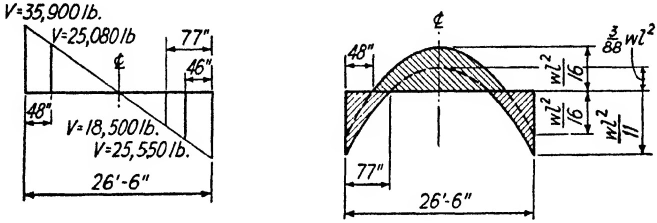


FIG. 39

support the maximum shear force $V = \dfrac{2705 \times 26.5}{2} = 35{,}900$ lb. (Fig. 39). At the support negative bending occurs and the value of $j = 0.87$ by Article 68.

$$\text{Maximum stress } v = \frac{V}{bjd} = \frac{35{,}900}{16 \times 0.87 \times 29.5} = 88 \text{ lb. per sq. in.}$$

Using ordinary anchorage and no web steel,

$$\text{Allowable } v = 0.02f'_c = 40 \text{ lb. per sq. in.} \quad \text{(A.C.I. Art. 305)}$$

Using ordinary anchorage and web steel, allowable $v = 0.06f'_c = 120$ lb. per sq. in. For $v = 88$ lb. per sq. in. use ordinary anchorage and web steel. We now definitely adopt a beam size of 16 in. by 32 in.

71. Steel Area for Bending Moment. *Positive Steel.*

$$A_p = \frac{M_p}{f_s jd} = \frac{1{,}420{,}000}{20{,}000 \times 0.885 \times 29} = 2.76 \text{ sq. in.}$$

This is the correct area if 1-in. bars are chosen. The actual depth d will vary slightly if other sizes are adopted. There are listed below possible steel sizes and number of bars to care for the moment of 1,420,000 in.-lb.

Steel	Depth d in.	Required Area sq. in.	Actual Area sq. in.	Number of Rows
2—$1\frac{1}{4}$-in. square	28.87	2.78	3.12	1
3—1-in. square	29.00	2.76	3.00	1
4—1-in. round	29.00	2.76	3.14	1
5—$\frac{7}{8}$-in. round	29.06	2.76	3.01	1

All these possibilities can be placed in one row. The minimum lateral spacings are given by A.C.I. Article 505. For instance, the minimum width of one row of three 1-in. square bars is

2 fireproofing clearances	4.00 in.
2 stirrup widths	1.00
3—1-in. square bars	3.00
2 spacings in the clear of 2 diameters	4.00
Minimum width	12.00 in.
Actual width	16.00 in.

Since the three 1-in. square bars are nearest the required area, adopt this size if bond stresses are satisfactory.

Negative Steel.

$$A_n = \frac{M_n}{f_s jd} = \frac{2{,}070{,}000}{20{,}000 \times 0.87 \times 29.5} = 4.03 \text{ sq. in.}$$

Possible steel sizes are

Steel	Depth d in.	Required Area sq. in.	Actual Area sq. in.	Number of Rows
3—$1\frac{1}{4}$-in. square	29.37	4.06	4.69	1
4—$1\frac{1}{8}$-in. square	27.75*	4.29	5.06	2
5—1-in. square	28.30	4.20	5.00	2
6—1-in. round	28.25	4.21	4.71	2
7—$\frac{7}{8}$-in. round	28.62	4.17	4.21	2
10—$\frac{3}{4}$-in. round	28.68	4.14	4.42	2

* Minimum d = 28.7 in. (p. 64). Compression steel must be used.

Seven $\frac{7}{8}$-in. round bars have an area closest to the required area but seven bars cannot be placed advantageously in two rows. It would be natural to place four bars in the top row, but, if placed symmetrically, the center bar in the bottom row is not in line with a bar above. When the concrete is poured this center bar will prevent the stone from flowing into the space below it and will produce "honeycombing." Do not use seven bars in two rows.

Ten $\frac{3}{4}$-in. round bars are too many to handle and will not be used unless the bond stresses of the larger bars are excessive.

72. Steel Checked for Bond. *Positive Steel.* Reference to the bending moment sketch for $\frac{wl^2}{16}$ in Figure 39 shows that the steepest portion of the curve, or most rapid change of positive bending moment, occurs at the point of inflection. Therefore bond will be checked at the point of inflection.

The discussion in Chapter 14 on continuous beams (Art. 340) shows that the sum of the maximum positive and negative bending moments for a uniformly distributed load must be $\frac{wl^2}{8}$, for the span at any instant. If the maximum positive moment is $\frac{wl^2}{16}$, the maximum negative moment is $-\frac{wl^2}{16}$ and the base line $M = 0$ is half way up on the $\frac{wl^2}{8}$ plot. Using Diagram 5 (Appendix) the point of inflection (M_p

= 0) is found at $0.15l$, or $0.15 \times 26.5 \times 12 = 48$ in. (Fig. 39). The shear force at 48 in. is

$$V_{48} = 35{,}900 - 2705 \times 4 = 25{,}080 \text{ lb.}$$

The purpose of determining the bond stress is to find out how many bars must pass the point of inflection to satisfy the bond requirements. The bond stress equation $u = \dfrac{V}{\Sigma ojd}$ will be rearranged to read

$$\text{Number of bars } \Sigma = \frac{V}{uojd}$$

With ordinary anchorage, A.C.I. Article 305 gives a maximum allowable bond stress $u = 0.05f'_c = 100$ lb. per sq. in. for deformed bars.

$$\Sigma_{48} = \frac{25{,}080}{100 \times o \times 0.885d} = \frac{283}{od} \text{ bars}$$

Checking the four possibilities of the list in Article 71, we obtain

Steel	Number of Bars Σ
$1\frac{1}{4}$-in. square	1.96
1-in. square	2.4
1-in. round	3.1
$\frac{7}{8}$-in. round	3.5

Since all these possibilities are safe for bond, adopt the three 1-in. squares as they are closest to the required area.

Negative Steel. The most rapid change of the negative bending moment occurs at the support. At this section

$$\text{Bond stress } u = \frac{V}{\Sigma ojd} = \frac{35{,}900}{\Sigma o \times 0.87 \times d}$$

Checking the four larger sizes listed in Article 71,

Steel	Bond Stress u
3—$1\frac{1}{4}$-in. square	94 lb. per sq. in.
4—$1\frac{1}{8}$-in. square	83 lb. per sq. in.
5—1-in. square	73 lb. per sq. in.
6—1-in. round	78 lb. per sq. in.

All these sizes are safe for bond, but none of them approaches closely the required area. Adopt the three $1\frac{1}{4}$-in. square bars, as they give the least number to handle and the actual area is only 15 per cent in excess.

73. Anchorage and Placing. A.C.I. Article 902 gives the requirements for ordinary anchorage.

Positive Steel. At least one fourth of this steel must run into the support for ten diameters, but two 1-in. square bars will be run into the support 10 in. for stirrup support. Bond requires that three bars run to the point of inflection, 48 in. out, and then be anchored 10 diameters, or 10 in.

Negative Steel. At the edge of the supports there are three $1\frac{1}{4}$-in. square bars resisting the maximum negative moment. The actual tensile stress in these bars equals

$$f_s = \frac{M}{A_s jd} = \frac{2{,}070{,}000}{4.69 \times 0.87 \times 29.37} = 17{,}300 \text{ lb. per sq. in.}$$

These bars must run far enough into the supporting column and the adjacent span to anchor for this stress. By equation 28 (Art. 57)

$$l = \frac{f_s a}{4u} = \frac{17{,}300 \times 1.25}{4 \times 100} = 54 \text{ in.}$$

In this span some of the bars must extend to the point of inflection; the others can be cut off after they are no longer needed to resist the bending moment or bond requirements. All these bars must extend 12 diameters beyond these critical sections. The point of inflection can be found by the use of Diagram 5. For the negative moment of $\frac{wl^2}{11}$ the point of inflection occurs at $0.24l$, or 77 in. out from the support. At this section the shear force equals 18,500 lb. (Fig. 39). The number of bars needed here for bond equals

$$\Sigma = \frac{V}{oujd} = \frac{18{,}500}{5 \times 100 \times 0.87 \times 29.37} = 1.5 \text{ bars}$$

The two outer bars will be carried 3 in. beyond the center line to meet the bars from the other support. They will serve as supports for the stirrups. The center bar can be cut off before it reaches the point of inflection. After it is cut off two bars remain which can carry a moment of

$$M_2 = f_s A_s jd = 20{,}000 \times 3.12 \times 0.87 \times 29.37 = 1{,}590{,}000 \text{ in.-lb.}$$

This moment is $\frac{1{,}590{,}000}{2{,}070{,}000} = 0.77$ of $\frac{wl^2}{11}$. On Diagram 5 the moment $\frac{wl^2}{11}$ is 73 per cent of $\frac{wl^2}{8}$, or 73 units of the 100 representing $\frac{wl^2}{8}$. The moment 1,590,000 in. lb. is represented by $0.77 \times 73 = 56$ units, or $73 - 56 = 17$ units from the base. The 17 abscissa line cuts the parabola at $0.05l = 16$ in. from the edge of the support. As far as bending moment is concerned the third bar can be anchored after passing this section.

Two bars can safely withstand bond stresses after the shear force decreases below a value of

$$V_2 = u\Sigma ojd = 100 \times 2 \times 5 \times 0.87 \times 29.37 = 25{,}550 \text{ lb.}$$

This occurs 46 in. from the support. This requirement is greater, so this third bar will run out 46 in. and then be anchored $12 \times 1.25 = 15$ in., ending 61 in. from the face of the support.

74. Summary for Tension Steel. *Positive Steel.* We need in the bottom of the beam

3 bars at the center line for bending moment
3 bars at 48 in. out for bond
1 bar at the support for anchorage (Use 2 bars)

Negative Steel. We need in the top of the beam

3 bars at the face of the support for bending moment (These are adequate for bond also)
2 bars are sufficient for bending moment and bond after passing the section 46 in. out

75. Diagonal Tension. Diagonal tension steel in the type 1 system is supplied entirely by vertical stirrups. We have assumed $\frac{1}{2}$-in. two-rod stirrups, which are tied to the outer bars both top and bottom. The hooks at the top will be made 6 in. long and are turned in for rectangular beams.

A.C.I. Article 305 allows a shear stress $v = 0.02f'_c = 40$ lb. per sq. in. for concrete when ordinary anchorage is used. Wherever v is less than 40 lb. per sq. in. the concrete can carry the diagonal tension. From Figure 40 this is seen to be for 144 in. in the center of the span. It is necessary to supply "web" steel for 87 in. at each end.

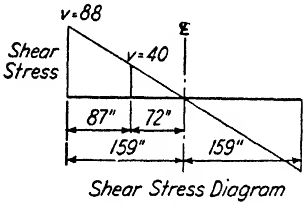


FIG. 40

The stirrup should be of such size that the closest spacings at the support are about 4 in. to 6 in. with gradual increase to the maximum allowed. Theoretically one can change the spacing for each stirrup, but this is not economical. It pays to use the same spacings for several stirrups and then change to one considerably greater, always using dimensions that space easily on the steel foreman's rule. In other words, do not use fractions of an inch, or 5 in., 7 in., 11 in., etc. The desirable spacings are multiples of 3 in. or 4 in.

Maximum Spacing. The maximum shear stress $v = 88$ lb. per sq. in. This is less than $v = 0.06f'_c = 120$ lb. per sq. in.; therefore, by A.C.I. Article 806, the maximum spacing equals the distance from "mid-depth of the beam to the longitudinal tension bars." Let us interpret this to mean

$$\text{Maximum spacing } s = \frac{d}{2} = \frac{29.37}{2} = 14.7 \text{ in.}$$

We are considering the section at the support where there is negative bending and we use the negative $d = 29.37$ in.

Commercial Spacing. Equation 32 (Art. 62) gives

$$v = v_c + v_s = 0.02f'_c + \frac{a_s f_s}{bs \sin \alpha}$$

where s is the minimum spacing at the section under investigation. Assuming $\frac{1}{2}$-in. round two-rod stirrups,

$$88 = 40 + \frac{2 \times 0.196 \times 20{,}000}{16 \times s \times 1}$$

Solving, $88 = 40 + \dfrac{490}{s}$, or minimum $s = 10.2$ in.

This choice of stirrup size is hardly satisfactory as the minimum spacing does not allow of much increase before the maximum is reached. Assuming $\frac{3}{8}$-in. round stirrups,

$$88 = 40 + \frac{275}{s} \qquad \text{and} \qquad \text{minimum } s = 5.7 \text{ in.}$$

This is better. Assuming the first crack to start at the support, the first stirrup can be 5.7 in. out. We shall start with a 4-in. spacing and increase to 6, 9, and 12 in. We can successively compute the maximum shear stress v that each spacing represents by use of

$$v = 40 + \frac{275}{s}$$

Listing these values:

Spacing s in.	Shear Stress v lb. per sq. in.	Distance from Face of Support x in.
4	..	..
6	86	4
9	71	31
12	63	45
	40	87

The values of x can be computed by similar triangles, or better still by scaling from a plot such as Figure 41*a*. The value of x is the minimum distance from the support at which that spacing s can be used.

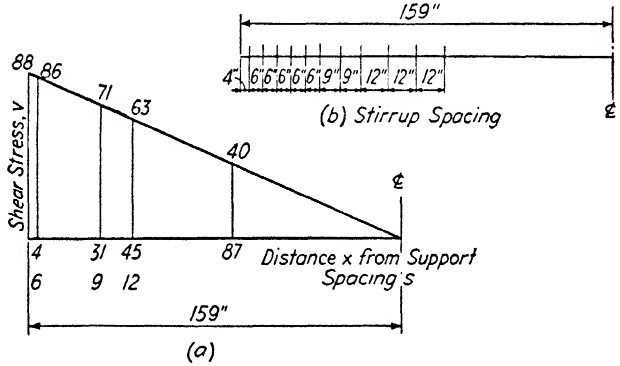


FIG. 41

Starting at the support we use

1—4-in. spacing until we reach $x = 4$ in.
5—6-in. spacings until we pass $x = 31$ in., and end 34 in. out
2—9-in. spacings until we pass $x = 45$ in., and end 52 in. out
3—12-in. spacings until we pass $x = 87$ in., and end 88 in. out

This can be done very rapidly on a plot such as Figure 41.

After passing $x = 87$ in. no web steel is needed. However, we usually add stirrups about 2 ft. apart to allow for the effect of moving loads or for unforeseen concentrated loads. Our assumption has been that the loading is uniformly distributed. In an actual building the live load frequently is not. A much smaller load concentrated in the center will give a higher shear diagram in the center, though the maximum shear is less (Fig. 42). A concentration as great as that of Figure 42*b* ought to be known in advance and the actual shear diagram used in computations. Machinery loads are frequently averaged as uniformly distributed, however, and a single machine in the center of the span may raise the shear diagram somewhat.

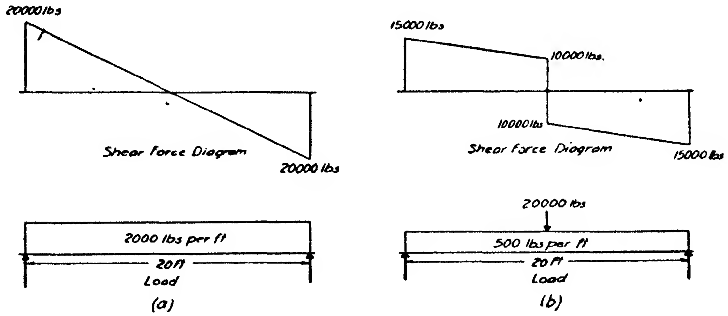


FIG. 42

76. Steel Assembly. The final detail sketch (Fig. 43) for the steel foreman should give all sizes and positions, so that he may cut, bend, and place the steel according to the design.

It will be noticed that, when the steel is wired together, it forms a continuous truss. The anchorage specification requires that some of the positive steel run into each support to complete the lower chord, some of the negative steel run on top to

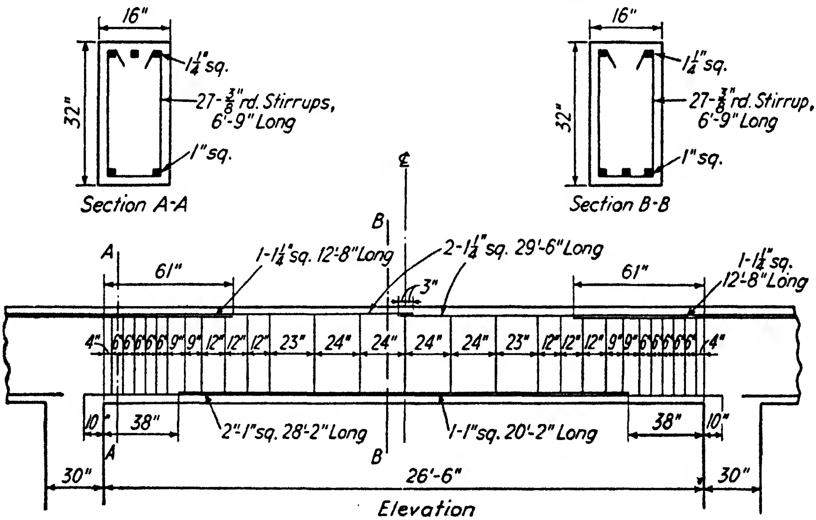


FIG. 43

the center line and then be wired to the bars from the adjacent support, while the stirrups form vertical supporting members until the concrete is poured around the steel frame.

It should be emphasized again that the positive steel is anchored in a compression area at or near the support and the negative steel is anchored in a compression area at or near the center.

PLASTIC THEORY

ILLUSTRATIVE PROBLEM 9

Let us solve the beam span of Problem 8, using the plastic theory. The previous computations gave a uniformly distributed load $w = 2705$ lb. per ft. Assuming a factor of safety of 2.5, substitute in the plastic theory formulae $0.4 \times 2000 = 800$ lb. per sq. in. for f'_c and $0.4 \times 50{,}000 = 20{,}000$ lb. per sq. in. for f_y. The maximum numerical moment equals

$$M_n = \frac{wl^2}{11} = 2{,}070{,}000 \text{ in.-lb.}$$

For balanced design, by equation 20

$$M = 2{,}070{,}000 = \frac{f'_c}{3} bd^2 = \frac{800}{3} bd^2$$

If $b = 14$ in.,

$$d = \sqrt{\frac{3 \times 2{,}070{,}000}{800 \times 14}} = 23.5 \text{ in.}$$

Assuming two rows of 1-in. square bars and $\frac{3}{8}$-in. stirrups

$$\text{Minimum } h = 23.5 + 1.5 + 0.38 + 0.5 + 1.5 = 27.38 \text{ in.}$$

Try a section 14 in. by 28 in. The revised load equals $w = 2585$ lb. per ft. The maximum shear stress equals

$$v = \frac{V}{bc} = \frac{2585 \times 13.25}{14 \times 0.732 \times 24.13} = 139 \text{ lb. per sq. in.}$$

Use special anchorage and web steel.

The revised negative moment equals 1,980,000 in.-lb. By equation 21 the necessary steel area is

$$A_n = \frac{M}{0.732 f_y d} = \frac{1{,}980{,}000}{0.732 \times 20{,}000 \times 24.13} = 5.60 \text{ sq. in.}$$

Use six 1-in. square bars in two rows.

The positive steel can be figured with the equations for an under-reinforced section. The bending moment equals 1,360,000 in.-lb. Assuming 1-in. round bars in one row, $d = 25.12$ in., $a = 6.54$ in., $c = 21.85$ in. $= 0.872d$.

$$A_p = \frac{1{,}360{,}000}{0.872 \times 20{,}000 \times 25.12} = 3.11 \text{ sq. in.}$$

Use four 1-in. round bars in one row.

The positive and negative steel are both safe for bond stresses. The placement of the steel and computation of the diagonal tension steel can be made as in Problem 8.

A rough comparison of the two designs is listed below.

Plastic theory design has

120 sq. in. less cross-section area
0.35 sq. in. more computed positive steel area
1.54 sq. in. more computed negative steel area.

An accurate cost comparison can be made from final detailed sketches, such as Figure 43.

CHAPTER 6

RECTANGULAR BEAMS REINFORCED WITH TENSION AND COMPRESSION STEEL

77. Compression Steel. The use of compression steel in rectangular beams may be regarded as an expedient to give satisfactory strength results in a portion of a beam which would otherwise require a deepening of the beam that elsewhere has satisfactory depth.

78. Economy. It has been stated in Article 66 that the economical rectangular beam has a section that stresses the extreme fiber in compression f_c and the steel f_s to the maximum allowable values. In certain cases, however, uneconomical sections may be tolerated for short distances in order to use an unchanged section with the consequent savings in formwork, etc. An example is that of the continuous beam supported by girders. The maximum numerical moment in the interior spans is $\frac{wl^2}{11}$, and the end span has a maximum value of $\frac{wl^2}{10}$. If the end span chances to be somewhat greater in length, the numerical value in the end span may be well in excess of the value in the interior spans. If there are many interior spans, it may pay to use the same beam size in the exterior span and care for the greater bending moment by an increase of tension steel and the addition of compression steel to assist the concrete. Another illustration of the use of a rectangular beam section which is in itself unecomonical is given in the discussion of tee beams in Article 110.

Until recently compression steel was uneconomical because it was rarely stressed to more than 10,000 lb. per sq. in., although the allowable stress is 20,000 lb. per sq. in. To carry a given force, about twice as much steel must be used than if it were to carry the same force in tension. Compression steel tends to buckle or bend when loaded, and frequent ties or stirrups are needed to assist the concrete to hold it in line. Tension steel, on the other hand, owing to its pull under load, remains straight.

The 1941 A.C.I. Code in Article 706 has doubled these compressive stresses, so the use of compression steel does not penalize the design as much as in the past.

The present wartime emergency has led to the recommendation that no compression steel be used, that a larger section be employed with a smaller steel requirement.

WORKING-STRESS STRAIGHT-LINE THEORY

79. Derivation of Formulae for Fiber Stresses. In addition to the nomenclature used for the rectangular beam (Art. 19), let

A'_s = area of compression steel

p' = the ratio $\dfrac{A'_s}{bd}$

f'_s = stress intensity in compression steel

C'_s = resultant compression force in compression steel

C = resultant of concrete force C_c and compression steel force C'_s, or $C = C_c + C'_s$

z = distance from extreme fiber in compression to resultant compression force C

The assumption that a plane section remains a plane section justifies the statement that the deformations are proportional to the distance from the neutral axis (Fig. 44). Therefore

$$\frac{d - kd}{kd} = \frac{e_s}{e_c} = \frac{f_s}{E_s} \times \frac{E_c}{f_c} = \frac{f_s}{nf_c} \tag{38}$$

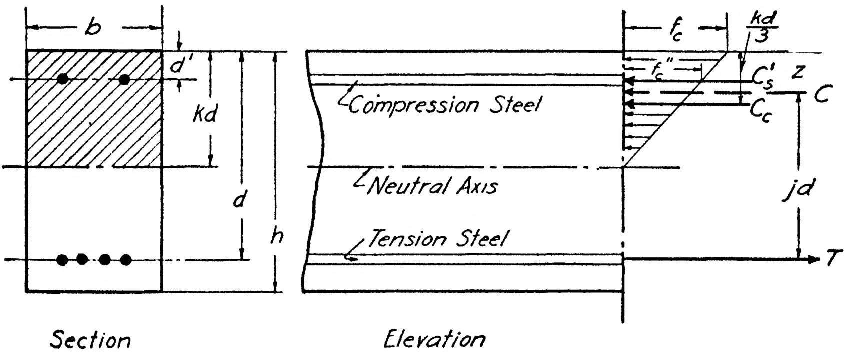


FIG. 44

As in the rectangular beam derivation (Art. 20) this gives

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \qquad (Designer's\ Equation) \tag{39}$$

The relation between the tensile steel stress f_s and the compression steel stress f'_s can also be obtained by this proportionality of deformations.

$$\frac{(kd - d')}{(d - kd)} = \frac{e'_s}{e_s} = \frac{f'_s}{E_s} \times \frac{E_s}{f_s} = \frac{f'_s}{f_s}$$

or

$$f'_s = f_s \left(\frac{kd - d'}{d - kd} \right) \tag{40}$$

Also the relation between f_c and f'_s can be found.

$$\frac{kd - d'}{kd} = \frac{e'_s}{e_c} = \frac{f'_s}{E_s} \times \frac{E_c}{f_c} = \frac{f'_s}{nf_c}$$

$$f'_s = nf_c \left(\frac{kd - d'}{kd} \right) \tag{41}$$

The resultant compression force C must be equal to the resultant tensile force T, or

$$C = C_c + C'_s = T \tag{42}$$

The concrete area is equal to $(bkd - A'_s)$, but it is customary to assume it equal to bkd. This is an error on the unsafe side, but the final results are only slightly affected. Thus, in equation 44 below, the only change for an exact solution is that the numerator of the right-hand term changes from k to $k - p'$. By the time this has followed through to the moment equation, the error is small. By equation 42,

$$\frac{f_c}{2} bkd + f'_s A'_s = f_s A_s \quad \text{(nearly)} \tag{43}$$

Substituting the values of f'_s from equation 40 and the relations $A_s = pbd$ and $A'_s = p'bd$ in equation 43 gives

$$\frac{f_c}{2} bkd + f_s \left(\frac{kd - d'}{d - kd} \right) p'bd = f_s pbd$$

$$\frac{f_s}{f_c} = \frac{k}{2\left[p - p' \left(\frac{kd - d'}{d - kd} \right) \right]} \tag{44}$$

Equating the ratio $\frac{f_s}{f_c}$ from equations 38 and 44

$$\frac{k}{2\left[p - p'\left(\frac{kd - d'}{d - kd}\right)\right]} = \frac{n(1-k)}{k}$$

$$k^2 = 2n(1-k)\left[p - p'\left(\frac{kd - d'}{d - kd}\right)\right]$$

$$k^2 + 2n(p + p')k = 2n\left(p + p'\frac{d'}{d}\right)$$

Completing the square,

$$k^2 + 2n(p + p')k + n^2(p + p')^2 = 2n\left(p + p'\frac{d'}{d}\right) + n^2(p + p)^2$$

The square root is

$$k = \sqrt{2n\left(p + p'\frac{d'}{d}\right) + n^2(p + p')^2} - n(p + p')$$

(*Checker's Equation*) (45)

This equation can be compared with the similar derivation for k for the rectangular beam (Art. 20). Equations 39 and 45 enable one to locate the neutral axis. The next step is to locate the resultant compression force C in order to obtain the moment arm jd.

Dealing with the compression forces only, the sum of the moments of the concrete force C_c and the compression steel force C'_s about the top of the cross section must equal the moment of the resultant C.

$$Cz = (C_c + C'_s)z = C_c\left(\frac{kd}{3}\right) + C'_s d'$$

$$\left[\frac{f_c}{2}bkd + f_s\left(\frac{kd - d'}{d - kd}\right)p'bd\right] = \frac{f_c}{2}bkd\left(\frac{kd}{3}\right) + f_s\left(\frac{kd - d'}{d - kd}\right)p'bdd'$$

$$\frac{\frac{f_c bk^2d^2}{6} + f_s p'bdd'\left(\frac{kd - d'}{d - kd}\right)}{\frac{f_c}{2}bkd + f_s p'bd\left(\frac{kd - d'}{d - kd}\right)}$$

For f_s substitute from equation 38: $f_s = nf_c\left(\frac{1-k}{k}\right)$

$$z = \frac{f_cbd\left[\frac{k^2d}{6} + np'd'\left(\frac{1-k}{k}\right)\left(\frac{k - \frac{d'}{d}}{1-k}\right)\right]}{f_cbd\left[\frac{k}{2} + np'\left(\frac{1-k}{k}\right)\left(\frac{k - \frac{d'}{d}}{1-k}\right)\right]}$$

Reducing

$$z = \frac{+ 2np'd'\left(k - \frac{d'}{d}\right)}{k^2 + 2np'\left(k - \frac{d'}{d}\right)} \tag{46}$$

This term z is a distance measured in inches. This is contrary to the previous practice for rectangular beams, namely, express distances as ratios of d, as kd or jd. Now

$$jd = d - z \tag{47}$$

In terms of steel stresses, the bending moment equals

$$M_s = Tjd = A_sf_sjd \tag{48}$$

or

$$f_s = \frac{M}{A_sjd}$$

From equation 38 the concrete stress can be found.

$$f_c = \frac{f_s}{n}\left(\frac{k}{1-k}\right) \tag{49}$$

The concrete stress can be found independently by

$$M_c = Cjd = (C_c + C'_s)jd = \left[\frac{f_c}{2}bkd + f'_sp'bd\right]jd \tag{50}$$

Substituting the value of f'_s from equation 41 in equation 50

$$M_c = \frac{f_cjbd^2}{2}\left[k + 2np'\left(1 - \frac{d'}{kd}\right)\right] \tag{51}$$

The concrete stress

$$f_c = \frac{2M}{jbd^2\left[k + 2np'\left(1 - \frac{d'}{kd}\right)\right]} \tag{52}$$

Equation 49 is usually the easier solution for f_c.

The stress in the compression steel f'_s can be found from the tension steel stress f_s by equation 40 or from the concrete stress f_c by equation 41.

The fiber stresses, or bending moments, cannot be computed unless we know the moment arm jd. This depends on the distance z (equation 46). The distance z depends on the neutral axis position k, the compression steel ratio p', its position d', and the depth d to the tension steel. To compute z, we must know or assume the beam size and the amounts of tension and compression steel. The equations derived above are better adapted to check a finished design than to make a new design.

80. Modification of Design Methods. A.C.I. Article 706 states: "The effectiveness of compression reinforcement in resisting bending may be taken at twice the values indicated from the calculations assuming a straight-line relation between stress and strain and the modular ratio given in Section 601, but not of greater value than the allowable stress in tension." [1] This states that the compression stress f'_s given by equation 41 should be doubled, unless the computed stress is greater than $f'_s = 10{,}000$ lb. per sq. in. There is no analytical justification for this change. It is based on a survey of tests of beams which include the effect of shrinkage and flow as well as the load effect. The shrinkage and flow strains add compression stresses to the steel in addition to whatever stress may be caused by the loads. For this reason the design of compression steel in beams or columns has become a matter of employing empirical equations based on tests.

The assumption that the stress in the compression steel should be considered as $2f'_s$ will affect the previous equations. Equation 43 will now become

$$\frac{f_c}{2}bkd + 2f'_sA'_s = f_sA_s \quad \text{(nearly)}$$

The result will be that equations 45, 46, 50, 51, and 52 may be used, if substitution of $2p'$ is made wherever p' occurs in these equations.

81. Alternative Solution of Fiber Stresses. The empirical requirement that the stress in the compression steel be taken as twice the computed value can be handled more easily by an alternate method. When

[1] This maximum is 20,000 lb. per sq. in. by the A.C.I. Code, but the 1940 Joint Committee Report limits the maximum to 16,000 lb. per sq. in.

compression steel is employed the cross-section dimensions are usually known, as well as the fact that the maximum stress in the concrete exceeds the allowable value.

The amount of tension steel A_s can be approximated by assuming a value of the ratio j and using equation 48.

The amount of compression steel A'_s can be determined by dealing with compression forces C_c in the concrete and C'_s in the compression steel separately. It is true by the previous discussion that the tensile force in the tension steel, $T = C_c + C'_s$. A part T_1 of the tensile force pairs with the compression force in the concrete C_c to form a couple equal to

$$M_1 = C_c\left(d - \frac{kd}{3}\right) = \frac{f_c}{2} bkd\left(d - \frac{kd}{3}\right) \quad \text{(nearly)} \tag{53}$$

Since we have determined the amount of tension steel to give a definite fiber stress f_s and we wish to add compression steel enough to give the allowable concrete fiber stress f_c in the extreme fiber, the values of f_c and f_s are known. By the relation $k = \dfrac{1}{1 + \dfrac{f_s}{nf_c}}$, which holds for rectangular beams whether there be compression steel or not, we determine the neutral axis position kd and hence this couple of equation 53. This couple is the moment that could be carried by the beam if the concrete is stressed to its working limit and the area of steel corresponding to the pull T_1 has a stress of f_s. The actual bending moment M is greater, however, and an additional couple must be provided; it is formed by using the rest of the area A_s of tensile steel at a stress of f_s and adding a compression steel area A'_s at the stress $2f'_s$. This additional couple has the magnitude

$$M'_s = 2f'_sA'_s(d - d') \tag{54}$$

where $2f'_s$ is substituted for the theoretical value of the compressive stress f'_s. Also

$$M'_s = M - M_c \tag{55}$$

The neutral axis position will not be changed if the final values of f_c and f_s are not changed. From equation 54 the compression steel area can be computed:

$$A'_s = \frac{M'_s}{2f'_s(d - d')} \tag{56}$$

The couple M'_s is known from equation 55; the moment arm $(d - d')$ is known; the compression steel stress f'_s can be found as follows.

The neutral axis is a distance kd below the extreme concrete fiber in compression. The compression stresses in the concrete are uniformly varying (Fig. 44). The compression stress f''_c in the concrete layer at the level of the compression steel is

$$f''_c = f_c\left(\frac{kd - d'}{kd}\right)$$

The compression steel must not slip and, therefore, has the same strain (unit deformation) as the adjacent concrete. With equal strains the stresses are proportioned to their moduli of elasticity and the compression steel stress f'_s equals

$$f'_s = nf''_c = nf_c\left(\frac{kd - d'}{kd}\right) \qquad 57)$$

This is, of course, equation 41 (Art. 79).

PLASTIC THEORY

82. Plastic Theory. The design of rectangular beams with compression steel can also be made for ultimate loads by the plastic theory discussed in Article 23 (Chapter 2). The general assumptions are the same and the stress approximation at failure is illustrated in Figure 45. If

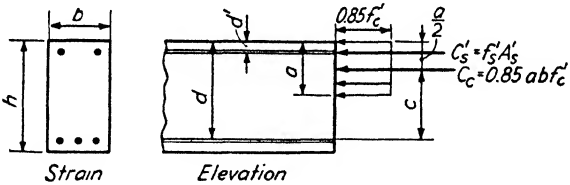


FIG. 45

the compression steel is held firmly by the concrete and closely spaced stirrups, it will resist as a column until the yield point is reached. It will be assumed that the compressive stress in this steel is equal to its yield point stress, if failure occurs on the compression side.

Failure on Compression Side of Section. It is assumed that the concrete and compression steel are both fully stressed at failure. Then

$$M = \frac{f'_c}{3}bd^2 + A'_s f_y(d - d') \qquad (58)$$

or

$$\frac{M}{bd^2} = \frac{f'_c}{3} + f'_y p'\left(1 - \frac{d'}{d}\right) \qquad (59)$$

where f'_y = yield point of compression steel.

Failure in Tension Steel. This case occurs if there is not sufficient tension steel compared with the concrete in compression and the compression steel. The assumption is made that the tension steel fails at its yield point stress f_y. As it elongates at this constant stress the compression steel is stressed to its yield point f'_y and then the area of concrete in compression is stressed to its ultimate. According to the analysis used in Article 81, a portion of the force in the tension steel forms a couple with the force in the compression steel, and the remainder forms a couple with the resultant compression force in the concrete. Assuming that all steel has the same yield point f_y, the ultimate moment of resistance equals

$$M = A'_s f_y(d - d') + (A_s - A'_s)f_y c \tag{60}$$

By equation 16 (Art. 23)

$$c = d - \frac{a}{2} = d - \frac{(A_s - A'_s)m}{2b} \tag{61}$$

Substituting $A_s = pbd$, etc.,

$$\frac{M}{bd^2} = p'f_y\left(1 - \frac{d'}{d}\right) + (p - p')f_y\left[1 - \frac{(p - p')m}{2}\right] \tag{62}$$

Solving for the ratio of tensile steel,

$$(p - p')^2 - (p - p')\frac{2}{m} = \frac{2p'}{m}\left(1 - \frac{d'}{d}\right) - \frac{2}{mf_y} \cdot \frac{M}{bd^2}$$

Completing the square on the left side and taking the square root

$$(p - p') - \frac{1}{m} = \pm\sqrt{\frac{1}{m^2} + \frac{2p'}{m}\left(1 - \frac{d'}{d}\right) - \frac{2}{mf_y} \cdot \frac{M}{bd^2}}$$

The minimum value of p equals

$$p = p' + \frac{1}{m} - \sqrt{\frac{2}{m}\left[\frac{1}{2m} + p'\left(1 - \frac{d'}{d}\right) - \frac{M}{f_y bd^2}\right]} \tag{62a}$$

The derivation is made for a deficiency in tension steel, yet the expression $(A - A'_s)$ implies that there is more tension steel than compression steel. For the unusual case of less tension steel than compression steel, the moment at failure can be written as

$$M = A_s f_y(d - d') \tag{63}$$

This assumes that the tension and compression steel form the moment of resistance together and that there are no compressive stresses in the concrete.

Use of Equations. For a given section it is not always possible to tell by inspection whether there is enough tension steel. In that case substitute the values given into equations 58 and 60 to determine which case holds.

EITHER THEORY

83. Shear, Bond, and Diagonal Tension. The greatest shear stress in a cross section occurs between the neutral axis and the tension steel. The concrete on the tension side is not used in maximum fiber stress computations for either the rectangular beam or the rectangular beam reinforced with compression steel. Therefore, the shear, bond of tension steel, and diagonal tension equations derived in Chapter 4 apply to the beam with compression steel, since conditions are the same between the tension steel and neutral axis.

There is also transfer of stress between the concrete and the compression steel. Since $T = C_c + C'_s$, the rate of change of the force C'_s in the compression steel will be less than the rate of change of the force T in the tension steel and the bond stresses will always be less when $A'_s = A_s$. Now that the compression steel stresses approach those used for the tension steel and the compression steel areas and perimeters are often comparatively small, the bond stress may need investigation. No allowance is made for the fact that the thrust in the compression steel bar will be supported in part by bearing on the end of the bar.

SOLUTION BY WORKING-STRESS STRAIGHT-LINE THEORY

ILLUSTRATIVE PROBLEM 10

84. Design of Rectangular Beam with Compression Steel. Design the exterior span AB of the beam (Fig. 46) given in Problem 8 (Chapter 5), using the same size beam, 16 in. by 32 in., adopted for the interior span CD, and the same 2000-lb. concrete.

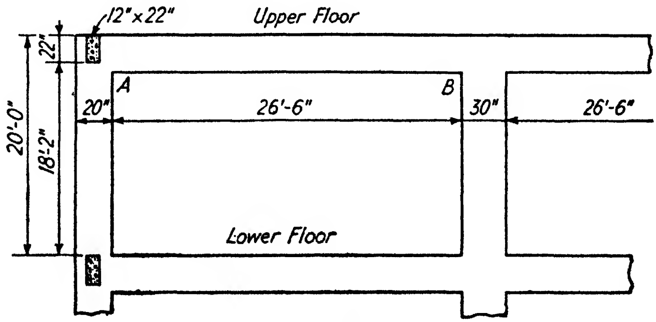


FIG. 46

85. Order of Procedure. The same general procedure used for rectangular beams (Art. 67) will be adopted for this problem. The beam dimensions are known but the concrete stresses are checked first to determine whether the section can be used at all. A logical order follows.

1. Check concrete fiber stresses and determine whether compression steel must be used.
2. Check shear stresses in concrete.
3. Figure tension steel areas for bending moment.
4. Check tension steel for bond. The dimensions d and d' are now known.
5. Figure compression steel areas for bending moment and check for bond stresses, if the steel is highly stressed.
6. Figure diagonal tension steel.

86. Bending Moments. This is a case of a continuous beam with uniform loading and equal spans supported by columns. The moment coefficients for such conditions are given in A.C.I. Article 701 (in the Appendix). The positive moment near the center of the span is given as $\frac{wl^2}{14}$ and the negative moment at the exterior face of the first interior column equals $\frac{wl^2}{10}$. No coefficient is given for the negative moment at the face of the exterior column. Therefore we shall adopt the recommendation of the 1928 A.C.I. Code. This code stated that two cases should be examined when beams were supported by columns. The criterion is the ratio of the beam stiffness $\frac{I}{l}$ to the exterior column stiffness $\frac{I}{h}$. Where $\frac{I}{l}$ is less than twice the sum of the $\frac{I}{h}$ for the exterior columns above and below the beam, the negative moment at the exterior support should be taken as $\frac{wl^2}{12}$. If $\frac{I}{l}$ is more than twice the sum of the two $\frac{I}{h}$ values, the negative moment at the exterior column should be taken as $\frac{wl^2}{16}$.

In our problem there is no column above. The column below will be taken as 16 in. by 20 in., with its center line 28 ft. 7 in. from the center line of the interior column. Since the steel in the column and beam is not known, the ratio of moment of inertia will be approximated by using the complete areas of column and beam.

$$\text{Moment of inertia of beam} = \frac{bh^3}{12} = \frac{16(32)^3}{12} = 43{,}690 \text{ (in.)}^4$$

$$\text{Moment of inertia of exterior column} = \frac{bh^3}{12} = \frac{16(20)^3}{12} = 10{,}670 \text{ (in.)}^4$$

The beam is continuous; therefore, use the clear span of 26 ft. 6 in.

$$\frac{I}{l} \text{ of beam} = \frac{43{,}690}{318} = 137 \text{ (in.)}^3$$

The distance from the top of the floor below to the top of this floor is 20 ft. (Fig. 46). If there is a perpendicular wall beam 12 in. by 22 in. in section, the unsupported length of the exterior column is 18 ft. 2 in.

$$\frac{I}{h} \text{ of lower column} = \frac{10{,}670}{218} = 49 \text{ (in.)}^3$$

The $\frac{I}{l}$ of the beam is greater than twice the sum of the $\frac{I}{h}$ of the upper and lower exterior columns, or $137 > 2 \times 49$. Therefore, the maximum negative moment at the exterior support will be taken as $M_n = \frac{wl^2}{16}$.

87. Fiber Stress in Concrete. *First Interior Support.* (Figure 47*c*.) The uniformly distributed load is $w = 2705$ lb. per ft. (Art. 68).

$$M_n = \frac{wl^2}{10} = \frac{2705 \times (26.5)^2 \times 12}{10} = 2{,}280{,}000 \text{ in.-lb.}$$

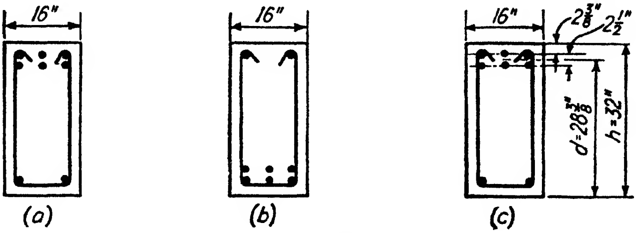


FIG. 47

The maximum allowable fiber stress in the concrete may be $f_c = 0.45f'_c = 900$ lb. per sq. in. and the steel stress may be $f_s = 20{,}000$ lb. per sq. in.

Assuming one row of $1\frac{1}{4}$-in. square bars (the largest size) and $\frac{3}{8}$-in. round stirrups, the height d will be less than the total height h by the amount

Protective covering	= 1.50 in.	(A.C.I. 507)
Stirrup diameter	= 0.38	
Half height of bar	= 0.62	
Total	= 2.50 in.	

therefore, $d = 32 - 2.50 = 29.50$ in.

If the maximum stress f_s and f_c are to be realized simultaneously, the constant $K = \frac{f_c}{2}jk = f_s pj = 157$ (Diagram 2, in the Appendix). Without compression steel the section can safely withstand a bending moment:

$$M_c = Kbd^2 = 157 \times 16 \times (29.50)^2 = 2{,}180{,}000 \text{ in.-lb.}$$

The actual bending moment is greater and compression steel must be used.

88. Fiber Stress in Concrete. *Center of Span.* (Figure 47*b*.) The maximum bending moment becomes

$$M_p = \frac{wl^2}{14} = 1{,}630{,}000 \text{ in.-lb.}$$

Assuming one row of $1\frac{1}{4}$-in. square bars (the largest size) and $\frac{3}{8}$-in. stirrups, the depth d will be less than the total depth h by the amount

Fireproofing	= 2.00 in.
Stirrup diameter	= 0.38
Half dimension of bar	= 0.62
Total	= 3.00 in.

Depth $d = 32 - 3 = 29$ in.

Without compression steel the section can safely withstand a bending moment:

$$M_c = Kbd^2 = 157 \times 16 \times (29)^2 = 2{,}120{,}000 \text{ in.-lb.}$$

No compression steel is needed.

Exterior Support. (Figure 47*a*.) The maximum moment here equals

$$M_n = \frac{wl^2}{16} = 1{,}425{,}000 \text{ in.-lb.}$$

The concrete can safely carry 2,180,000 in.-lb. No compression steel is needed.

89. Shear Stresses. The discussion of maximum shear forces for continuous beams given in Chapter 14 (Art. 349) states that the shear force diagrams for an exterior span vary somewhat as different applications of the live load is made on this span or adjacent ones. A.C.I. Article 701 recommends that the two limiting diagrams be taken as shown in Figure 48. The non-symmetrical shear diagram of

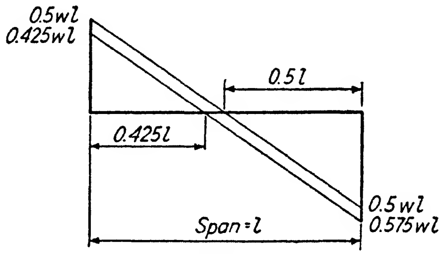


FIG. 48

$0.425wl$ to $-0.575wl$ will occur when the negative moment of $-\dfrac{wl^2}{10}$ is present at the first interior support. The maximum shear force in this span equals

$$V = 0.575wl = 0.575 \times 2705 \times 26.5 = 41{,}200 \text{ lb.}$$

At the face of the interior column

$$\text{Maximum shear stress} = v = \frac{V}{bjd} = \frac{41{,}200}{16 \times 0.87 \times 29.5} = 100 \text{ lb. per sq. in.}$$

This is within the allowable of $v = 120$ lb. per sq. in. for ordinary anchorage with diagonal tension reinforcement (A.C.I. Article 305).

90. Tension Steel. *First Interior Support.* Previous computations show that this section must have compression steel. When this steel is supplied the concrete stress f_c should be just under $f_c = 900$ lb. per sq. in. We intend to supply tension steel enough so that the tensile stress $f_s = 20{,}000$ lb. per sq. in. In that case the neutral axis ratio

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} = \frac{1}{1 + \dfrac{20{,}000}{15 \times 900}} = 0.403$$

$$\text{Neutral axis } kd = 0.403 \times 29.50 = 11.90 \text{ in.}$$

The compression force C_c acts $\frac{kd}{3}$ from the bottom, or $\frac{11.90}{3} = 3.97$ in. If there were no compression steel, the moment arm ratio

$$j = 1 - \frac{k}{3} = 1 - \frac{0.403}{3} = 0.866$$

When $1\frac{1}{4}$-in. square bars are used, the compression steel force C'_s acts $d' = 3.00$ in. from the bottom, or $d' = 2.00 + 0.38 + 0.63 = 3.00$ in. The resultant compression force C will act somewhere between C_c and C'_s, or between 3.00 in. and 3.97 in. It seems reasonable to assume $j = 0.88$, a value greater than that for C_c alone, since C'_s acts nearer the bottom of the beam.

Assuming, therefore, $j = 0.88$ and $d = 29.50$ in., the minimum area of tension steel equals

$$A_s = \frac{M}{f_s jd} = \frac{2{,}280{,}000}{20{,}000 \times 0.88 \times 29.50} = 4.39 \text{ sq. in.}$$

This is negative steel and is placed at the top of the beam (Fig. 47*c*).

Bond. Negative bending steel is checked for bond at the face of the support which is the section of maximum rate of change of bending moment.

$$u = \frac{V}{\Sigma ojd} = \frac{41{,}200}{\Sigma o \times 0.88d}$$

The allowable bond stress for ordinary anchorage is $u = 0.05 \times 2000 = 100$ lb. per sq. in. The possible commercial sizes of steel are tabulated below.

Steel	Actual Height d in.	Computed Area A_s sq. in.	Actual Area A_s sq. in.	Number of Rows	Bond Stress u lb. per sq. in.
3—$1\frac{1}{4}$-in. square	29.50	4.39	4.69	1	106
4—$1\frac{1}{8}$-in. square	29.56	4.38	5.06	1	88
5—1-in. square	28.42	4.55	5.00	2	82
6—1-in. round	28.37	4.56	4.71	2	88

The $1\frac{1}{4}$-in. bars cannot be used as the allowable bond stress is exceeded. Adopt the 1-in. round bars because they have the least excess area and the bond stress is safe (see Fig. 47*c*). The actual tensile stress equals

$$f_s = \frac{M}{A_s jd} = \frac{2{,}280{,}000}{4.71 \times 0.88 \times 28.37} = 19{,}400 \text{ lb. per sq. in.}$$

91. Tension Steel. *Center of Span.* The minimum tension steel area equals

$$A_s = \frac{M}{f_s jd} = \frac{1{,}630{,}000}{20{,}000 \times 0.87 \times 29} = 3.23 \text{ sq. in.}$$

This is positive steel and is placed at the bottom of the beam.

Bond. The maximum positive bending moment occurs for the live-load placement that gives the shear force of $0.5wl$ at each end of the beam (Fig. 48). Positive steel is checked for bond at the point of inflection, as that is the section with the maximum rate of change of positive bending moment. By consulting Diagram 5 in the Appendix, the point of inflection ($M = 0$) for $M_p = \frac{wl^2}{14}$ is found to be at $0.13l$, or $0.13 \times 26.5 \times 12 = 41$ in. from the face of the support. The shear at this section

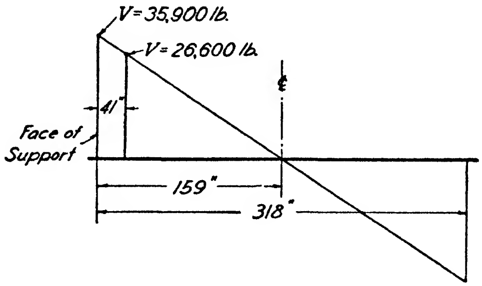


FIG. 49

equals 26,600 lb. (Fig. 49). The allowable bond stress $u = 0.05 \times 2000 = 100$ lb. per sq. in. The number of bars that must run to this section is

$$\Sigma = \frac{V}{uojd} = \frac{26{,}600}{100 \times o \times 0.87d} \text{ bars}$$

Possible commercial sizes are tabulated below.

Steel	Actual Depth d in.	Computed Area A_s sq. in.	Actual Area A_s sq. in.	Number of Rows	Number of Bars
3—$1\frac{1}{8}$-in. square	29.06	3.22	3.80	1	2.4
4—1-in. square	29.12	3.22	4.00	1	2.6
5—1-in. round	29.12	3.22	3.93	1	3.4
6—$\frac{7}{8}$-in. round	28.09	3.34	3.61	2	4.0

Adopt the six $\frac{7}{8}$-in. round bars because they have the least excess area (see Fig. 47*b*).

92. Tension Steel. *Exterior Support.* The maximum negative bending moment at the face of this support is $M_n = \frac{wl^2}{16} = 1{,}425{,}000$ in.-lb. Assuming $j = 0.87$ and $d = 29.50$ in., the minimum steel area equals

$$A_s = \frac{M}{f_s jd} = \frac{1{,}425{,}000}{20{,}000 \times 0.87 \times 29.50} = 2.78 \text{ sq. in.}$$

Bond. Bond will be checked at the face of the support; the shear force $V = 0.5wl$ that gives maximum shear (Fig. 48) will be used.

$$u = \frac{V}{\Sigma ojd} = \frac{0.5 \times 2705 \times 26.5}{\Sigma o \times 0.87 \times d}$$

Possible commercial steel sizes are tabulated below.

Steel	Actual Height d in.	Computed Area A_s sq. in.	Actual Area A_s sq. in.	Number of Rows	Bond Stress u lb. per sq. in.
2—$1\frac{1}{4}$-in. square	29.50	2.78	3.12	1	140
3—1-in. square	29.62	2.77	3.00	1	116
4—1-in. round	29.62	2.77	3.14	1	111
5—$\frac{7}{8}$-in. round	29.68	2.76	3.01	1	101
7—$\frac{3}{4}$-in. round	Not recommended				
10—$\frac{5}{8}$-in. round	29.00	2.83	3.07	2	73
3—$\frac{7}{8}$-in. round 3—$\frac{3}{4}$-in. round	28.77	2.85	3.13	2	94

The allowable bond stress for ordinary anchorage, $u = 0.05 \times 2000 = 100$ lb. per sq. in., determines the bar sizes to use. Ten bars are a great many to handle, so we shall adopt the combination of three $\frac{7}{8}$-in. bars in the top row and three $\frac{3}{4}$-in. bars in a lower second row (Fig. 47*a*).

93. Compression Steel. *First Interior Support.* A small amount of compression steel is needed at the first interior support. The tension steel consists of six 1-in. round bars with a height $d = 28.37$ in. There is available for compression steel some of the six $\frac{7}{8}$-in. round positive bars from the center of the span. The lower row of this steel is located at a height of 2.82 in. from the bottom of the beam. The maximum allowable stress in the concrete is $f_c = 900$ lb. per sq. in. and the actual stress in the tension steel equals $f_s = 19{,}400$ lb. per sq. in. (Art. 90).

Total bending moment	$M = 2{,}280{,}000$ in.-lb.
Maximum moment taken by the concrete	$M_c = 2{,}180{,}000$ in.-lb.
Moment to be taken by the compression steel	$M'_s = 100{,}000$ in.-lb.

If all the compression steel can be placed in the lower row, $d' = 2.82$ in.

$$\text{Neutral axis ratio } k = \frac{1}{1 + \dfrac{f_s}{nf_c}} = \frac{1}{1 + \dfrac{19{,}400}{15 \times 900}} = 0.41$$

$$\text{Neutral axis } kd = 0.41 \times 28.37 = 11.62 \text{ in.}$$

The theoretical compression steel stress equals

$$f'_s = nf_c \frac{kd - d'}{kd} = 15 \times 900 \frac{(11.62 - 2.82)}{11.62} = 10{,}200 \text{ lb. per sq. in.}$$

A.C.I. Article 706*b* recommends that compression steel area A'_s be computed using a stress of $2f'_s$ or 20,000 lb. per sq. in., whichever is the less. Use 20,000.

$$A'_s = \frac{C'_s}{20{,}000} = \frac{M'_s}{20{,}000(d - d')} = \frac{100{,}000}{20{,}000(28.37 - 2.82)} = 0.20 \text{ sq. in.}$$

Two $\frac{7}{8}$-in. bars will run into this support anyway and give considerably more than the required compression steel area. The actual stress is later found to be $2f'_s = 17{,}100$ lb. per sq. in. (Art. 98).

94. Placing Steel. *Steel in Bottom of Beam.* (Figure 50.) This steel serves as positive tension steel in the center portion of the beam and as compression steel at the first interior support. It consists of six $\frac{7}{8}$-in. round bars in two rows. It must be so placed that there are

(*a*) 6 bars for the maximum bending moment at the center
(*b*) 4 bars for bond at a section 41 in. from the support
(*c*) 2 bars for anchorage running 10 diameters into each support (A.C.I. Art. 902*b*)
(*d*) 2 bars at the first interior support to act as compression steel

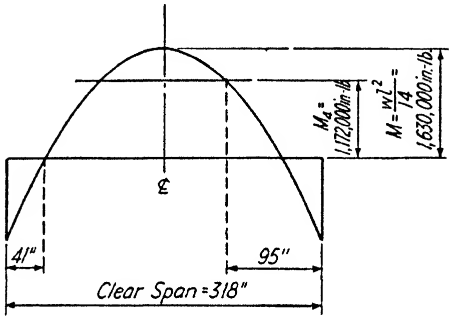


FIG. 50

These requirements permit two bars to be cut off before they reach the point of inflection (41 in.). The center bars of each row can be cut as soon as four bars can withstand the decreasing bending moment. The remaining four bars can carry a moment of

$$M_4 = f_sA_sjd = 20{,}000 \times 4 \times 0.601 \times 0.87 \times 28.09 = 1{,}172{,}000 \text{ in.-lb.}$$

This occurs at a section $0.3l = 95$ in. from the supports. With anchorage of 10 diameters (9 in.) these bars end at 86 in. from the supports.

The remaining two bars on the top row will run 9 in. beyond the point of inflection, or $41 - 9 = 32$ in. from each support. The remaining two bars on the bottom row should run 9 in. into the supports for anchorage. At the interior support these bars should also run in far enough to anchor the compressive steel stress of 17,100 lb. per sq. in. This anchorage is figured by equation 28 (Art. 57) as

$$l = \frac{17{,}800}{4 \times 100} \times \frac{7}{8} = 38 \text{ in.}$$

The bar lengths are

2 bars	12 ft. 2 in. long
2 bars	21 ft. 2 in. long
2 bars	30 ft. 5 in. long

95. Placing Steel. *Steel in Top of Beam. First Interior Support.* (Figure 51.) This steel serves as negative tension steel over the first interior support and consists of six 1-in. round bars in two rows. It has been figured for the moment of $-\frac{wl^2}{10}$ at the exterior face but must also serve for the moment of $-\frac{wl^2}{11}$ in the adjacent span. This adjacent span is an interior span and this moment amounts to 2,070,000 in.-lb. and the shear force equals $0.5wl = 35{,}900$ lb. (Problem 8, Chapter 5). At the point of inflection, 77 in. out, the shear force equals 18,500 lb. (Art. 73). The number of bars needed at this section to satisfy bond is

$$\Sigma = \frac{V}{oujd} = \frac{18{,}500}{3.14 \times 100 \times 0.87 \times 28.37} = 2.4 \text{ bars}$$

In the exterior span we have a uniform load supported at the interior end by a force of $0.575wl$ and restrained by a moment of $0.1wl^2$. Solving, the point of inflection occurs at $0.214l = 68$ in. The maximum shear force equals 41,200 lb. and the

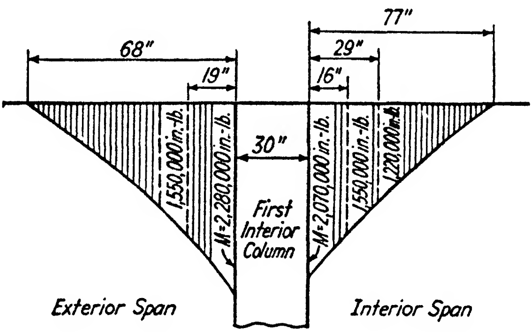


FIG. 51

shear force reduces to zero in $0.575l = 183$ in. (Fig. 52). The shear force at the point of inflection equals 25,900 lb. The number of bars needed at this section for bond is 3.2 bars.

The requirements to be met by this steel are

(a) 6 bars at the first interior support for bending moment and bond.
(b) 3 bars in the interior span and 4 bars in the exterior to the points of inflection. These bars will extend 12 diameters (12 in.) beyond the point of inflection for anchorage. They will extend $68 + 12 = 80$ in. into the exterior span and $77 + 12 = 89$ in. into the interior span.
(c) 3 bars in the interior span and 2 bars in the exterior can be cut before they reach the point of inflection. Such bars must fulfill bending moment and bond requirements before they end.

The two outer bars in the upper row will be carried to the center line of both spans and lapped 3 in. beyond with the negative tension steel coming from the exterior support and the second interior support. The remaining bar on the top row will be carried out beyond the points of inflection. The middle bar in the lower row will run beyond the points of inflection. The two end bars will be cut together to preserve the symmetry of the steel arrangement.

In the exterior span, if two bars are cut, four remain to resist the bending moment. The moment four bars can carry is

$$M_4 = f_s A_s jd = 20{,}000 \times 3.14 \times 0.87 \times 28.37 = 1{,}550{,}000 \text{ in.-lb.}$$

By the bending moment equation for the exterior span two bars are not needed for bending moment beyond 19 in. from the interior support (Fig. 51).

By the use of Diagram 5 (Appendix) for the interior side of this column, two bars are not needed $0.05l = 16$ in. out and the third bar is not needed after $0.09l = 29$ in. out from the face of this column (Fig. 51).

The check for bond can be made by computing the shear force the remaining bars can withstand. If four bars remain,

$$V_4 = u\Sigma ojd = 100 \times 4 \times 3.14 \times 0.87 \times 28.37 = 31{,}000 \text{ lb.}$$

The shear force diagrams adjacent to the first interior column is shown in Figure 52. The two outer bars in the lower row are not needed for bond beyond 46 in.

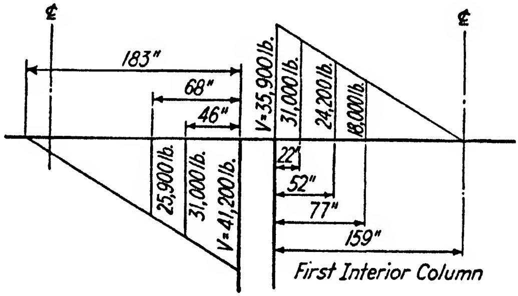


FIG. 52

in the exterior span and 22 in. in the interior. The center bar is not needed beyond 68 in. and 52 in. respectively.

Assembling the bending moment and bond data, the two outer bars are not needed for 46 in. out in the exterior span and 22 in. out in the interior. The center bar in the lower row is not needed 68 in. out in the exterior and 52 in. in the interior span. All these distances were determined by the bond requirement. In addition each bar extends 12 diameters (12 in.) beyond these points for anchorage (A.C.I. Art. 902).

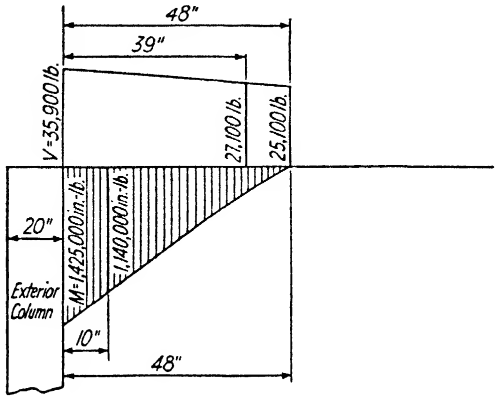
Lower Row	The two center bars are at least 10 ft. 2 in. long.
	The center bar is at least 14 ft. 6 in. long.
Upper Row	The two outer bars are 29 ft. 6 in. long.
	The center bar is 16 ft. 7 in. long.

96. Placing Steel. *Steel in Top of Beam. Exterior Support.* (Figure 53.) This steel consists of three $\frac{7}{8}$-in. round bars in the top row and three $\frac{7}{8}$-in. round bars in the lower row. The bond stresses are computed for a maximum shear force of $0.5wl$

(Fig. 48). The point of inflection for the maximum negative moment of $-\frac{wl^2}{16}$ occurs at $0.15l$. At this section the sum of the bar perimeters needed for bond is

$$\Sigma o = \frac{V}{ujd} = \frac{25{,}100}{100 \times 0.87 \times 28.59} = 10.0 \text{ in.}$$

This can be supplied by the three $\frac{7}{8}$-in. bars plus one $\frac{3}{4}$-in. bar.



Shear Force and Bending Moment Diagrams
Exterior Support

FIG. 53

The requirements to be met by this tension steel follow.

(a) Six bars at the face of the exterior support for bending moment and bond. This steel has an actual stress of 18,300 lb. per sq. in. and must be anchored in the support for

$$l = \frac{f_s}{4u}a = \frac{18{,}300}{4 \times 100} \times a = 46 \text{ diameters}$$

The $\frac{7}{8}$-in. bars must run into the support 40 in. and the $\frac{3}{4}$-in. bars for 35 in.

(b) Three $\frac{7}{8}$-in. bars and one $\frac{3}{4}$-in. bar must run at least 12 diameters beyond the point of inflection (48 in.).

The two outer $\frac{7}{8}$-in. bars will be carried out 3 in. beyond the center line of the span and lap with the two 1-in. bars coming from the first interior support. The center bar in each row will run 12 diameters beyond the point of inflection. The remaining two $\frac{3}{4}$-in. bars can be cut off before reaching the point of inflection, providing bond and bending moment requirements are satisfied. When the two $\frac{3}{4}$-in. bars are cut the remaining bars can carry a bending moment of

$$M_4 = f_sA_sjd = 20{,}000 \times 2.24 \times 0.87 \times 29.42 = 1{,}140{,}000 \text{ in.-lb.}$$

The bending moment reduces to this value at $0.03l = 10$ in. out (Diagram 5). The remaining bars can carry a shear force to satisfy bond of

$$V_4 = u\Sigma ojd = 100 \times 10.61 \times 0.87 \times 29.42 = 27{,}100 \text{ lb.}$$

The shear force becomes less than this amount at 39 in. (Fig. 53). This is greater than 10 in. so these two $\frac{3}{4}$-in. bars will be cut off at $39 + 9 = 48$ in. from the face of the support.

Upper Row	The two outer bars are at least 16 ft. 10 in. long.
	The center bar is at least 8 ft. 3 in. long.
Lower Row	The center bar is at least 7 ft. 8 in. long.
	The two outer bars are at least 6 ft. 11 in. long.

97. Diagonal Tension. We shall again use stirrups for diagonal tension steel, assuming the same $\frac{3}{8}$-in. two-rod stirrup used for the interior span (Art. 75). The stirrup spacings will be different in the exterior side than in the interior side since the shear force diagrams differ (Fig. 48).

$$\text{Maximum spacing } s = \frac{d}{2} = \frac{28.37}{2} = 14.2 \text{ in.}$$

Shear stress at exterior column:

$$v = \frac{V}{bjd} = \frac{35{,}900}{16 \times 0.87 \times 28.59} = 90 \text{ lb. per sq. in.}$$

Shear stress at exterior force of interior column:

$$v = \frac{41{,}200}{16 \times 0.88 \times 28.37} = 103 \text{ lb. per sq. in.}$$

The stirrup spacings are figured by the equation

$$v = 40 + \frac{2 \times 0.11 \times 20{,}000}{16 \times s \times 1} = 40 + \frac{275}{s}$$

Listing desirable spacings and corresponding distances:

Spacing s	Shear Stress v	Distance from Face of Exterior Support	Distance from Face of Interior Support
in.	lb. per sq. in.	in.	in.
4	..	..	..
6	86	7	30
9	71	34	57
12	63	48	71
	40	88	112

The stirrup spacings in the exterior portion of the beam are practically the same as those of the interior spans, the variation being due to differences in the value of d. The actual stirrup spacings are shown in the final sketch of Figure 54.

Note that A.C.I. Article 706a requires that wherever compression steel is needed stirrups must be used at a spacing not exceeding

$$16 \text{ bar diameters} = 16 \times \tfrac{7}{8} = 14 \text{ in.}$$

$$48 \text{ tie diameters} = 48 \times \tfrac{3}{8} = 18 \text{ in.}$$

At the interior support no compression steel is needed when the moment has decreased to 2,180,000 in.-lb., which occurs about 4 in. out. The above design fulfills this requirement.

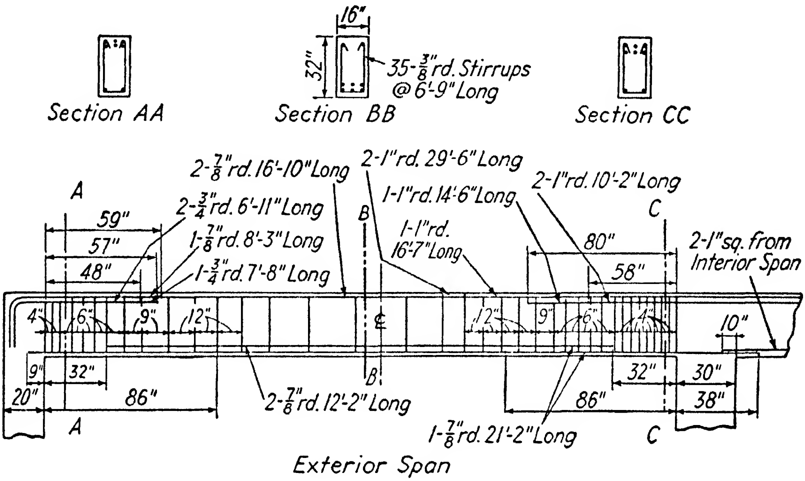


FIG. 54

ILLUSTRATIVE PROBLEM 11

98. To Check a Completed Design. The use of the equations derived in Article 79 to check designs using compression steel will be illustrated by reviewing the design just completed; refer to the final sketch of Figure 54.

First Interior Support. We note from the sketch and the loading data that

Width of section	$b = 16$ in.
Height to tensile steel	$d = 28.37$ in.
Height to compression steel	$d' = 2.82$ in.
Tensile steel ratio	$p = \dfrac{A_s}{bd} = \dfrac{6 \times 0.785}{16 \times 28.37} = 0.0104$

The force in the compression steel is assumed to be $2f'_sA'_s$ (A.C.I. Art. 706). In order to figure f'_s by the theoretical equations it will be necessary to use the area of the compression steel as $2A'_s$.

$$\text{Compression steel ratio } p' = \frac{2A'_s}{bd} = \frac{2 \times 2 \times 0.601}{16 \times 28.37} = 0.0053$$

$$\text{Maximum bending moment } \frac{wl^2}{10} = 2{,}280{,}000 \text{ in.-lb.}$$

By equation 45 we solve for the neutral axis ratio k:

$$k = \sqrt{2 \times 15\left(0.0104 + 0.0053 \times \frac{2.82}{28.37}\right) + (15)^2(0.0104 + 0.0053)^2} - 15(0.0104 + 0.0053)$$

$$k = \sqrt{0.328 + 0.055} - 0.236 = 0.619 - 0.236 = 0.383$$

The neutral axis height $kd = 0.383 \times 28.37 = 10.85$ in.
Using equation 46, the distance z equals

$$z = \frac{\dfrac{(0.383)^3 \times 28.37}{3} + 2 \times 15 \times 0.0053 \times 2.82\left(0.383 - \dfrac{2.82}{28.37}\right)}{(0.383)^2 + 2 \times 15 \times 0.0053\left(0.383 - \dfrac{2.82}{28.37}\right)}$$

$$z = \frac{0.532 + 0.127}{0.147 + 0.045} = \frac{0.659}{0.192} = 3.43 \text{ in.}$$

The moment arm $jd = d - z = 28.37 - 3.43 = 24.94$ in.

Note that the moment arm ratio $j = \dfrac{24.94}{28.37} = 0.879$ agrees with the value of $j = 0.88$ assumed by the designer (Art. 90).

The maximum compressive stress in the concrete is found by equation 52:

$$f_c = \frac{2 \times 2{,}280{,}000}{0.88 \times 16 \times (28.37)^2\left[0.383 + 2 \times 15 \times 0.0053\left(1 - \dfrac{2.82}{10.85}\right)\right]}$$

$$f_c = \frac{4{,}560{,}000}{11{,}330[0.383 + 0.118]} = \frac{4{,}560{,}000}{5680} = 803 \text{ lb. per sq. in.}$$

The allowable stress is $f_c = 0.45 \times 2000 = 900$ lb. per sq. in. Safe.
The tensile stress equals

$$f_s = \frac{M}{A_s jd} = \frac{2{,}280{,}000}{(6 \times 0.785) \times 0.88 \times 28.37} = 19{,}400 \text{ lb. per sq. in.}$$

The allowable stress is $f_s = 20{,}000$ lb. per sq. in. The tension steel is safe and is economically designed.

The compression steel stress is given by equation 41:

$$f'_s = 15 \times 803\left(\frac{10.85 - 2.82}{10.85}\right) = 8900 \text{ lb. per sq. in.}$$

The compression steel is designed for twice this value, or 17,800 lb. per sq. in., which is less than the allowable of 20,000 lb. per sq. in. It will be noticed that, though there is an excess of compression steel area, the stress is not reduced in proportion to the areas.

99. Solution by Plots. The algebraic solution for the compression steel area A'_s by the method of Article 81 is so direct that plots do not save time. The solution deals with the terms the designer needs in order to complete the design; namely, the depths d and d', the compression steel stress f'_s and area A'_s.

On the other hand, the algebraic equations developed for checking designs are complicated in form, and time can be saved by the use of plots. Diagrams 6 to 8, in the Appendix, give values of the neutral axis ratio j and the bending moment constant K', where

$$M_c = \frac{f_c jbd^2}{2}\left[k + 2np'\left(1 - \frac{d'}{kd}\right)\right] = K'f_c bd^2$$

or

$$K' = \frac{j}{2}\left[k + 2np'\left(1 - \frac{d'}{kd}\right)\right]$$

The diagrams require that the terms np and np' be known or assumed.

ILLUSTRATIVE PROBLEM 12

100. To Check Designs by Diagrams. Let us check the design of Problem 10 by means of plots.

First Interior Support. As before, we note from Figure 54 that

Width of section	$b = 16$ in.
Height to tensile steel	$d = 28.37$ in.
Height to compression steel	$d' = 2.82$ in.
Tensile steel ratio	$\frac{A_s}{bd} = p = 0.0104$
Compression steel ratio	$\frac{2A'_s}{bd} = p' = 0.0053$
Maximum bending moment	$M = \frac{wl^2}{10} = 2{,}280{,}000$ in.-lb.
Moduli of elasticity ratio	$n = 15$

Then

$$np = 15 \times 0.0104 = 0.156$$

$$np' = 15 \times 0.0053 = 0.0795$$

$$\frac{d'}{d} = \frac{2.82}{28.37} = 0.1 \text{ (nearly)}$$

By Diagram 7 in the Appendix, for $\frac{d'}{d} = 0.10$

$$k = 0.38 \qquad j = 0.879 \qquad K' = 0.22$$

$$f_c = \frac{M}{K'bd^2} = \frac{2{,}280{,}000}{0.22 \times 16 \times (28.37)^2} = 800 \text{ lb. per sq. in.}$$

If allowance is made for the less accurate values taken from a plot, the values of j, k, f_c check the values laboriously computed in Problem 11. The stresses f_s and f'_s can be now obtained. If the ratio $\frac{d'}{d}$ were not closely 0.10, it would be necessary to interpolate between two diagrams to get the values of k, j, and K'. These plots save much time in checking designs.

PLASTIC THEORY

ILLUSTRATIVE PROBLEM 13

101. Compression Steel by Plastic Theory. Using the plastic theory, let us make a parallel solution to that of Problem 10. The design of the interior span by the plastic theory in Problem 9 gave a section 14 in. by 28 in., which was "balanced design" for a moment of 1,980,000 in.-lb. Let us adopt the same section for the exterior span whose load w = 2585 lb. per ft. With a factor of safety of 0.4 use f'_c = 800 lb. per sq. in. and f_y = 20,000 lb. per sq. in.

Tension Steel. There is no advantage using more steel than that called for by balanced design. The moment $M = \frac{wl^2}{10} = 2{,}180{,}000$ in.-lb. and, by equation 21 (Art. 25),

$$A_s = \frac{M}{0.732f_yd} = \frac{2{,}180{,}000}{0.732 \times 20{,}000 \times 23.62} = 6.28 \text{ sq. in.}$$

Try four $1\frac{1}{4}$-in. square bars in two rows.

Failure on Compression Side of Section. By equation 58 (Art. 82), assuming the compression steel to be $1\frac{1}{8}$-in. square bars

$$2{,}180{,}000 = \frac{800}{3} \times 14 \times (23.62)^2 + A'_s \times 20{,}000 \times (23.62 - 2.94)$$

$$2{,}180{,}000 = 2{,}085{,}000 + 413{,}600A'_s$$

$$A'_s = 0.23 \text{ sq. in.}$$

The steel used is computed below.

Failure in Tension Steel. The compression steel will be supplied by the positive tension steel from the center of the span. The positive moment $\frac{wl^2}{14} = 1{,}560{,}000$ in.-lb. is less than the "balanced design" moment of the interior span. This positive steel in the exterior span will be "under-reinforced" steel and can be computed by use of equation 17 (Art. 24).

$$1{,}560{,}000 = 20{,}000 \times p \times \left(1 - \frac{29.4p}{2}\right) \times 14 \times (25.06)^2$$

where $m = \frac{f_y}{0.85f'_c} = \frac{20{,}000}{0.85 \times 800} = 29.4$. Solving,

$$p = 0.0105 \qquad \text{and} \qquad A_s = pbd = 3.72 \text{ sq. in.}$$

Use three $1\frac{1}{8}$-in. square bars. Two of these bars will run into the support and be available for use as compression steel; therefore, at the interior support A'_s = 2.54 sq. in. At this section the steel ratio

$$p = \frac{6.24}{14 \times 23.62} = 0.0188 \qquad \text{and} \qquad p' = \frac{2.54}{14 \times 23.62} = 0.0077$$

Substituting in equation 62 (Art. 82),

$$\frac{M}{bd^2} = \frac{77}{(10)^4} \times (2 \times 10^4)\left(1 - \frac{2.94}{23.62}\right) + \left(\frac{188 - 77}{10^4}\right)(2 \times 10^4)\left[1 - \frac{(188 - 77)29.4}{2 \times 10^4}\right]$$

Solving, $\dfrac{M}{bd^2} = 154 \times 0.876 + 222 \times 0.837 = 321$. But $\dfrac{M}{bd^2} = 279$ actually, so there is sufficient tension and compression steel at the section. Equation 62 could be solved for the minimum value of p', but the method above, using available steel, is simpler. The solution by plastic theory gives a smaller section and requires more tension steel than the straight-line stress theory, the minimum compression steel area A'_s being very small for both solutions.

ILLUSTRATIVE PROBLEM 14

102. The Check of a Design by Plastic Theory. Let us check the straight-line theory design of Problem 10 at the first interior column by the plastic theory. The essential data are

Width of section	$b = 16$ in.
Height to tension steel	$d = 28.37$ in.
Height to compression steel	$d' = 2.82$ in.
Tension steel area	$A_s = 4.71$ sq. in.
Compression steel area	$A'_s = 1.20$ sq. in.
	$m = 29.4$
Maximum moment	$M_N = 2{,}280{,}000$ in.-lb.

Failure on Compression Side of Section. By equation 59 (Art. 82)

$$M = 16 \times (28.37)^2 \left[\frac{800}{3} + \frac{20{,}000 \times 1.20}{16 \times 28.37}\left(1 - \frac{2.82}{28.37}\right)\right]$$

$$M = 3{,}340{,}000 + 614{,}000 = 3{,}954{,}000 \text{ in.-lb.}$$

By the plastic theory it is evident that no compression steel is needed to aid the concrete.

Failure in Tension Steel. By equation 61

$$c = 28.37 - \frac{(4.71 - 1.20)29.4}{2 \times 16} = 25.14 \text{ in.}$$

By equation 60,

$$M = 1.20 \times 20{,}000(28.37 - 2.82) + (4.71 - 1.20)20{,}000 \times 25.14$$

$$M = 20{,}000(30.7 + 88.2) = 2{,}378{,}000 \text{ in.-lb.}$$

This moment is slightly greater than the actual moment of 2,280,000 in.-lb. and the plastic theory checks this particular design by the "straight-line working-stress" theory. Both theories give equations that are better adapted to check a design than for initial design.

CHAPTER 7

TEE BEAMS

The usual reinforced concrete floor system is poured as a unit—slabs and beams at the same time. Since the slab steel runs over the beam and the prongs of the beam stirrups project into the slab, the slab is closely tied to the beam and deflects with it. It was early recognized that in such a case the beam was much stiffer than the same size rectangular beam having no assistance from the slab. It was stiffer and tests showed that the concrete and tension steel had lower stresses. Therefore, it is customary to count in a portion of the slab adjacent to the beam as forming a part of the beam section.

103. Tee Beams. Beams whose compression areas are tee-shaped are called *tee beams.* In Figure 55 the sections represented are subjected

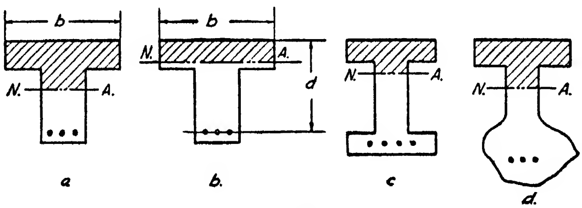


FIG. 55

to positive bending and their compression areas are shown shaded. Sections *a, c,* and *d* are tee beams. Section *b* is a rectangular beam. It will be designed for a width *b* of compression area and depth *d* to the tension steel by the usual rectangular beam equations. See also discussion in Article 18.

WORKING-STRESS STRAIGHT-LINE THEORY

104. Tee-Beam Theory. *Compression in Web Neglected.* Given a beam of tee-shaped cross section (Fig. 56). It is subjected to a positive bending moment *M* of such a magnitude that the neutral axis is located in the stem at a distance *kd* from the top greater than the thickness *t* of the flange. The flange has a width *b* and the stem a width of *b′*. The tension steel has an area A_s and is located a distance d from the extreme fiber in compression.

For this case assume the neutral axis to be not far below the bottom of the flange. The area of the stem above the neutral axis is small, and the compressive stresses on this area are small. If we neglect this stem area entirely the total compression force C will not be much reduced and the design will be on the safe side since we use a smaller force than actually exists. The resulting equations will be much simpler.

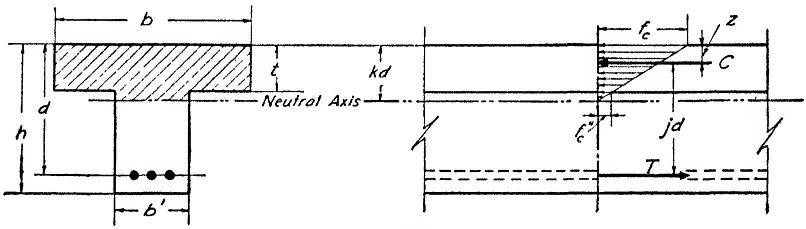


FIG. 56

The first assumption of the beam theory that a plane section remains plane justifies the statement that the strains are proportional to the distance from the neutral axis. If the strain in the extreme compression fiber is e_c and the strain in the tension steel is e_s, it follows that

$$\frac{e_s}{e_c} = \frac{d - kd}{kd} = \frac{1 - k}{k}$$

By Hooke's law the strain equals the ratio of stress to the modulus of elasticity, or

$$\frac{e_s}{e_c} = \frac{\dfrac{f_s}{E_s}}{\dfrac{f_c}{E_c}} = \frac{E_c f_s}{E_s f_c} = \frac{f_s}{nf_c} = \frac{d - kd}{kd} = \frac{1 - k}{k} \tag{64}$$

This reduces to

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \tag{65}$$

This is the same result that is given for rectangular beam and for rectangular beams with compression steel. It gives the neutral axis ratio k if the actual steel and concrete stresses are known as well as the concrete strength.

Similarly, the strain e''_c at the bottom of the flange equals

$$\frac{e''_c}{e_c} = \frac{\dfrac{f''_c}{E_c}}{\dfrac{f_c}{E_c}} = \frac{f''_c}{f_c} = \frac{kd - t}{kd}$$

Therefore the concrete stress f''_c at the bottom of the flange equals

$$f''_c = f_c \frac{kd - t}{kd} \tag{66}$$

The resultant compression force C can be found by using the average compressive stress on the flange:

$$C = \frac{(f_c + f''_c)}{2} bt = \frac{f_c}{2}\left[1 + \frac{kd - t}{kd}\right] bt = \frac{f_c bt}{2kd}(2kd - t) \tag{67}$$

The compressive force C must equal the resultant tension force T.

$$\frac{f_c bt}{2kd}(2kd - t) = f_s A_s$$

Therefore

$$\frac{f_s}{f_c} = \frac{bt(2kd - t)}{2A_s kd} \tag{68}$$

From equations 64 and 68 we get

$$\frac{f_s}{f_c} = \frac{n(d - kd)}{kd} = \frac{bt(2kd - t)}{2A_s kd}$$

Multiplying by $2A_s kd$ gives

$$2nA_s(d - kd) = bt(2kd - t)$$

$$kd = \frac{2ndA_s + bt^2}{2nA_s + 2bt} \tag{69}$$

The resultant compression force C acts at a distance z from the top. The moment of the resultant force C about an axis will equal the sum of the moments of the uniformly varying stresses acting on the rectangle bt. These varying stresses may be considered as a constant stress of f''_c plus a stress varying uniformly from zero at the bottom to $(f_c - f''_c)$ at the top of the flange.

Taking moments about the top and using the value of C in equation 67,

$$Cz = \frac{f_c bt}{2kd}(2kd - t)z = f''_c bt\left(\frac{t}{2}\right) + \frac{(f_c - f''_c)}{2}bt\left(\frac{t}{3}\right)$$

Substituting the value of f''_c from equation 66,

$$\frac{f_c bt}{2kd}(2kd - t)z = f_c\left(\frac{kd - t}{kd}\right)\frac{bt^2}{2} + \frac{f_c}{2}\left[1 - \frac{(kd - t)}{kd}\right]\frac{bt^2}{3}$$

Multiplying by $\dfrac{6kd}{f_c bt}$ gives

$$3(2kd - t)z = 3(kd - t)t + t^2$$

$$z = \frac{t(3kd - 2t)}{3(2kd - t)} \tag{70}$$

From Figure 56 it is evident that

$$jd = d - z \tag{71}$$

In terms of steel stresses, the moment of resistance M_s equals

$$M_s = Tjd = A_s f_s jd \tag{72}$$

This is the same equation derived for rectangular beams. The moment arm ratio j, however, is usually greater for tee beams than for rectangular beams. It usually varies from 0.91 to 0.94. It is an advantage to have j large as a greater moment can be carried for the same depth and same allowable stress.

In terms of concrete stresses, the moment of resistance M_c equals

$$M_c = Cjd = \frac{f_c bt(jd)}{2(kd)}(2kd - t) \tag{73}$$

From equation 64 it is possible to obtain the concrete stress f_c, if the steel stress f_s is known, by the relation

$$f_c = \frac{f_s}{n}\left(\frac{kd}{d - kd}\right) \tag{74}$$

105. Tee-Beam Theory. *Compression in Web Considered.* A tee beam with a thin flange, or one whose flange width b is not much greater than the stem width b', or a tee beam heavily loaded may have the neutral axis so far below the bottom of the flange that this area should be considered in computations.

Using the same nomenclature as in Article 104, it is again true that the stress relation equals

$$\frac{f_s}{nf_c} = \frac{d - kd}{kd} = \frac{1 - k}{k} \tag{75}$$

From which the neutral axis ratio k comes:

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \tag{76}$$

As before

$$f''_c = f_c\left(\frac{kd - t}{kd}\right)$$

The resultant compression force C equals the force in the flange, as before, plus the additional force in the stem:

$$C = \frac{f_c bt}{2kd}(2kd - t) + \frac{f''_c b'}{2}(kd - t)$$

$$C = \frac{f_c bt}{2kd}(2kd - t) + \frac{f_c b'}{2kd}(kd - t)^2 = \frac{f_c}{2kd}[bt(2kd - t) + b'(kd - t)^2]$$

Equating this to the tension force T gives

$$\frac{f_c}{2kd}[bt(2kd - t) + b'(kd - t)^2] = f_s A_s$$

$$\frac{f_s}{f_c} = \frac{[bt(2kd - t) + b'(kd - t)^2]}{2kdA_s} \tag{77}$$

Solving equation 75 for the ratio $\dfrac{f_s}{f_c}$ and equating it to equation 77

$$n\left(\frac{d - kd}{kd}\right) = \frac{[bt(2kd - t) + b'(kd - t)^2]}{2kdA_s}$$

Multiplying by $2kdA_s$ and expanding the terms,

$$2nA_s d - 2nA_s kd = 2kdbt - bt^2 + b'(kd)^2 - 2b'tkd + b't^2$$

$$b'(kd)^2 + 2[nA_s + (b - b')t]kd = 2nA_s d + (b - b')t^2$$

Divide by b' and complete the square of the terms on the left by adding $\left[\frac{nA_s + (b - b')t}{b'}\right]^2$ to both sides.

$$(kd)^2 + 2\frac{[nA_s + (b - b')t]kd}{b'} + \left[\frac{nA_s + (b - b')t}{b'}\right]^2 =$$
$$\frac{2nA_sd + (b - b')t^2}{b'} + \left[\frac{nA_s + (b - b')t}{b'}\right]^2$$

Taking the square root,

$$kd + \frac{nA_s + (b - b')t}{b'} = \sqrt{\frac{2nA_sd + (b - b')t^2}{b'} + \left[\frac{nA_s + (b - b')t}{b'}\right]^2}$$

Therefore

$$kd = \sqrt{\frac{2nA_sd+(b-b')t^2}{b'} + \left[\frac{nA_s+(b-b')t}{b'}\right]^2} - \frac{nA_s+(b-b')t}{b'} \tag{78}$$

Taking moments about the top of the beam and equating the sum of the moments of the stresses to the moment of the resultant force C (Fig. 56):

$$Cz = \frac{f_c}{2kd}[bt(2kd - t) + b'(kd - t)^2]z =$$
$$f''_c\frac{bt^2}{2} + \frac{(f_c - f''_c)bt^2}{6} + \frac{f''_cb'}{2}(kd - t)\left(t + \frac{kd - t}{3}\right)$$

$$\frac{f_c}{2kd}[bt(2kd - t) + b'(kd - t)^2]z =$$
$$f_c\left(\frac{kd - t}{kd}\right)\frac{bt^2}{2} + f_c\left[1 - \frac{kd - t}{kd}\right]\frac{bt^2}{6} + \frac{f_cb'}{2kd}(kd - t)^2\left(t + \frac{kd - t}{3}\right)$$

Multiply by $\frac{6kd}{f_c}$:

$$3[bt(2kd - t) + b'(kd - t)^2]z = 3(kd - t)bt^2 + bt^3 + b'(kd - t)^2(kd + 2t)$$

Solving for z gives

$$z = \frac{b'(kd)^3 + t^2(b - b')(3kd - 2t)}{3[b'(kd)^2 + t(b - b')(2kd - t)]} \tag{79}$$

The moment arm jd equals

$$jd = d - z \tag{80}$$

The moment of resistance in steel stress terms equals

$$M_s = Tjd = A_s f_s jd \tag{81}$$

The moment of resistance in concrete stress terms equals

$$M_c = Cjd = \frac{f_c(jd)}{2(kd)}[bt(2kd - t) + b'(kd - t)^2] \tag{82}$$

It is also true that

$$f_c = \frac{f_s}{n}\left(\frac{kd}{d - kd}\right) \tag{83}$$

PLASTIC THEORY

106. Plastic Theory of Tee Beams. The plastic theory for rectangular beams assumes for balanced design that the compression stresses are constant at a value of $0.85f'_c$ for a depth of $a = 0.537d$ (Art. 25). Whenever the flange of a tee beam has a thickness t greater than $0.537d$ the rectangular beam equations can be applied unchanged. Whenever the flange of a tee beam has a thickness t not much less than $0.537d$ the section can be designed as a rectangular beam of width b and depth t with the constant stress $0.85f'_c$ acting on it. In other words, $a = t$. In this case equation 20 (Art. 25) becomes

$$M = 0.85f'_c bt\left(d - \frac{t}{2}\right) = 0.85f'_c \frac{t}{d}\left(1 - \frac{t}{2d}\right) bd^2 \tag{84}$$

and equation 21, for the steel area, equals

$$A_s = \frac{M}{f_y\left(d - \frac{t}{2}\right)} \tag{85}$$

If the slab thickness t is much less than $0.537d$, the area in the stem to a depth of $0.537d$ may be included. All this area is assumed to have an average stress of $0.85f'_c$. This compression area A_c equals

$$A_c = (b - b')t + b'(0.537d)$$

Taking moments about the top of the beam, its center of gravity z equals

$$z = \frac{d}{2}\,\frac{\left[(b - b')\left(\frac{t}{d}\right)^2 + 0.288b'\right]}{\left[(b - b')\left(\frac{t}{d}\right) + 0.537b'\right]} \tag{86}$$

The bending moment equals

$$M = 0.85f'_c(d - z)A_c \tag{87}$$

The steel area equals

$$A_s = \frac{M}{f_y(d - z)} \tag{88}$$

If these equations are used, the tests available indicate that the width of flange b should not be taken greater than $b = 8t + b'$, instead of present A.C.I. Code value of $b = 16t + b'$.

107. Shear Stresses in Tee Beams. It has been shown (Art. 51) that the maximum shear stresses in a section occur between the neutral axis and the tension steel. This is entirely within the stem and, therefore, the maximum shear stress v for a tee beam will be given by the relation

$$v = \frac{V}{b'jd}$$

The beam of Figure 55*b* is a rectangular beam. The shear force on any layer is constant between the tension steel and the neutral axis which is now in the flange. The layers with the least width and hence greatest stress will be in the stem, and here again the maximum shear stress v equals

$$v = \frac{V}{b'jd}$$

108. Bond Stress in Tee Beams. The discussion of bond in Article 54 shows that the bond or adhesion between the bar and the concrete depends on the rate of change of bending moment. This is independent of the shape of the cross section. Therefore the bond stress is computed by the previously derived relation:

$$u = \frac{V}{\Sigma ojd}$$

109. Width of Flange. When the slab and beam are poured simultaneously there is no definite demarcation between the two. If the beam is heavily loaded and deflects, as at B in Figure 57, it drags some of the slab down with it, as shown by the dashed lines. The portion of the

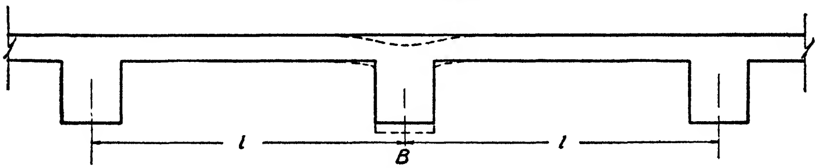


FIG. 57

slab near the beam assists in resisting the deflecting loads. Tests confirm the statement that the adjacent slab can be considered as part of the beam. In a commercial design it is necessary that there be restrictions so that the designer does not assign too great a width of the slab as flange of the tee beam.

A.C.I. Article 705*a* (see Appendix) specifies that the flange width b shall not exceed one fourth the span length of the beam, or $b < \frac{l}{4}$. It is evident that short-span beams will not deflect as much as long-span beams and, therefore, will not drag down as great a width of slab.

It is also specified that the overhanging width on either side shall not exceed eight times the thickness of the slab. In other words the flange width $b < 16t + b'$. Since the flange is used as though it were a cantilever from the stem the thinner slabs cannot have as great a projection.

The third restriction requires that the overhanging width shall not be greater than half the clear distance to the next beam, or $b <$ spacing. This specification merely states that no part of the slab shall be considered as flange for two beams.

The flange width used in computations is the least of these three restrictions. The flange can be used whenever the slab and beam are poured simultaneously and suitable steel is provided to tie the two together. In Figure 55 sections a and b both use the flange width b as the width of the compression area.

It may seem unwise to figure the slab as fully stressed in compression for both the slab computations and the tee-beam design. It should be noted, however (Fig. 58), that the slab is figured for a cross section such

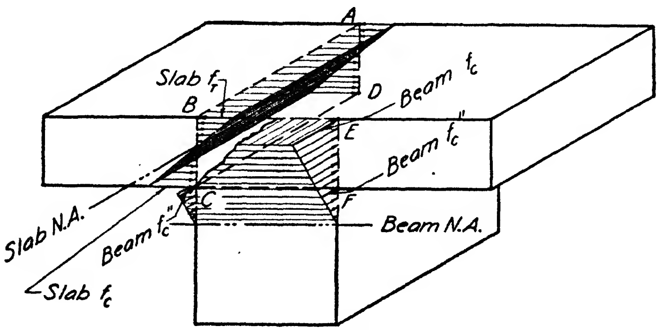


FIG. 58

as $ABCD$ and is in negative bending. The particles at C at the bottom of the slab have the maximum compression stress f_c in plane $ABCD$ due to the slab loads, while the particles at B and A are in tension. The beam shown will be a tee beam only for sections in heavy positive bending. Let us say that the section *BEFC* is the section of maximum positive bending moment. The particles *B* and *E* now have the maximum fiber stresses f_c in plane *BEFC* by the tee-beam computations, while particle *C* has only $f''_c = f_c \dfrac{kd - t}{kd}$. If plane *BEFC* is a plane of maximum bending moment, the shear force is zero and the shear stresses are zero for all particles. Therefore, plane *BEFC* is a principal plane and the stress f''_c is the minimum stress for particle *C*, and the slab stress f_c on the perpendicular plane *ABCD* must be the maximum stress on the particle *C*. Other cross sections through the beam will have smaller bending moments and smaller fiber stresses but there will also be shear forces in the section. The principal planes will be inclined to the beam cross section, but the principal stresses will not exceed the value f_c at point *C* on plane *ABCD* or the value f_c due to beam computations for particles *B* and *E*. Therefore, the use of the slab in both slab and beam computations does not give a fiber stress greater than the allowable value of f_c.

110. Economical Size for Tee Beams. It is possible, but not usual, to have a slab and beam floor which has only one span. If the supports are not restrained, the bending moment will be wholly positive and the beam sections will have the broad compression areas of Figures 55*a* and 55*b*. The most economical size will be the one which gives the maximum allowable stresses f_c and f_s in concrete and steel. The formulae for kd and z in Articles 104 and 105 require a knowledge of the steel area A_s, depth d, and the flange width b and depth t. In other words, the equations are checker's or inspector's equations. After designing the slab the designer can most easily proceed by assuming a beam stem dimension and the tension steel area and solve for the concrete and steel stresses f_c and f_s. Three or four trials will give an economical beam section and area of steel.

The more usual slab and beam floor system is one that has several continuous spans, both of slabs and beams. In this case the beam is a continuous beam of constant cross section. For uniformly distributed loads the maximum positive bending moments will be of the magnitude of $\dfrac{wl^2}{16}$ and $\dfrac{wl^2}{14}$. The compression area, shown shaded in Figure 59*a*, will be broad and not deep, tending to give a high value of the moment arm ratio j and requiring comparatively small values of the depth d and the stem width b' to give satisfactory concrete stress f_c and steel stress f_s. There will be negative bending, also, in these continuous beams, and the maximum negative bending moments will be fully as great in magnitude, having values such as $\frac{wl^2}{10}$, $\frac{wl^2}{11}$, and $\frac{wl^2}{12}$. These moments must be resisted by a rectangular beam of the shape shown in Figure 59*b*, whose compression area is narrow and must therefore be deep with low values of the moment arm ratio j and the necessity of using comparatively large values for the depth d and stem width b' to keep within the maximum allowable values of the concrete stress f_c and the steel stress f_s.

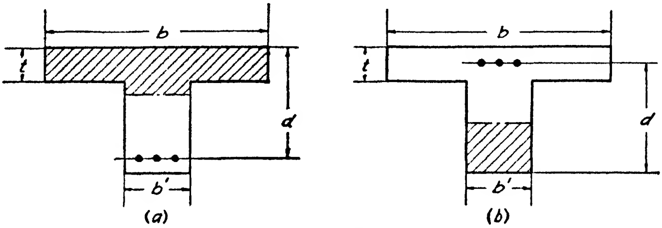


FIG. 59

A section, which at the center is fully stressed with $f_c = 0.45f'_c$ and $f_s = 20{,}000$ lb. per sq. in. for positive bending, will give values of f_c vastly greater than the allowable $f_c = 0.45f'_c$ at the support; in fact, the concrete stress at that section will approach the ultimate. On the other hand, a design which gives a section safely stressed at the support will require a section so large that the concrete stress in the center would be very low, which is not economical.

The economical solution is a compromise. The intermediate size used may be aided at the heavily loaded section at the support with compression steel, so that the final result gives a maximum concrete stress at the center of $f_c = 0.2f'_c$, or thereabouts, and a maximum concrete stress at the support of $f_c = 0.45f'_c$.

As the section at the support is the critical one, there will be for each stem width b' (compression area width) a corresponding most economical depth d. Of the possible stem widths the most practical will be adopted. When the placing of the concrete for the floor system commences the beam stem is the only portion of the beams projecting below the general slab level. The stem must not be too narrow, or the concrete cannot be placed about the steel without "honeycombing" showing at exposed surfaces when the forms are stripped. The stem must not be too shallow, for that requires an excessive amount of steel. Commercial stems are usually not over a 1 : 2 ratio of width b' to depth $(h - t)$.

111. Economical Depth. The cost analysis of a beam and slab floor system should compare possible column spacings and possible beam and girder arrangements. Increasing the distance between columns requires a substitution of fewer and heavier beams for the original layout. Increasing the spacing between beams increases the depth of the slab and the size of the beam. Comparative analyses of this sort in order to be complete should include the costs of columns, walls, slabs, and beams. It is not proposed to discuss such a comprehensive analysis at this time. After the economical column spacing and the framing plan for the girders and beams have been adopted, the slab thickness t can be computed. There then remains the necessity of determining the size of beam stem which shall give the cheapest beam for the given loads. The following discussion of economical size is based on fiber stress computations. Beams which are very long tend to have excessive deflections, and economy will be sacrificed for stiffness. Beams of short span and very heavy loads may have excessive shear stresses, and economy will be sacrificed to give safe shear conditions. The beam with a reasonable span and usual loads should be designed for economical proportions.

112. Cost of Concrete in Beam Stem. Let c equal the cost of the concrete per cubic foot. This unit cost should be supplied by the estimating department of the firm by which the designer is employed. It includes the cost of cement, sand, and stone, also the costs of mixing and placing the concrete. The cost of the concrete in the stem equals $cb'(h - t)$ per foot of beam length.

113. Cost of Steel in Beam. The estimating department furnishes a unit cost s per pound of steel. This includes the purchase of the steel, the bending, and the placing of the steel.

The designer wishes to select an economical size but he does not yet know the total steel in the beam. The firm by which the designer is employed has doubtless adopted a standard type of steel placement. Three such types are shown in Figures 28, 29, and 30 (Art. 60). If the standard type is always used there is a nearly constant ratio a between the weight of the total steel in the beam per foot of length and the weight of the positive steel area A_p per foot. Let us call $s_p = sa$ the unit cost of the steel per pound of positive steel. Also note that a bar 1 sq. in. in section and 1 ft. long weighs 3.40 lb. The designer can now figure approximately the amount of positive steel A_p and thereby estimate the cost of the total steel. Thus the total cost of steel per foot of beam length equals

$$\frac{s\text{ (weight of total steel in beam)}}{l} = 3.40 s_p A_p$$

where

$$s_p = as \tag{89}$$

The total cost of the steel per foot length of beam equals

$$3.40 a s A_p = 3.40 s_p A_p$$

114. Determination of Steel Ratio a. Referring to Problem 17 below, the total steel for intermediate beams BF or CG (Fig. 81) with Type II steel amounts to

Positive steel, 32 linear ft. $\frac{3}{4}$-in. rd. bars at 1.50 lb. per ft.	=	48.0 lb.
Negative steel, 62.5 linear ft. $\frac{1}{2}$-in. sq. bars at 0.85 lb. per ft.	=	53.1 lb.
Stirrups, 14.7 linear ft. $\frac{1}{4}$-in. rd. bars at 0.17 lb. per ft.	=	2.5 lb.
Neglecting spacer bars, ties, etc. Total	=	103.6 lb.

The total steel per foot of beam length $= \dfrac{103.6}{13.33} = 7.77$ lb. per ft.

The positive steel per foot $= 2 \times 1.50 = 3.00$ lb. per ft.

$$a = \frac{7.77}{3.00} = 2.59$$

Similar computations for the cross beam AE (Fig. 85) with Type II steel give a value of the ratio $a = \dfrac{8.03}{3.00} = 2.68$.

An analysis of the girder AD with Type III steel gives a value of the ratio $a = 2.05$ (Fig. 93).

This indicates that the ratio a varies from 2.0 to 2.7 for the author's methods of design when Types II and III systems of reinforcement and special anchorage are used. Ordinary anchorage of the steel will give lower ratios of a, say 1.9 to 2.2.

115. Cost of Forms. The estimating department usually reports the cost of formwork for beam and slab floor systems as f cents per square foot of floor surface. This includes the cost of forms for slabs and beams and all posts, ledgers, braces, wedges, etc., necessary to hold up the floor system until the concrete has reached a suitable strength. It is difficult to isolate the cost of the beam stem, and it is not necessary as a variation of 2 to 6 in. in stem depth makes little difference in the total form cost for the entire floor system. For our discussion we shall assume that the cost of the forms has little effect on the relative cost of different sizes of beam stems and may be neglected in an analysis.

ILLUSTRATIVE PROBLEM 15

116. Economical Depth by Computation. Given a beam and slab floor system with a slab thickness $t = 6$ in. The beam is subjected to a maximum positive bending moment of 1,400,000 in.-lb. Determine the cheapest section for a stem $b' = 12$ in.

Cost of concrete $c = 40$ cents per cubic foot

Cost of steel $s = 5$ cents per pound

The type of steel reinforcing used gives a steel ratio $a = 2$ for ordinary anchorage, or cost of steel $s_p = 10$ cents per pound of positive steel. The estimating department states that it has allowed about 15 cents per foot of beam as the form cost assigned to the beam stem.

Let us determine the cost of various beam stems, each of which has a width $b' = 12$ in. The following computation is made for the beam stem 12 in. by 24 in. (total depth = 30 in.).

Assume the positive steel to be in one row, 3 in. from the bottom, and assume that the moment arm ratio $j = 0.92$ approximately.

$$\text{Steel area } A_p = \frac{M}{f_s jd} = \frac{1{,}400{,}000}{20{,}000 \times 0.92 \times 27} = 2.82 \text{ sq. in.}$$

We shall use this minimum area A_p for our comparisons, so that the non-coincidence of commercial areas with the minimum may not affect this discussion.

$$\begin{aligned}
&\text{Cost of concrete} = \frac{cb'(h - t)}{144} = \frac{0.40 \times 12 \times 24}{144} &&= \$0.80\\
&\text{Cost of steel} = 3.40 \times s_p \times A_p = 3.40 \times (2 \times 0.05) \times 2.82 &&= \$0.96\\
&\text{Cost of forms} &&= \$0.15\\
&\quad\text{Total cost per foot length} &&= \$1.91
\end{aligned}$$

The tabulation below summarizes similar computations for a series of beam stems each 12 in. wide.

	Beam depth h inches						
$b' = 12$ in.	21	24	27	30	33	36	39
	Stem height = $(h - t)$ inches						
	15	18	21	24	27	30	33
Depth d (one row)	....	21	24	27	30	33	36
Depth d (two rows)	17	....	....	....	....	....	....
$A_p = \dfrac{1{,}400{,}000}{20{,}000 \times 0.92 \times d}$	4.48	3.63	3.14	2.82	2.54	2.31	2.12
Cost of steel	\$1.52	\$1.24	\$1.07	\$0.96	\$0.86	\$0.79	\$0.72
Cost of concrete	0.50	0.60	0.70	0.80	0.90	1.00	1.10
Cost of forms	0.15	0.15	0.15	0.15	0.15	0.15	0.15
Total cost per foot	\$2.17	\$1.99	\$1.92	\$1.91	\$1.91	\$1.94	\$1.97

In Figure 60 these costs are plotted against stem depth and indicate that the minimum cost occurs for the 12-in. by 27-in. stem. An old "rule of thumb" criterion that "the cheapest section is that for which the cost of concrete and forms equals the cost of the steel" agrees fairly well with this result, as it calls for the 12-in. by 24-in. stem whose total cost is also \$1.91 per foot length.

The computations given above give the cheapest stem for a 12-in. width. The process must be repeated for 8-in., 10-in., 14-in., and 16-in. stems in order to determine the cheapest possible stem. The complete computations are too long, and the

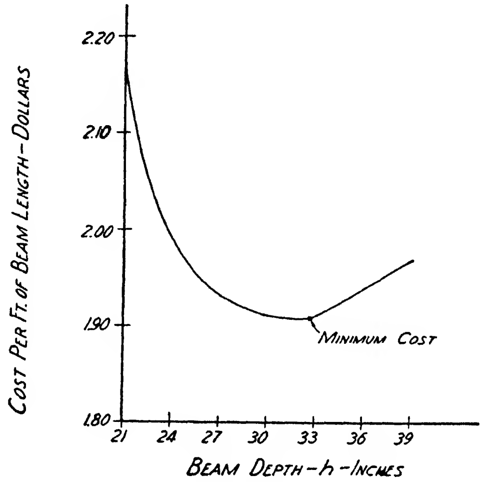


Fig. 60

problem is given only to illustrate the effect of too great depth and of a shallow beam. The determination of the economical size can be handled better by an algebraic analysis of the factors involved.

If a steel ratio $a = 3$ for special anchorage is used, the cheapest section in the tabulation above is the 12-in. by 33-in. stem. However, the 12-in. by 27-in. stem costs only 1 cent per foot more.

117. Economical Depth Equation. The total cost of the beam stem per foot length will be the sum of the concrete, form, and steel costs. It has been previously stated that the variation in form cost may be neglected if we are comparing different stem sizes to carry the same bending moment. Variation in the cost will be produced by the concrete and steel only.

$$\text{Total cost of stem per foot} = \frac{cb'(h-t)}{144} + 3.40s_pA_p + F$$

where F is the constant form cost and

$$A_p = \frac{M_p}{f_sjd} = \frac{M_p}{f_s\left(d - \dfrac{t}{2}\right)}$$

using $jd = d - \dfrac{t}{2}$, approximately.

Let $h = d + e$, where e is the fireproofing clearance.

$$\text{Total cost per foot} = \frac{cb'(d + e - t)}{144} + \frac{3.40 s_p M_p}{f_s\left(d - \frac{t}{2}\right)} + F$$

The maximum and minimum values of this equation are obtained by equating to zero the first differential in respect to the variable depth d and solving for depth d.

$$\frac{d(\text{Cost})}{d(d)} = \frac{cb'}{144} - \frac{3.40 s_p M_p}{f_s} \times \frac{1}{\left(d - \frac{t}{2}\right)^2} + 0 = 0$$

$$\left(d - \frac{t}{2}\right)^2 = \frac{3.40 \times 144 s_p M_p}{cb' f_s} = \frac{rM_p}{f_s b'}$$

where

$$r = \frac{3.40 \times 144 s_p}{c} = \frac{490 s_p}{c} \tag{90}$$

$$d - \frac{t}{2} = \sqrt{\frac{rM_p}{f_s b'}} \quad \text{or} \quad d = \sqrt{\frac{rM_p}{f_s b'}} + \frac{t}{2} \tag{91}$$

The depth d is the cheapest depth for a given stem width b'. A trial of two or three stem widths will indicate the stem depth and width that are nearly a 2 : 1 ratio.

118. Alternate Methods of Determining Section Dimensions. Proper application of the economical depth equations requires that the estimating department have up-to-date costs obtained from other jobs under comparable conditions. Such information is not always available or the designer may not have easy access to the costs. In the drive to get work done designers often prefer to obtain the section dimensions by some other means.

Size by Shear Stress. A favorite means, because it is easy, is to select the section by using the maximum allowable shear stress. It is customary to use the ordinary anchorage maximum, as this practice grew up before special anchorage was defined; also the dimensions given by the maximum shear stress for special anchorage are much too small. Thus in Problem 15 the section dimensions can be obtained, assuming $f'_c = 2000$ lb. per sq. in.

Ordinary Anchorage. The allowable shear stress $v = 0.06f'_c = 120$ lb. per sq. in. The section at the support is similar to that of Figure 59*b* and the value of j is approximately 0.87. If the clear span is taken as 26 ft. and the positive bending moment $= \dfrac{wl^2}{16} = 1{,}400{,}000$ in.-lb., the uniformly distributed load $w = 2760$ lb. per ft. The maximum shear force $V = \dfrac{wl}{2} = 2760 \times 13 = 35{,}900$ lb.

$$b'd = \frac{V}{vj} = \frac{35{,}900}{120 \times 0.87} = 344 \text{ sq. in.}$$

If $b' = 12$ in., $d = 28.7$ in., and $h = 32$ in. The beam stem is 12 in. by 26 in., which is close to the economical size.

On the other hand, had the span been only 22 ft., the following values would obtain: $w = 3870$ lb. per ft., $V = 42{,}600$ lb., and $bd = 407$ sq. in. If $b' = 12$ in., $d = 33.8$ in., and $h = 37$ in. The beam stem is now 12 in. by 31 in., which is not economical.

Special Anchorage. The maximum shear stress now equals $v = 0.12f'_c = 240$ lb. per sq. in. Using the dimensions given above, for a 26-ft. span, the beam stem would be 12 in. by 12 in. The 22-ft. span calls for a beam stem of 12 in. by 14 in. Both sizes are evidently very uneconomical. Tee-beam sizes should not be determined for shear by the use of the special anchorage maximum.

Size Determined by Negative Moment. Some designers prefer to determine the size by the choice of one which requires no compression steel at the supports. For Problem 15 let the negative moment be taken as 2,040,000 in.-lb. The section is similar to that of Figure 59*b* and the compression area is a rectangle. By the rectangular beam theory,

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{2{,}040{,}000}{157 \times 12}} = 32.9 \text{ in.}$$

The total depth $h = 36$ in. and the beam stem is 12 in. by 30 in. This is not economical for the costs used in Problem 15.

The author recommends that the economical depth equation be used, providing that suitable costs can be obtained.

ILLUSTRATIVE PROBLEM 16

119. Economical Depth Equation. Let us check the results of Problem 15 by the equation for economical depth. From that problem we note that the concrete cost $c = 40$ cents per cubic foot and the steel special cost $s_p = as = 2 \times 5 = 10$ cents per pound of positive steel.

From equation 90, $r = \dfrac{490 \times 10}{40} = 123$. This result may also be obtained from Diagram 9 (in the Appendix). From equation 91,

$$d = \sqrt{\frac{123 \times 1{,}400{,}000}{20{,}000 \times 12}} + \frac{6}{2} = 26.8 + 3 = 29.8 \text{ in.}$$

The total depth h of the beam equals

$$h = d + 3 = 29.8 + 3 = 32.8 \text{ in.}$$

This agrees with the results plotted in Figure 60.

The trial sizes given by the three methods are listed below for a beam stem 12 in. wide.

	Beam Stem Depth (in.)
Span = 26 ft.	
Economical depth (r = 123)	27
Shear, ordinary anchorage	26
No compression steel	30
Span = 22 ft.	
Economical depth (r = 123)	27
Shear, ordinary anchorage	31
No compression steel	30

120. Design of Tee Beams. *Order of Procedure.* The efficient order of procedure to employ in designing tee beams concentrates on selecting the proper beam size before selecting any steel areas. The recommended procedure is

1. Determine economical depth and select the desired values of stem width b' and total depth h.
2. Check the section for shear.
3. Check the sections at the supports for fiber stress f_c. To give the allowable value of the concrete stress f_c compression steel is often needed. The concrete stress f_c at the section of maximum positive bending moment seldom needs to be checked.
4. Determine tension steel areas A_s for bending moment requirements.
5. Check tension steel for bond.
6. Determine anchorage requirements.
7. Place tension steel.
8. Design diagonal tension reinforcement.
9. Reinforce flange, if necessary (A.C.I. Art. 705c in the Appendix).

STRAIGHT-LINE WORKING-STRESS THEORY

ILLUSTRATIVE PROBLEM 17

121. Design of Tee Beams. Given the column spacings of Problem 8 (Chapter 5) whose elevation is shown in Figure 36. After a preliminary cost analysis the designer has adopted a concrete beam and slab floor system whose plan view is given in Figure 61. The live load is 130 lb. per sq. ft.

A typical interior panel such as *ADHE* (Fig. 61) is to be designed; a 2000-lb. concrete[1] will be used. The cost of a concrete testing 2000 lb. per sq. in. at the age of 28 days is 45 cents per cubic foot. Forms cost 13 cents per square foot. Steel costs 4 cents per pound. Assume a positive steel to total steel ratio $a = 2.5$.

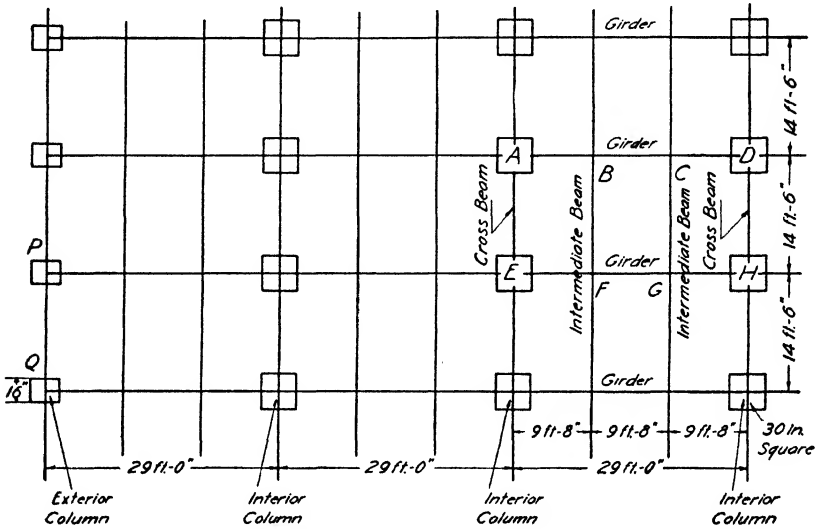


FIG. 61

122. Division of Loads. The floor system, which has column spacings of 29 ft. by 14 ft. 6 in., has been divided by the girder and beams into panels which are 14 ft. 6 in. by 9 ft. 8 in. These are markedly rectangular in shape so the slab will be designed with one-way steel spanning the short direction over the beams.

The loads brought to the cross beams will be carried by them directly to the columns. The intermediate beams carry their loads to the girders, which in turn bring their loads to the columns.

The greater part of the slab loads will be brought by the one-way steel to the beams; but a square foot of slab at some such position as *a* (Fig. 62) close to the girder, in fact part of its flange, is undoubtedly principally supported by the girder. Some other square foot of slab, such as *b*, close to the beam is wholly taken by the beam. The load on an area such as *c* is probably partly carried to the beam and partly to the girder.

We can draw boundary lines such as *An*, *Bn*, *Bo*, *Co*, etc., which separate the areas whose loads are principally carried by the girder or by the beam. The boundary lines of Figure 62 are drawn at 45° as a reasonable division. Such a division of the slab area would assign to the beam an area *FtnBouF*. Shown as a beam of single span supported at the ends, the loading diagram is given in Figure 63, and the shear force and bending moments are given as full lines in Figure 64, where *w* is

[1] See footnote of Article 32.

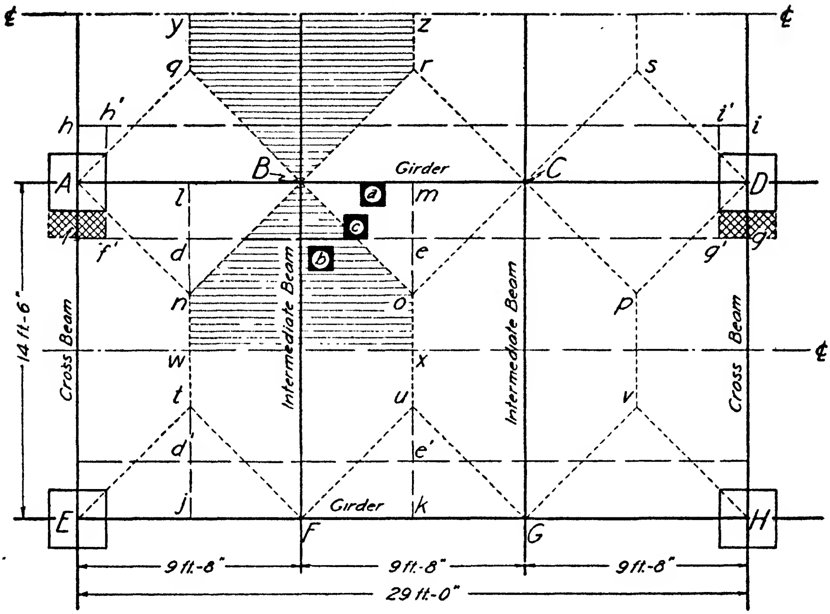

Girder
Intermediate Beam
Intermediate Beam
Cross Beam
Cross Beam
Girder
14 ft.-6"
9 ft.-8"
9 ft.-8"
9 ft.-8"
29 ft.-0"

FIG. 62

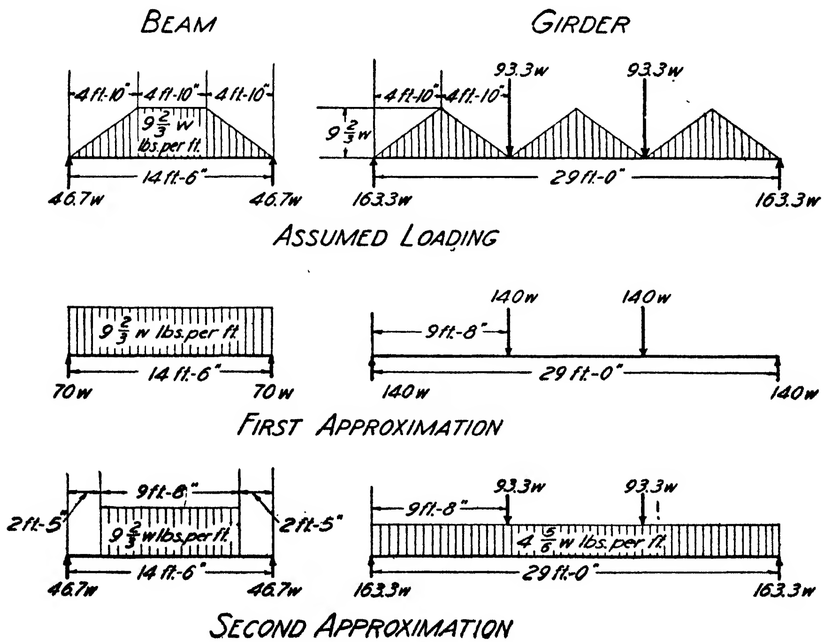

BEAM
GIRDER
4 ft.-10"
4 ft.-10"
4 ft.-10"
$9\frac{2}{3}w$ lbs. per ft.
14 ft.-6"
46.7w
46.7w
93.3w
93.3w
$9\frac{2}{3}w$
29 ft.-0"
163.3w
163.3w
ASSUMED LOADING
$9\frac{2}{3}w$ lbs. per ft.
14 ft.-6"
70w
70w
140w
140w
9 ft.-8"
29 ft.-0"
140w
140w
FIRST APPROXIMATION
2 ft.-5"
9 ft.-8"
2 ft.-5"
$9\frac{2}{3}w$ lbs. per ft.
14 ft.-6"
46.7w
46.7w
93.3w
93.3w
9 ft.-8"
$4\frac{5}{6}w$ lbs. per ft.
29 ft.-0"
163.3w
163.3w
SECOND APPROXIMATION

FIG. 63

the total load per square foot of slab area. The shear force diagram, shown for the half span only, is unchanged if the beam is continuous (Art. 349), and the bending moment diagram, also for the half span, is unchanged in shape, the base line ($M = 0$) being shifted upwards.

The floor area carried directly by the girder AD (Fig. 62) consists of the three diamond-shaped areas $AqBn$, etc. The girder also receives from each intermediate beam a concentrated load of the areas $wnBox$ and $yzrBq$. The result is the loading

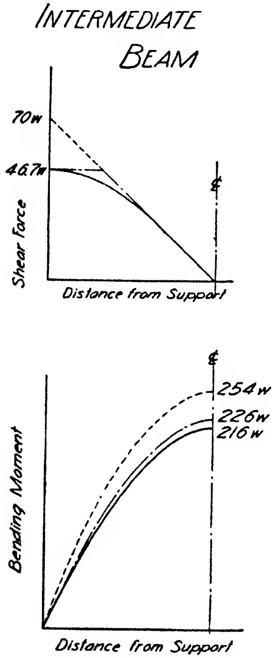


FIG. 64

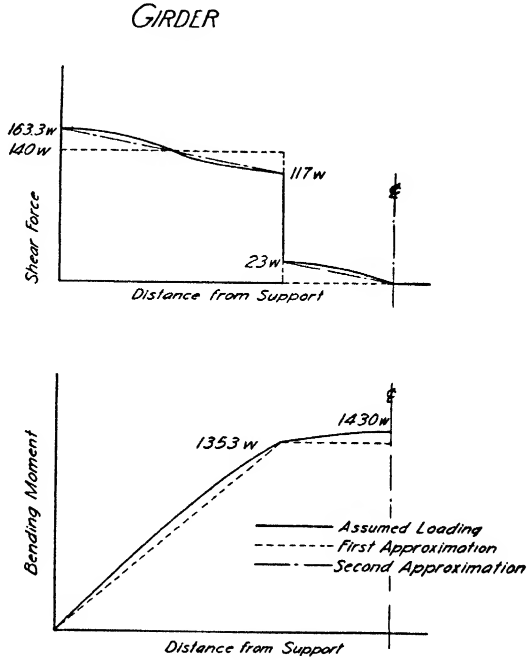


FIG. 65

shown in Figure 63. The shear force and bending moment diagrams are shown in full lines in Figure 65.

Both these loading diagrams give uniformly varying loads over all or part of the span. The solution of shear force and bending moments for uniformly varying loads is more difficult than for uniform loading. The designer is accustomed, therefore, to approximate these assumed loadings in order to deal with uniformly distributed loads or concentrated loads. Such approximations should be on the safe side and should not give shear force or bending moment diagrams diverging greatly from the desired ones.

An approximation often made with one-way steel is that the beam carries all the slab load, or the area $jlmk$ (Fig. 62). In this case the loading for the beam and girder is shown as the first approximation in Figure 63. The shear force and bending moment diagrams are shown as dash lines in Figures 64 and 65. The shear force diagram for the beam coincides with the original assumption in the center portion but is higher near the supports with a maximum of $70w$ instead of $46.7w$. The bending moment values are greater throughout the span with a maximum of $254w$ instead of $216w$. The margin of safety is rather excessive for our problem. If the column spacings were such that the beam had a greater span this margin would be much less. The girder shear force diagram has a maximum of $140w$ which is less than the $163.3w$ given by the loading originally assumed. The bending moment diagram gives less values throughout the span, the maximum being $1353w$ instead of $1430w$. The first assumption gives shear and bending moment diagrams which are on the safe side for the beam and on the unsafe side for the girder. The first approximation would not be safe for the girder design.

Another approximation that might be made is that the areas assigned to the girder are averaged to a strip of constant width. The width of this strip is sometimes reduced by a requirement that this strip shall be wholly within the flange width b of the girder. The remaining slab area is taken by the beams. In Figure 63 the loading for this second approximation is shown. The girder is assumed to take a strip *higf* (Fig. 62), of constant width of 4 ft. 10 in. This area is the same as the sum of the three diamond-shaped areas $AqBn$, etc. The flange of the girder should have a width b of at least 58 in. The beam carries the floor area $d'e'ed$, which is 9 ft. 8 in. wide and 9 ft. 8 in. long. The area *dexw* is brought to the girder by the beam from each side. The shear force and bending moment diagrams for the second approximation are shown in Figures 64 and 65 as dot and dash lines. The shear force for the beam is constant at the maximum of $46.7w$ for 2 ft. 5 in., and then the diagram coincides with the first approximation and later with the original assumption of loading. The bending moment diagram is slightly higher than the original with a maximum value of $226w$ instead of the $216w$ of the original. The shear diagram for the girder has the same maximum value of $163.3w$ as the original assumption. It substitutes a straight line for the reverse curve of the original, but the other boundary values of $117w$ and $23w$ are also identical with the original. The bending moment diagram for this case closely coincides with the original diagram and cannot be plotted separately. The curve for the original loading reverses slightly, just as does more markedly its shear force diagram. The curve for the second assumption is alternately slightly above or below the original values. The girder will be designed for the loading of the second approximation.

The second approximation gives good results. At the same time the designer deals with concentrated loads or uniformly distributed loads. To be sure, the load for the beam does not extend over the whole span, but its bending moment diagram is a close approximation to a parabola and the same relative shifting of the base line ($M = 0$) can be used if the beam is continuous.

The fact that the uniformly distributed load for the beam does not extend over the full span prevents the adoption of the second approximation for the beam design by many designers. They prefer the simplicity of the first assumption though in this case both the maximum shear force and the maximum bending moment are considerably in excess of the values that one wishes to approximate. The present problem is an extreme case because the girder span is so great compared with the beam span. If the column spacings had been 29 ft. in both directions the loading diagram for the beam would be that shown in Figure 66. The corresponding shear force and bending moment diagrams for the beam are plotted on Figure 67 with the assumed loading with full lines and the first approximation with dash lines. The excess of shear force and of bending moment is not great, and the first approximation would be adopted without question.

In our problem the maximum shear force by the first approximation is 50 per cent in excess of the assumed loading. The bending moments at every section are nearly 20 per cent in excess. The saving in time for the designer would be too dearly purchased, so we adopt the second assumption for the beam design in this particular problem.

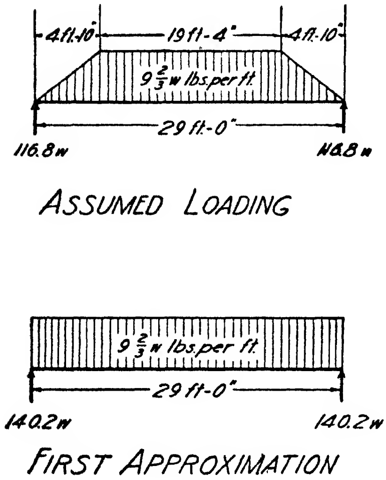


FIG. 66

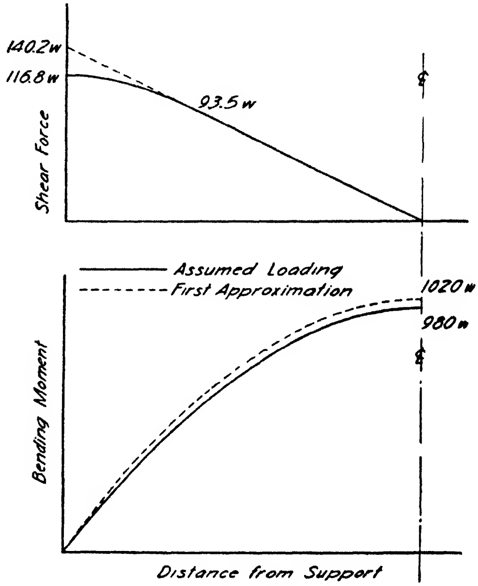


FIG. 67

123. Depth of Slab. The slab steel is one-way and spans 9 ft. 8 in. over the cross and intermediate beams. We shall design a strip 1 ft. wide, assuming a 4-in. thickness.

Live load	=	130 lb. per sq. ft.
Weight of slab	=	50 lb. per sq. ft.
Total load w	=	180 lb. per sq. ft.

The panel *ADHE* (Fig. 61) is an interior panel. The three spans of the slab in this panel are all interior spans. By A.C.I. Article 701 (see Appendix) the maximum positive bending moment equals $\frac{wl^2}{16}$ and the negative moment equals $\frac{wl^2}{12}$. The rectangular slab section will be designed for the negative moment, using the clear span. The designer estimates that the cross and intermediate beams will be 8 in. to 10 in. wide. He uses the smaller value so that the slab may be too large rather than too small, and the design may be on the safe side. The clear span is 9 ft. 8 in. minus 8 in. A.C.I. Article 701 states that the maximum negative moment for slabs with spans not exceeding 10 ft. shall be

$$M_n = \frac{wl^2}{12} = \frac{180 \times (9)^2 \times 12}{12} = 14{,}580 \text{ in.-lb.}$$

$$\text{Depth to steel } d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{14{,}580}{157 \times 12}} = 2.78 \text{ in.}$$

The protective covering is $\frac{3}{4}$ in. for slabs (A.C.I. Art. 507*b*). Assume that the steel is not larger than $\frac{1}{2}$ in. The minimum slab thickness t equals

Minimum $t = 0.75 + 0.25 + 2.78 = 3.78$ in. Use 4 in. slab.

Before we compute the slab steel we shall check the assumption that the beam stem width $b' = 8$ in.

124. Size of Cross and Intermediate Beams. *Loading.* The cross and intermediate beam will be made the same size. The architect approves of this for general appearance, the engineer desires the same depth so that sprinkler pipes, conduits, shafting, etc., can be readily carried down the building, and the contractor has uniform sizes for his slab and beam forms.

There are two intermediate beams and one cross beam in each bay. We shall compute the economical size for the intermediate beams as there are twice as many of them. The economical depth will be determined from the maximum positive bending moment.

We have decided to use the second approximation of Article 122 which uses the slab area *dee'd'* of Figure 62. The intermediate beams are continuous and we shall use the clear span. The designer estimates that the girders are 12 in. to 14 in. wide. He will assume 12 in. in order to use the greater span as a margin of safety. The clear span is 14 ft. 6 in. minus 12 in., or 13 ft. 6 in. Near its center the beam carries $9\frac{2}{3}$ sq. ft. of slab for each foot of length.

Slab load $= 180 \times 9\frac{2}{3} = 1740$ lb. per ft. length. In addition the beam must carry its own stem weight for the entire span. If the beam is 8 in. wide, the maximum stem will be 8 in. by 16 in.

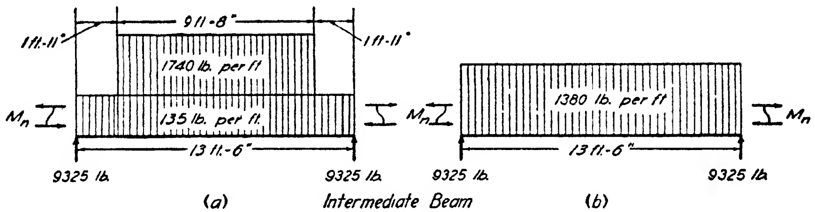


FIG. 68

Stem weight $= \dfrac{8 \times 16}{144} \times 150 = 135$ lb. per ft. length. The loading is shown in Figure 68*a*. Taking the origin at the left support, the shear and bending moment equations for a beam are

From $x = 0$ to $x = 1.92$

$$\text{Shear force} = 9325 - 135x$$

$$\text{Bending moment} = 9325x - \frac{135x^2}{2} + M_n$$

From $x = 1.92$ to $x = 6.75$ (center line)

$$\text{Shear force} = 9325 - 135x - 1740(x - 1.92)$$

$$\text{Bending moment} = 9325x - \frac{135x^2}{2} - \frac{1740}{2}(x - 1.92)^2 + M_n$$

The shear force and bending moment diagrams are shown in Figure 69 as full lines. The bending moment is plotted for a beam freely supported at the ends ($M_n = 0$).

Some designers would now propose a third approximation. This approximation would substitute a uniformly distributed load equal to the total load of Figure 68*a*. The load w equals 1380 lb. per ft.

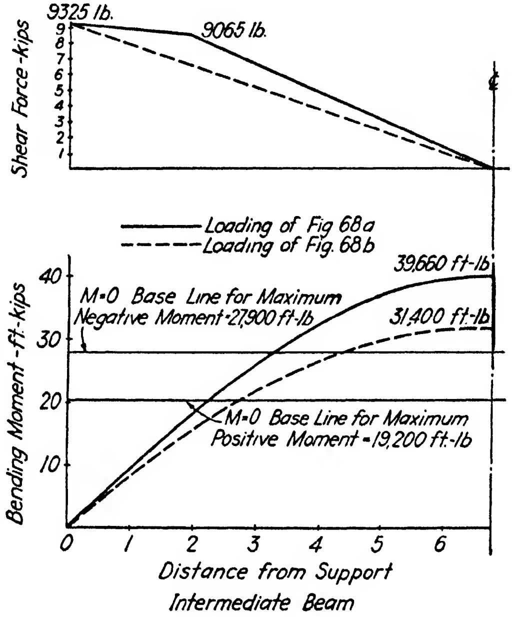


FIG. 69

This loading is shown in Figure 68*b*. The shear force and bending moment diagrams are shown in Figure 69 as dash lines. Both the shear force and bending moment values are too small throughout the span, and this approximation will be rejected. We shall continue to design by the second approximation.

The beams are continuous and we wish to design an interior span. Assume that the beam with the load of Figure 68*a* has the same relative fixity as that given by the A.C.I. moment coefficients for uniformly distributed loads.

The intermediate beams are supported by the girder. The moment coefficients of A.C.I. Article 701 for uniform loads are the basis for reference and give a maximum negative moment M_n of $-\dfrac{wl^2}{11}$ for an interior span. A beam with fixed ends has an end moment of $-\dfrac{wl^2}{12}$. The maximum positive bending moment M_p equals $+\dfrac{wl^2}{16}$, which must accompany a minimum negative moment $M_n = -\dfrac{wl^2}{16}$, in order that the total change may be $\dfrac{wl^2}{8}$.

Paralleling this for the loading of Figure 68*a*, solve for the fixed end moment. Using one of the methods discussed in Chapters 12 and 13 with the knowledge that the slope is zero at the ends for a fixed beam, computations give

$$\text{Fixed-end moment } M_n = -25{,}590 \text{ ft.-lb.}$$

Summarizing for the usual uniformly distributed load,

$$\text{Fixed-end moment} \qquad = -\frac{wl^2}{12}$$

$$\text{Interior span, maximum negative moment } M_n = -\frac{wl^2}{11}$$

$$\text{Interior span, maximum positive moment } M_p = +\frac{wl^2}{16}$$

Using the loading of Figure 68*a*,

$$\text{Fixed-end moment} \qquad = -25{,}590 \text{ ft.-lb.}$$

$$\text{Interior span, } M_n = \tfrac{12}{11} \times 25{,}590 \qquad = -27{,}900 \text{ ft.-lb.}$$

$$\text{Interior span, } M_p = \tfrac{12}{16} \times 25{,}590 \qquad = +19{,}200 \text{ ft.-lb.}$$

In Figure 69 the base line ($M = 0$) for the negative moment is located at the 27,900 ft.-lb. level. The base line ($M = 0$) for the positive moment of 19,200 ft.-lb. is located at 39,660 − 19,200 = 20,460 ft.-lb. from the bottom of the diagram.

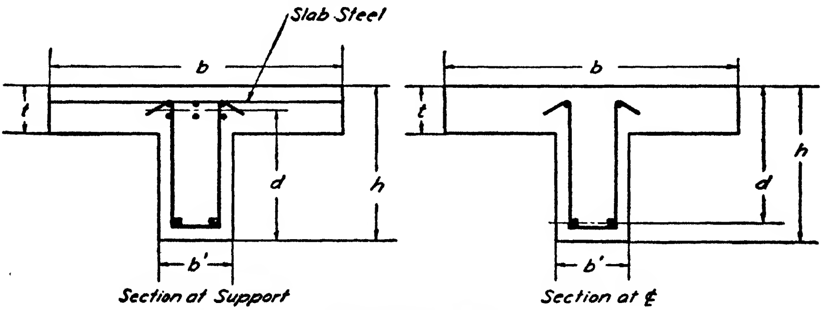
125. Economical Depth. Intermediate Beam. The cost of concrete $c = 45$ cents per cubic foot. The cost of the steel $s = 4$ cents per pound, and the special unit cost $s_p = as = 2.59 \times 4 = 10.36$ cents per pound (Art. 114).

$$\text{Cost constant } r = \frac{490 s_p}{c} = \frac{490 \times 10.36}{45} = 113$$

The maximum positive bending moment $M_p = 19{,}200$ ft.-lb.
By equation 91,

$$\text{Economical depth } d = \sqrt{\frac{113 \times 19{,}200 \times 12}{20{,}000 \times b'}} + \frac{4}{2} = \frac{35.4}{\sqrt{b'}} + 2$$

Let us examine values of stem width b' of 6, 8, and 10 in. Adopt the steel reinforcement of Type II (Art. 60) and assume the positive steel consists of one row of 1-in.



Intermediate Beam

FIG. 70

bars. Use stirrups not larger than $\frac{3}{8}$-in. (Fig. 70*b*). The distance from the bottom of the beam to the center of the steel is

Fireproofing clearance	= 2.00 in.
Stirrup diameter	= 0.38
Half dimension of bar	= 0.50
$h - d$	= 2.88 in.

Stem Width b'	Depth to Tension Steel d	Total Depth h	Depth of Stem $(h - t)$
6	14.5	16.5	
	2.0	2.9	
	16.5	19.4	15
		Use 19	
8	12.53	14.53	
	2.0	2.88	
	14.53	17.41	13
		Use 17	
10	11.22	13.22	
	2.0	2.88	
	13.22	16.10	12
		Use 16	

The depths are economical ones, not minimum, so we may select sizes larger or smaller as we prefer. The 8-in. by 17-in. beam has a stem ratio less than 2 to 1 so we shall tentatively adopt it.

Selection of Size by Shear Stress. As a basis of comparison let us also determine the size of beam by the allowable shear stress method, using ordinary anchorage. The maximum shear force is 9325 lb. Assuming the distance from the top of the beam to the center of gravity of the negative steel to be about 3 in., the minimum size to satisfy shear becomes

$$b'd = \frac{V}{vj} = \frac{9325}{120 \times 0.87} = 89.2 \text{ sq. in.}$$

If $b' = 6$ in., $d = 14.85$ in., $h = 18$ in., and $h - t = 14$ in.

If $b' = 8$ in., $d = 11.13$ in., $h = 15$ in., and $h - t = 11$ in.

By the shear stress method we would use an 8-in. stem with a total depth 2 in. less than that of the economical size.

Selection of Size for No Compression Steel. The maximum negative moment is 27,900 ft.-lb. At the support we have compression at the bottom of the beam (Fig. 59*b*) and a rectangular compression area.

$$d = \sqrt{\frac{M}{Kb'}} = \sqrt{\frac{27{,}900 \times 12}{157 \times b'}} = \frac{46.1}{\sqrt{b'}}$$

If $b' = 6$ in., $d = 18.85$ in., $h = 22$ in., and $h - t = 18$ in.

If $b' = 8$ in., $d = 16.3$ in., $h = 20$ in., and $h - t = 16$ in.

This method would call for an 8-in. stem with a total depth which is 3 in. deeper than that given by the economical size.

126. Shear. Intermediate Beam. Since we have adopted the economical size, we have one larger than that called for by the shear stress method and the shear stress is safe. It will be necessary to use web steel.

127. Size of Girder. Economical Depth. *Maximum Positive Bending Moment.* We have adopted the loading of the second approximation (Fig. 63) for the girder. We are designing an interior span of a continuous beam, whose clear span is 26 ft. 6 in. The loading is shown in Figure 71. This loading assigns to the girder as

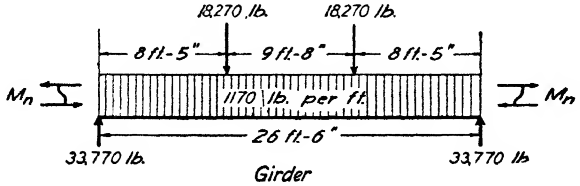


FIG. 71

direct slab load a strip of 4.83 ft. of slab area for each foot length of girder. We wish to limit such direct loads to loads on the girder flange. The girder flange width b (A.C.I. Art. 705*a*) cannot exceed

$$b = \frac{l}{4} = \frac{26.5 \times 12}{4} = 80 \text{ in.}$$

$$b = 16t + b' = 16 \times 4 + 12 = 76 \text{ in.}$$

$$b = \text{spacing} = 14 \text{ ft. } 6 \text{ in.} = 174 \text{ in.}$$

Use flange width $b = 76$ in. The desired direct slab load for a width of 4.83 ft. = 58 in. can be adopted.

The uniformly distributed load for each foot of length of the girder equals 4.83 sq. ft. of slab plus the weight of the girder stem. We have previously assumed a 12-in. stem width and we now assume the maximum depth of 24 in.

$$\text{Slab load} = 180 \times 4.83 = 870 \text{ lb. per ft.}$$

$$\text{Girder stem} = \frac{12 \times 24}{144} \times 150 = 300 \text{ lb. per ft.}$$

$$\text{Total uniformly distributed load } w = 1170 \text{ lb. per ft.}$$

The concentrated load consists of the slab load of $93.3w$ plus the weight of the beam stem for the clear span of the beam.

$$\text{Slab load} = 93.3 \times 180 = 16{,}800 \text{ lb.}$$

$$\text{Beam stem} = 110 \times 13.5 = 1{,}470 \text{ lb.}$$

$$\text{Total concentrated load} = 18{,}270 \text{ lb.}$$

The girder loads of Figure 71 can be handled separately as a uniformly distributed load and two symmetrically placed concentrated loads. It is necessary to determine the maximum positive bending moment for the interior span. This will occur at the center section for both loadings.

The maximum positive bending moment due to the uniformly distributed load $w = 1170$ lb. per ft. will be

$$M_p = \frac{wl^2}{16} = \frac{1170 \times (26.5)^2 \times 12}{16} = 616{,}000 \text{ in.-lb.}$$

The two concentrated loads $W = 18{,}270$ lb. are symmetrically placed but are 5 in. from the "third points," which are the sections at $\frac{l}{3} = 106$ in. from the face of the column. The maximum moment will be somewhat smaller than the values recommended in Table J (Chapter 14) for concentrated loads at the third points, but near enough so that the relative displacement of the base line ($M = 0$) from the fully fixed position may be assumed to be the same.

Figure 72*a* shows two equal loads of W at the third points of the span. If the ends are fixed, the couples M_n at the supports can be found by the methods

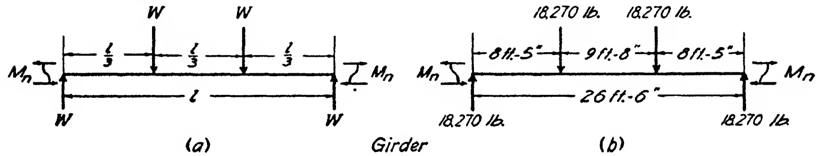


FIG. 72

discussed in Chapter 14. Using the fact that the change of slope is zero for a fixed-end beam, the bending moment at the fixed end equals

$$M_n = -\tfrac{2}{9}Wl$$

In Table J (Chapter 14) it is recommended that a maximum positive bending moment $M_p = \frac{3}{16}Wl$ be used for interior spans of continuous beams. This is accompanied by the minimum negative moment $M_n = \frac{Wl}{3} - \frac{3}{16}Wl = -\frac{7}{48}Wl$, since $M_p + M_n = \frac{Wl}{3}$. This moment is two thirds the value of the fixed-end moment $M_n = -\frac{2}{9}Wl$, for the negative moment of $-\frac{7}{48}Wl$ accompanying the recommended positive moment is 0.66 of the fixed-end moment $-\frac{2}{9}Wl$.

Turning now to the actual loading of Figure 72*b*, the fixed-end moment can be obtained by a similar computation. It equals

$$M_n = -105{,}000 \text{ ft.-lb.}$$

Using the same relative shifting of the base line ($M = 0$), the minimum negative moment $= \frac{2}{3} \times 105{,}000 = -70{,}000$ ft.-lb. The maximum positive moment M_p at the center sections equals

$$\text{Maximum } M_p = 18{,}270 \times 8.42 - 70{,}000 = 84{,}000 \text{ ft.-lb.}$$

The total maximum positive bending moment due to the complete loading of Figure 71 equals

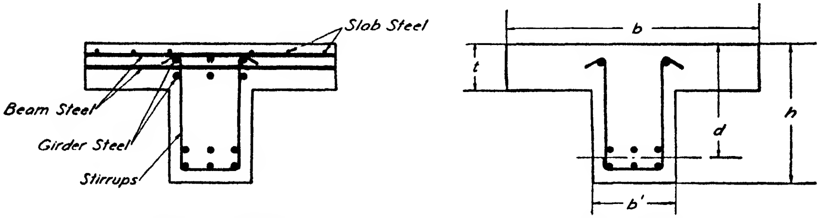
$$M_p = 616{,}000 + 84{,}000 \times 12 = 1{,}624{,}000 \text{ in.-lb.}$$

Assume the value of $a = 2.5$ and the cost constant $r = 109$.[2]

$$\text{Economical depth } d = \sqrt{\frac{109 \times 1{,}624{,}000}{20{,}000b'}} + \frac{4}{2}$$

$$d = \frac{94.1}{\sqrt{b'}} + 2.0$$

We shall examine stem widths b' of 12 in., 14 in., and 16 in. It is customary to use the same type of reinforcement for all beams and girders, but in this problem we shall use for the girder Type III reinforcement with truss bars (Art. 60) in order



Section at Support (a) Girder Section at ℄ (b)

FIG. 73

to illustrate its design. Assume the positive steel to be two rows of bars not larger than $\frac{3}{4}$-in. rounds in size (Fig. 73b). We shall again assume $\frac{3}{8}$-in. rounds for the stirrups. The steel clearance $h - d$ equals

Fireproofing clearance	= 2.00 in.
Stirrup diameter	= 0.38 in.
Distance to center of lower row	= 0.37 in.
Half distance to upper row	= 0.88 in.
$h - d$	= 3.63 in.

Stem Width b'	Depth to Tension Steel d	Total Depth h	Depth of Stem $(h - t)$
12	27.2	29.2	28
	2.0	3.6	
	29.2	32.8	
		Use 32	
14	25.1	27.1	26
	2.0	3.6	
	27.1	30.7	
		Use 30	
16	23.5	25.5	25
	2.0	3.6	
	25.5	29.1	
		Use 29	

[2] This proved to be a generous assumption, as the final design gives a value of the steel ratio $a = 2.05$ (Art. 114). The final design indicates that a smaller section is the economical one.

The section, 14 in. wide by 30 in. deep, gives a stem ratio just under 2 to 1, and this will be tentatively adopted.

Selection of Size by Shear Stress. As a matter of interest, let us determine the 14-in. beam section by the shear stress method. The maximum shear force V equals

$$V = 1170 \times 13\tfrac{1}{4} + 18{,}270 = 33{,}770 \text{ lb.}$$

Assuming ordinary anchorage, the minimum size is

$$b'd = \frac{V}{vj} = \frac{33{,}770}{120 \times 0.87} = 323 \text{ sq. in.}$$

$$\text{If } b' = 14 \text{ in., } d = 23.1,\ h = 27 \text{ in., and } h - t = 23 \text{ in.}$$

This calls for a section 3 in. shallower than the economical.

Selection of Size for No Compression Steel. The maximum negative moment due to the uniformly distributed load equals

$$M_n = \frac{wl^2}{11} = -895{,}000 \text{ in.-lb.}$$

The negative moment due to the two concentrated loads will be taken as their fixed-end moment of 105,000 ft.-lb. The total negative moment at the support is

$$\text{Maximum } M_n = -895{,}000 - 105{,}000 \times 12 = -2{,}155{,}000 \text{ in.-lb.}$$

Since we have a rectangular compression area in the bottom of the stem at the support, the depth to the steel equals

$$d = \sqrt{\frac{M}{Kb'}} = \sqrt{\frac{2{,}155{,}000}{157 \times 14}} = 31.3 \text{ in.}$$

This calls for a total depth of $h = 35$ in. and a stem depth $h - t = 31$ in. Using the section computed for economy, we have a beam 5 in. shallower and we must use compression steel at the support.

128. Shear. Girder. The corrected stem weight equals

$$\text{Stem weight} = \frac{14 \times 26}{144} \times 150 = 380 \text{ lb. per ft.}$$

This is more than the assumed weight of 300 lb. per ft. The shear stress will still figure somewhat under the limit of 120 lb. per sq. in. for ordinary anchorage. Use special anchorage with web steel. The results so far call for ordinary anchorage for the girder and beams; however, in order to illustrate the application of special anchorage requirements, it will be used for girder and beam design. This is no hardship as the increased allowable stresses for bond and diagonal tension are advantageous.

The girder stem size of 14 in. by 26 in. is adopted. The designer now knows very closely the dimensions of the floor system. He should then compute his column stacks for size and steel to verify the assumption of column size before figuring the steel for the floor system. In this problem we will accept the 30-in. column as a satisfactory size for this floor, and start the final design of each member.

SLAB

129. Steel for Slab. The preliminary design of the slab gives us the following data.

Beam stem width = 8 in.

Clear span = 9 ft.

Load w = 180 lb. per sq. ft.

$$\text{Maximum positive bending moment } M_p = \frac{wl^2}{16} = 10{,}900 \text{ in.-lb.}$$

$$\text{Maximum negative bending moment } M_n = \frac{wl^2}{12} = 14{,}600 \text{ in.-lb.}$$

Thickness of slab $t = 4$ in.

The fire-protective covering for the positive steel is 1.5 in. for glacial gravel. Assuming $\frac{1}{2}$-in. bars, the depth $d = 4 - 1.75 = 2.25$ in.

$$\text{Positive steel } A_p = \frac{M}{f_s jd} = \frac{10{,}900}{20{,}000 \times 0.87 \times 2.25} = 0.28 \text{ sq. in.}$$

This area is given by $\frac{1}{2}$-in. rounds spaced $8\frac{1}{2}$ in. on centers. This spacing is too large for a 4-in. slab. Assuming $\frac{3}{8}$-in. rounds with a depth $d = 2.31$ in., the area $A_p = 0.28$ sq. in. Use $\frac{3}{8}$-in. rounds spaced at $4\frac{1}{2}$ in. on centers.

The negative steel area A_n can be computed, assuming $\frac{3}{8}$-in. rounds and a height $d = 4.00 - 0.75 - 0.19 = 3.06$ in.

$$A_n = \frac{14{,}600}{20{,}000 \times 0.87 \times 3.06} = 0.28 \text{ sq. in.}$$

With the steel arrangement of Figure 15 the slab steel arrangement is sketched in Figure 74. Every second bar is a Type A bar and is bent up at one fifth of the clear span (22 in.). The Type B bars are straight and run three spans (29 ft.).

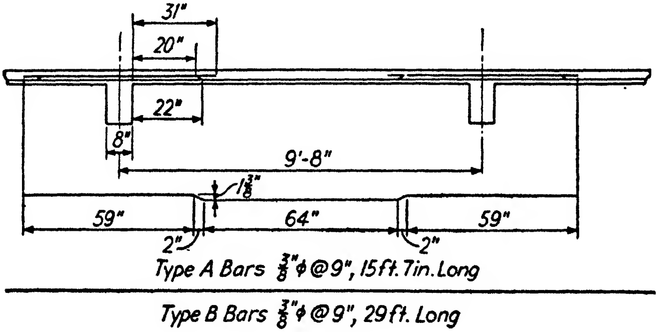


FIG. 74

By this arrangement we meet the requirements that for

Positive bending: (*a*) at least one half the steel beyond 27 in.
(*b*) some positive steel runs into the support.
Negative bending: (*c*) 0.28 sq. in. at supports.
(*d*) at least one half the steel until 11 in. out.
(*e*) some negative steel to run 12 diameters (5 in.) beyond point of inflection (26 in.).

The shrinkage-temperature steel is supplied as a steel ratio $p = 0.002$ (A.C.I. Art. 707).

$$\text{Temperature steel } A_t = 0.002 \times 12 \times 2.31 = 0.06 \text{ sq. in.}$$

Use $\frac{1}{4}$-in. rounds spaced 10 in. on centers. This steel is parallel to the intermediate beams and will run two spans, or 29 ft.

INTERMEDIATE BEAM

130. Design of Intermediate Beam. *Shear Force and Bending Moment.* The design of the intermediate beam will be made before that of the cross beam as its span is longer and the moments and shears will be greater. Note from the previous computations that

$$\text{Slab load} = 1740 \text{ lb. per ft.}$$
$$\text{Corrected stem weight} = \frac{8 \times 13}{144} \times 150 = 110 \text{ lb. per ft.}$$
$$\text{Clear span} = 13 \text{ ft. } 4 \text{ in.}$$
$$\text{Stem width } b' = 8 \text{ in.}$$
$$\text{Total depth } h = 17 \text{ in.}$$

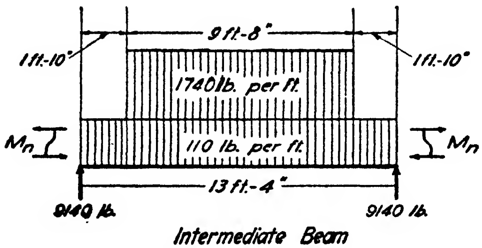


FIG. 75

The corrected loading diagram is given in Figure 75. The change in load and span is so slight that the designer would continue with the previous loadings of Figure 68 which are on the safe side. We shall use, however, the corrected loading.

The fixed-end moment can be computed as before, giving

$$M_n = -24{,}800 \text{ ft.-lb.}$$

By the same procedure as used in Article 124 the maximum moments are

$$\text{Interior span, negative } M_n = \tfrac{12}{11} \times 24{,}800 = -27{,}100 \text{ ft.-lb.}$$

$$\text{Interior span, positive } M_p = \tfrac{12}{16} \times 24{,}800 = +18{,}600 \text{ ft.-lb.}$$

The shear force diagram is plotted in Figure 76, and the two bending moment diagrams in Figure 77.

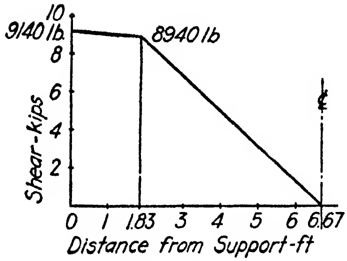


FIG. 76

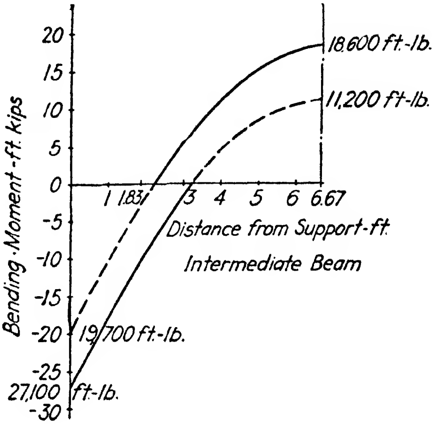


FIG. 77

131. Positive Tension Steel. Intermediate Beam. We are using the Type II system of reinforcement for the beams (Art. 60) and special anchorage. The positive steel consists of straight bars. Assume a moment arm ratio $j = 0.92$, and a depth to steel $d = 17 - 2.88 = 14.12$ in. (Fig. 70*b*).

$$\text{Steel area } A_p = \frac{M}{f_s jd} = \frac{18{,}600 \times 12}{20{,}000 \times 0.92 \times 14.12} = 0.86 \text{ sq. in.}$$

Bond is checked at the point of inflection. From Figure 77 the point of inflection is 2.2 ft. from the face of the support. At this section the shear force $V = 8250$ lb. For special anchorage the allowable bond stress $u = 0.075f'_c = 150$ lb. per sq. in. The number of bars needed at 2.2 ft. for bond is

$$\Sigma = \frac{V}{uojd} = \frac{8250}{150 \times o \times 0.92 \times d} = \frac{60}{od} \text{ bars}$$

The positive steel should be a few large-sized bars in one row. It is important that they should be in one row if they are employed as compression steel at the supports. Tabulating possibilities,

Number	Size in.	Steel Depth d	Actual Area A_s	Computed Area A_s	Number of Bars for Bond	In One Row
2	$\frac{3}{4}$ rd.	14.25	0.88	0.86	1.8	Yes
3	$\frac{5}{8}$ rd.	14.31	0.92	0.85	2.2	No

The three $\frac{5}{8}$-in. round cannot be placed in one row as assumed in the computations. Since the two $\frac{3}{4}$-in. bars have less area, they would be adopted anyway.

132. Negative Tension Steel. Intermediate Beam. Some of the negative tension steel is to be bent down and used as diagonal tension reinforcement. It should be bent as near to the support as possible in order to eliminate stirrups in a region where they are closely spaced. The steel will be designed first for its use as tension steel and then used as diagonal tension steel wherever bending conditions permit.

Assume that a moment arm ratio $j = 0.87$ is conservative, and that the height to the steel $d = 17 - 3.13 = 13.87$ in., using 1.5 in. for clearance (Fig. 70*a*).

$$\text{Steel area } A_n = \frac{M}{f_s jd} = \frac{27{,}100 \times 12}{20{,}000 \times 0.87 \times 13.87} = 1.35 \text{ sq. in.}$$

The bond stress is a maximum at the support. All bars are present at this section, and the bond stress will be computed instead of the necessary number of bars.

$$\text{Bond stress } u = \frac{V}{\Sigma ojd} = \frac{9140}{\Sigma o \times 0.87 \times d} = \frac{10{,}500}{\Sigma od}$$

The negative steel should be four or more bars, so that there may be enough to bend down as diagonal tension steel. Tabulating possibilities,

Number	Size in.	Steel Height d	Actual Area A_s	Computed Area A_s	Bond Stress u	Number of Rows
4	$\frac{3}{4}$ rd.	13.87	1.77	1.35	80	2
6	$\frac{1}{2}$ sq.	14.12	1.50	1.33	62	2

Adopt the six $\frac{1}{2}$-in. square bars. They will be located as shown in Figure 78. When the two outer bars in the lower row are bent down they will be inclined toward

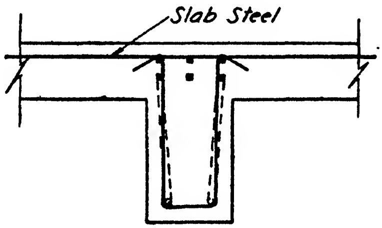


FIG. 78

the section center line (shown by dash lines) to preserve the fireproofing clearance on the beam stem.

The point of inflection for the negative bending moment occurs at 3.2 ft. from the girder face (Fig. 77). The shear force at this section equals $V = 6410$ lb. (Fig. 76). The number of bars required at this section for bond is

$$\Sigma = \frac{V}{uojd} = \frac{6410}{150 \times 2 \times 0.87 \times 14.12} = 1.8 \text{ bars}$$

133. Fiber Stresses in Concrete. Intermediate Beam. *Negative Bending.* The maximum negative bending moment is 27,100 ft.-lb. = 325,000 in.-lb. Without the aid of compression steel at the support the rectangular beam can care for a bending moment $M_c = Kbd^2 = 157 \times 8 \times (14.12)^2 = 251{,}000$ in.-lb. Therefore, compression steel is needed.

Maximum moment	M	$= 325{,}000$ in.-lb.
Concrete fully stressed carries	M_c	$= 251{,}000$ in.-lb.
Moment taken by compression steel	M'_s	$= 74{,}000$ in.-lb.

The stress in the tension steel $f_s = 17{,}500$ lb. per sq. in. The neutral axis ratio k equals

$$k = \frac{1}{1 + \frac{f_s}{nf_c}} = \frac{1}{1 + \frac{17{,}500}{15 \times 900}} = 0.435$$

The neutral axis lies at $kd = 0.435 \times 14.12 = 6.14$ in. above the bottom of the beam. The compression steel will be supplied from two $\frac{3}{4}$-in. bars used as positive steel. These bars are 2.75 in. from the bottom. The stress f'_s in the compression steel equals

$$f'_s = nf_c \frac{kd - d'}{kd} = 15 \times 900 \times \frac{(6.14 - 2.75)}{6.14} = 7450 \text{ lb. per sq. in.}$$

By A.C.I. Article 706 the compression steel area A'_s is figured for twice this stress

$$A'_s = \frac{M'_s}{2f'_s(d - d')} = \frac{74{,}000}{2 \times 7450 \times (14.12 - 2.75)} = 0.44 \text{ sq. in.}$$

In order to preserve the symmetry of steel placement use the two $\frac{3}{4}$-in. bars. This gives a considerable excess of area but, if one checks the actual stress using twice the area of these bars, it is found to be reduced to $f'_s = 5800$ lb. per sq. in. (see Problem 11, Chapter 6). As positive bending steel these bars should run into the support and terminate in a hook (A.C.I. Art. 903); as compression steel with a stress of 2×5800 lb. per sq. in. these bars must run into the support a distance

$$l = \frac{(2f'_s)}{4u} a = \frac{11{,}600}{4 \times 100} \times \frac{3}{4} = 21.7 \text{ in.}$$

The standard hook will be used with a 180° bend and a radius of 4 in. (A.C.I. Art. 906). Entrance into the support, plus the hook, plus the extension of four diameters should give an anchorage of at least 22 in.

Positive Bending. The concrete area for maximum positive bending is either a wide rectangle in the flange or a tee beam if the neutral axis is in the stem. The designer knows that the fiber stress will be small, and in commercial computations he will not check it.

However, if we desire to ascertain the maximum fiber stresses, we investigate by first solving for the flange width b.

$$\text{Maximum } b = \frac{\text{span}}{4} = \frac{13.33 \times 12}{4} = 40 \text{ in.}$$

$$\text{Maximum } b = 16t + b' = 16 \times 4 + 8 = 72 \text{ in.}$$

$$\text{Maximum } b = \text{spacing} = 9 \text{ ft. } 8 \text{ in.} = 116 \text{ in.}$$

Use $b = 40$ in. and $d = 14.25$ in. Let us assume that the neutral axis is in the flange and that the compression area is rectangular.

$$\text{Steel ratio } p = \frac{A_s}{bd} = \frac{0.88}{40 \times 14.25} = 0.00155$$

From Diagram 1 in the Appendix, the intersection of $p = 0.00155$ and $n = 15$ gives values of the neutral axis ratio $k = 0.20$ and the moment arm ratio $j = 0.935$. The neutral axis is a distance $kd = 0.20 \times 14.25 = 2.85$ in. from the top. This is within the slab, which is 4 in. deep, and the compression area is rectangular.

From Diagram 2 in the Appendix, using values of

$$K = \frac{M}{bd^2} = \frac{223{,}000}{40 \times (14.25)^2} = 28 \qquad \text{and} \qquad p = 0.00155$$

Actual $f_c = 300$ lb. per sq. in. and actual $f_s = 19{,}200$ lb. per sq. in.

The assumption of $j = 0.92$ is safe, but we shall now use the correct $j = 0.935$.

134. Placing Steel. Intermediate Beam. *Positive Steel.* In placing the straight positive bars we must meet the following requirements.

1. Two $\frac{3}{4}$-in. round bars at center line for bending moment.
2. Two bars at 2.2 ft. = 27 in. from support for bond.
3. Two bars running 22 in. into the supporting girder for anchorage of compression steel.

Negative Steel. The following requirements must be met by the designer as he places the negative steel.

1. Six $\frac{1}{2}$-in. square bars at the face of the support for bending moment.
2. Bond requirements at the support are satisfied by these six bars.
3. Two bars at the point of inflection for bond or anchorage (A.C.I. Art. 903). The point of inflection is 3.20 ft. = 39 in. from the face of the support (Fig. 77).

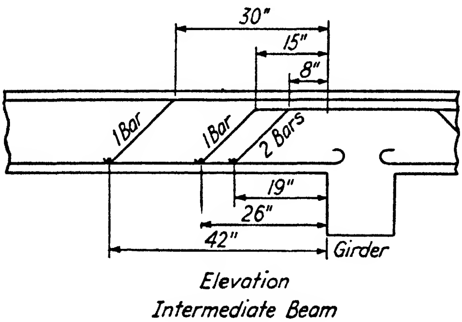


FIG. 79

Extend the two outside bars in the upper row to 3 in. beyond the center line. The stirrups can be wired to these bars (Fig. 79). We wish to bend down the other four bars as soon as possible to act as diagonal tension steel, keeping the steel arrangement symmetrical about the vertical axis of the cross section. Bend down the two outside bars in the lower row first, then the center bar, and the center bar in the upper row last of all.

2 bent: $M_4 = 20{,}000 \times 1.00 \times 0.87 \times 14.50 = 252{,}000$ in.-lb.

3 bent: $M_3 = 20{,}000 \times 0.75 \times 0.87 \times 14.87 = 194{,}000$ in.-lb.

4 bent: $M_2 = 20{,}000 \times 0.50 \times 0.87 \times 14.87 = 129{,}000$ in.-lb.

If these moments are changed to foot-pound units, the bends can be made by use of Figure 77.

$M_4 = 21{,}000$ ft.-lb. Bend two bars at 0.65 ft. = 8 in.

$M_3 = 16{,}150$ ft.-lb. Bend one bar at 1.2 ft. = 15 in.

$M_2 = 10{,}750$ ft.-lb. Bend one bar at 1.8 ft. = 22 in.

Since the number of bars is reduced, the points of bending should be checked for bond. For the successive bends the maximum shear force is

2 bent: $V_4 = u\Sigma ojd = 150 \times 4 \times 2 \times 0.87 \times 14.50 = 15{,}150$ lb.

3 bent: $V_3 = 150 \times 3 \times 2 \times 0.87 \times 14.87 = 11{,}650$ lb.

4 bent: $V_2 = 150 \times 2 \times 2 \times 0.87 \times 14.87 = 7{,}760$ lb.

The maximum shear in this span equals 9140 lb., so bond does not affect the location of the first two bends. The shear force reduces to 7760 lb. at a section 2.47 ft. 30 in. out. This dictates the position of the third bend.

These three bending points are minimum values. It is not permissible to bend these bars nearer the support but, if desired for diagonal tension, the distances may be increased. The bars will be bent down at 45° and hooked about a cross bar wired to the positive steel (Fig. 79). The bars in the lower row drop about $10\frac{1}{2}$ in. and hence extend horizontally the same distance; the bar from the top row drops 12 in. The top and bottom points of bending are listed in Figure 79. The diagonal tension analysis may result in bends farther out from the supports.

135. Diagonal Tension. Intermediate Beam. Diagonal tension reinforcement will be supplied by bent bars and stirrups. The corrected maximum shear stress at the support equals

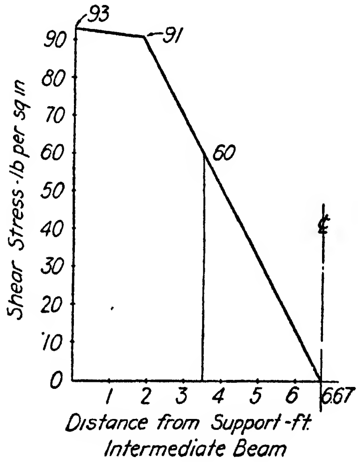


FIG. 80

$$v = \frac{V}{bjd} = \frac{9140}{8 \times 0.87 \times 14.12} = 93 \text{ lb. per sq. in.}$$

This value is less than $v = 0.06f'_c = 120$ lb. per sq. in. The shear stress diagram is shown in Figure 80.

Maximum Spacing.

$$\text{Maximum spacing} = \frac{d}{2} = \frac{1}{2} \times 14.12 = 7.06 \text{ in. (stirrups)}$$

A bar bent at 45° will meet a 45° crack at "mid-depth" if its maximum spacing equals $d = 14.12$ in.

The diagonal tension steel must be designed to cover a variation of negative bending moment from a maximum negative moment of 27,100 ft.-lb. to 19,700 ft.-lb. as the live load is shifted about on the floor. When the negative moment equals 27,100 ft.-lb. the diagonal cracks will tend to form as in Figure 24. When the moment equals 19,700 ft.-lb. the crack diagram approaches more nearly that of Figure 22. For maximum negative bending the tension cracks tend to appear on the top of the beam and the bent bars cross them close to the support, while the slab tends to prevent crack formation. When the tension cracks tend to appear on the bottom of the beam, the bent bars are farther from the support as they cross possible crack lines and there is no aid from the slab. The diagonal tension steel will be designed for the case of tension cracks at the bottom of the beam due to maximum positive bending.

Let us assume that the first crack starts at the bottom of the section at the support (Fig. 22). The first pair of bent bars are 19 in. from the support. This is too far away as it exceeds their maximum spacing of 14 in. Stirrups must be used near the support.

Stirrups. We have assumed a two-rod $\frac{3}{8}$-in. round stirrup. The minimum spacing is computed by using equation 32 (Art. 62):

$$93 = 60 + \frac{2 \times 0.11 \times 20{,}000}{8 \times s \times 1} = 60 + \frac{550}{s}$$

$$\text{Minimum } s = \frac{550}{33} = 16.6 \text{ in.}$$

This exceeds the maximum spacing of 7 in., and the $\frac{3}{8}$-in. bar is too large.

Assuming a two-rod $\frac{1}{4}$-in. round stirrup we compute

$$\text{Minimum } s = \frac{250}{33} = 7.6 \text{ in.}$$

This is still rather large and is more than the maximum, but there is no smaller commercial size. Since this is a smaller size than previously assumed, all clearances are safe. Wherever stirrups are used they must be limited in spacing to 7 in.

Bent Bars. Let us endeavor to use the bent bars at their maximum spacing of 14 in. One $\frac{1}{2}$-in. square bar bent at 45° can be used at a spacing of 14 in. when the shear stress is less than

$$v = 60 + \frac{1 \times 0.25 \times 20{,}000}{8 \times 14 \times 0.707} = 60 + 63 = 123 \text{ lb. per sq. in.}$$

The maximum shear stress is only 93 lb. per sq. in., so the bent bars may be used at their maximum spacing in any part of the span.

Placing. The first pair of bent bars cannot reach the bottom of the beam before 19 in. from the support. They can prevent the appearance of diagonal cracks up to the section $19 - 14 = 5$ in. from the support. Therefore, use one strirup at 7-in. spacing, and then move out the first pair of bent bars to $7 + 14 = 21$ in. The single bar from the bottom row can be placed at $21 + 14 = 35$ in., and the center top bar at $35 + 14 = 49$ in. out. The shear stress $v_c = 60$ lb. per sq. in. occurs at 42 in. (Fig. 80). Beyond this section no diagonal steel is needed. The designer may add a few widely spaced stirrups near the center of the span, but the computations do not require their presence. The complete steel arrangement for this span is shown in Figure 81.

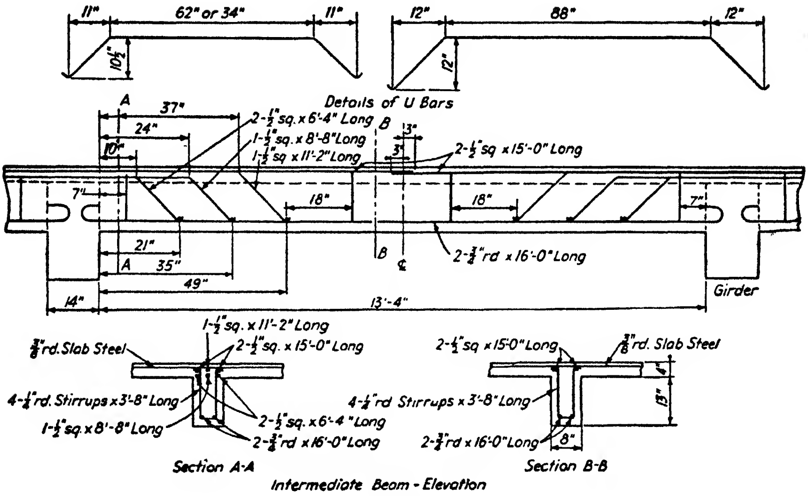


FIG. 81

Note that wherever compression steel is used stirrups must be spaced closer than

16 bar diameters = 12 in.

48 tie diameters = 12 in.

The bending moment reduces to M_c = 251,000 in.-lb. about 8 in. out. The stirrup spacing of 7 in. is safe.

CROSS BEAM

136. Design of Steel. Cross Beam. The cross beam will be made the same size as the intermediate beams. Since the girder load is figured for its clear span only, the slab area *higf* (Fig. 62) carried directly by the girder is reduced to *h'i'g'f'*, and the slab areas shown heavily shaded near the columns *A* and *D* must be carried by the cross beams. The loading diagram is given in Figure 82*a*; the clear span of 12 ft. is used. The shear force diagram is shown as the dash line *AB* and full line

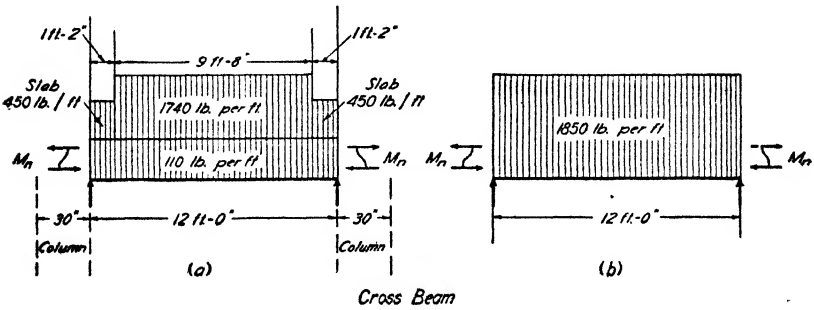


FIG. 82

BC in Figure 83. The bending moment diagram for a simple beam ($M_n = 0$) is shown in dash lines with a maximum of 32,500 ft.-lb.

The reduction of load near the support is not great. If we approximate this loading by assuming that the uniformly distributed load in the center part $w = 1740 + 110 = 1850$ lb. per ft. acts the full span, we get the loading diagram of Figure 82*b*. The shear force diagram (Fig. 83) is the full line DBC. The bending moment diagram for a simple beam ($M_n = 0$) is shown as a full line with a maximum of 33,300

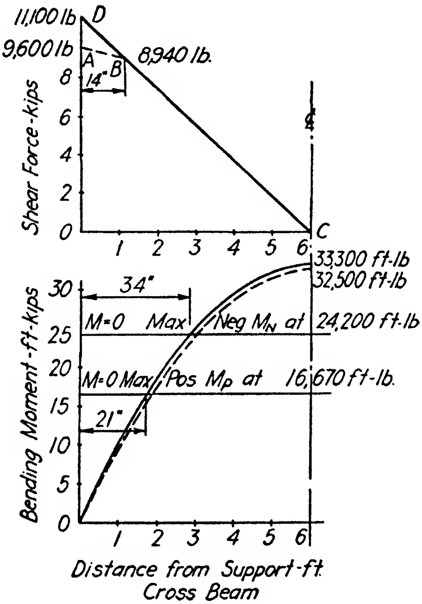


FIG. 83

ft.-lb. Both the shear force and bending moment diagrams are good approximations of those first adopted, and we shall design the cross beam by using the loading of Figure 82*b*.

The design is made for an interior span of a beam loaded with a uniformly distributed load $w = 1850$ lb. per ft. The maximum positive bending moment $M_p = \frac{wl^2}{16}$ and the maximum negative bending moment $M_n = \frac{wl^2}{11}$.

$$M_p = \frac{wl^2}{16} = \frac{1850 \times (12)^2}{16} = 16{,}670 \text{ ft.-lb.} = 200{,}000 \text{ in lb.}$$

$$M_n = \frac{wl^2}{11} = 24{,}200 \text{ ft.-lb.} = 291{,}000 \text{ in.-lb.}$$

These moments are about 90 per cent of those used for the intermediate beam. The base lines ($M = 0$) are drawn on Figure 83 at $M_n = 33{,}300 - 16{,}670 = 16{,}670$ ft.-lb. for the maximum positive moment (minimum negative), and at 24,200 ft.-lb. for the maximum negative moment.

137. Tension Steel. Cross Beam. *Positive Bending.* We are again using the Type II system of reinforcement (Art. 60). Assume $j = 0.92$, $\frac{7}{8}$-in. round bars, and $\frac{1}{4}$-in. stirrups.

$$A_p = \frac{200{,}000}{20{,}000 \times 0.92 \times 14.31} = 0.76 \text{ sq. in.}$$

Bond will be checked at the point of inflection which is 1.75 ft. = 21 in. from the face of the column (Fig. 83), where the shear force $V = 7860$ lb.

$$\Sigma = \frac{V}{uojd} = \frac{7860}{150 \times o \times 0.92d} = \frac{57.0}{od}$$

Number	Size in.	Steel Depth d	Actual Area A_s	Computed Area A_s	Number of Bars for Bond	Number of Rows
Use 2	$\frac{3}{4}$ rd.	14.37	0.88	0.76	1.7	1
3	$\frac{5}{8}$ rd.	13.90	0.92	0.79	2.1	2

Three bars cannot be placed advantageously in two rows. Use $\frac{3}{4}$-in rounds.

Negative Bending. This steel will be in the form of inverted U bars. Assume two rows of $\frac{5}{8}$-in. bars, $j = 0.87$.

$$A_n = \frac{291{,}000}{20{,}000 \times 0.87 \times 14.13} = 1.19 \text{ sq. in.}$$

$$u = \frac{V}{\Sigma ojd} = \frac{11{,}100}{\Sigma o \times 0.87 \times d} = \frac{12{,}750}{\Sigma od}$$

Number	Size in.	Steel Depth d	Actual Area A_s	Computed Area A_s	Number of Bars for Bond	Number of Rows
4	$\frac{5}{8}$ rd.	14.13	1.23	1.19	114	2
Use 5	$\frac{1}{2}$ sq.	14.40	1.25	1.16	88	2

The point of inflection for maximum negative bending is at a section 2.86 ft. = 34 in. from the column face (Fig. 83), where the shear force $V = 5850$ lb.

$$\Sigma = \frac{V}{uojd} = \frac{5850}{150 \times 2.00 \times 0.87 \times 15.00} = 1.5 \text{ bars}$$

138. Fiber Stress in Concrete. Cross Beam. *Negative Bending.* The maximum negative bending moment $M_n = 291{,}000$ in.-lb. Without compression steel the rectangular beam at the support can care for a moment $M_c = Kbd^2 = 157 \times 8 \times (14.40)^2 = 261{,}000$ in.-lb. The necessary compression steel $A'_s = 0.19$ sq. in. is found by the procedure used for the intermediate beam in Article 133. Employing the two $\frac{3}{4}$-in. positive bars that run into the support, the compression steel stress $2f'_s = 6600$ lb. per sq. in. and its anchorage must be at least 13 in.

139. Placing Steel. Cross Beam. *Positive Steel.* The following requrements govern the placing of the straight positive bars.

1. Two $\frac{3}{4}$-in. round bars at the center line for bending moment.
2. Two bars at 21 in. from support for bond.
3. Two bars at the support for anchorage and compression steel.

Negative Steel. The following requirements govern the placing of the negative tension steel

1. Five $\frac{1}{2}$-in. square bars at the face of the column for bending moment.
2. These five bars satisfy bond requirements at the support.
3. Two bars at the point of inflection for bond. The point of inflection is 2.86 ft. = 34 in. from the column face.

The two outside $\frac{1}{2}$-in. bars in the upper row will be carried to a section 3 in. beyond the center line. They support the stirrups, and fulfill the anchorage requirement by substituting an exceptional length for the standard hook. The other three bars can be bent down before they reach the point of inflection. Bend the pair in the lower row first and then the middle bar of the upper row.

If two bars are bent down, the remaining three can carry a bending moment:

$$\text{2 bent:}\quad M_3 = 20{,}000 \times 0.75 \times 0.87 \times 15.00 = 195{,}500 \text{ in.-lb.}$$

$$\text{3 bent:}\quad M_2 = 130{,}300 \text{ in.-lb.}$$

If these moments are changed to foot-pound units, the bends can be made by using Figure 83.

$M_3 = 16{,}300$ ft.-lb. Bend two bars at 0.77 ft. = 10 in.

$M_2 = 10{,}850$ ft.-lb. Bend one bar at 1.36 ft. = 17 in.

These bends have been determined by bending moment considerations. The allowable stress in bond should not be exceeded and it is essential to insure that there are bar perimeters enough at all sections until the point of inflection is reached. For successive bends the maximum shear force is:

$$\text{2 bent:}\quad V_3 = 150 \times 3 \times 2.00 \times 0.87 \times 15.00 = 11{,}730 \text{ lb.}$$

$$\text{3 bent:}\quad V_2 = 7{,}830 \text{ lb.}$$

The first bend can occur at any section as the maximum shear force in this span is 11,100 lb. The second bend can occur at a section 1.76 ft. = 22 in. out. The two bars in the lower row can be bent at 10 in. and the middle bar in the top row at 22 in. from the column face.

140. Diagonal Tension. Cross Beam. *Figure 84.* Diagonal tension reinforcement will be supplied by a combination of bent bars and stirrups.

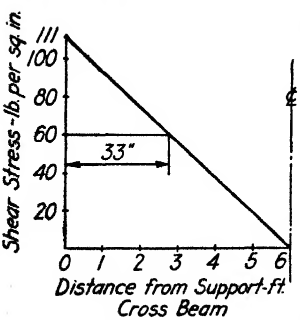


FIG. 84

$$\text{Maximum } v = \frac{11{,}100}{8 \times 0.87 \times 14.40} = 111 \text{ lb. per sq. in.}$$

$$\text{Maximum stirrup spacing} = \frac{d}{2} = \frac{14.40}{2} = 7.2 \text{ in.}$$

Maximum bent bar spacing = 14.4 in.

The first pair of bent bars can reach the bottom row at 10 + 11 = 21 in. from the column face, which exceeds the maximum spacing of 14 in. Use two-rod $\frac{1}{4}$-in. round stirrups.

Minimum stirrup spacing:

$$v = 111 = 60 + \frac{2 \times 0.05 \times 20{,}000}{8 \times s \times 1}$$

$$\text{Minimum } s = \frac{250}{51} = 4.9 \text{ in.}$$

STIRRUP SPACING DATA (Fig. 84)

Spacing s in.	Shear Stress v lb. per sq. in.	Distance from Support x in.
4	...	..
5	110	1
6	102	6
7	96	10
	60	33

A single ½-in. square bar bent down at 45° can resist diagonal tension stresses safely at its maximum spacing of 14 in., if the shear stress is less than

$$v = 60 + \frac{1 \times 0.25 \times 20{,}000}{8 \times 14 \times 0.707} = 60 + 63 = 123 \text{ lb. per sq. in.}$$

The bent bars can be used anywhere in the span at their maximum spacing.

The first pair of bent bars cannot reach the bottom of the beam until 21 in. out. This pair prevents crack formation for 14 in., or to the section 7 in. out. Therefore, use stirrup spacings of 4 and 5 in. Then use the pair of bent bars for 14 in.

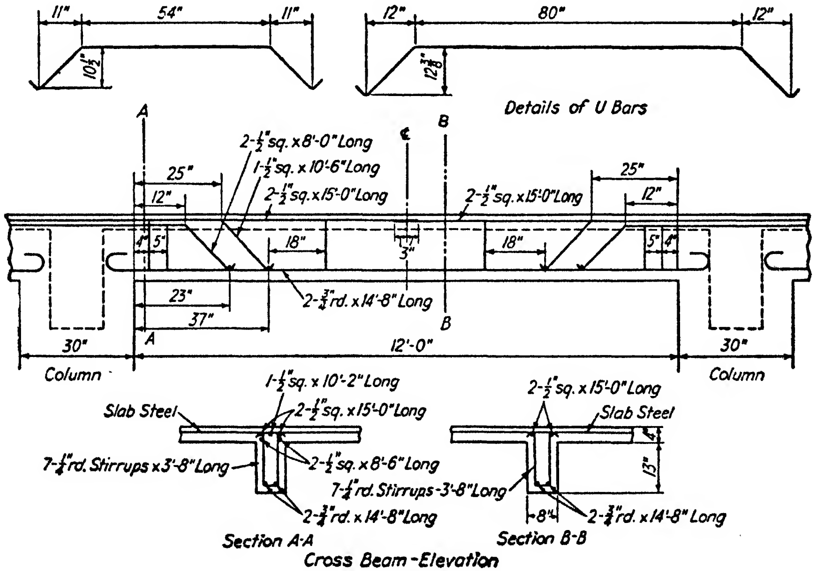


FIG. 85

to a section 23 in. out. The single bent bar from the top row can reach the bottom at $22 + 12 = 34$ in.; however, when spaced at the maximum spacing of 14 in., it reaches the bottom at $23 + 14 = 37$ in. out. The shear stress reduces to $v = 60$ lb. per sq. in. at the section 33 in. out, so no more diagonal tension steel is required. Figure 85 shows the complete steel for this span.

GIRDER

141. Design of Girder. The preliminary design of the girder gives us the following information (Art. 127):

Width of stem	$b' = 14$ in.
Depth of stem	$h - t = 26$ in.
Corrected uniformly distributed load	= 1250 lb. per ft.
Concentrated load	= 18,270 lb.

The corrected loading diagram is shown in Figure 86.

Maximum shear force at support $V = 34{,}850$ lb.

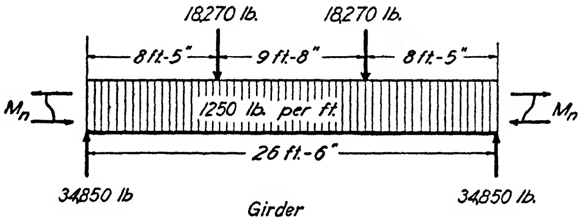


FIG. 86

With an origin at the left support the shear force and bending moment equations are

From $x = 0$ to $x = 8.42$ ft.

$$\text{Shear force} \qquad V = 34{,}850 - 1250x$$

$$\text{Bending moment } M = 34{,}850x - \frac{1250x^2}{2} + M_n$$

The shear force and bending moment diagrams are shown in Figures 87 and 88. Corrected maximum positive bending moment due to uniformly distributed load:

$$M_p = \frac{wl^2}{16} = 660{,}000 \text{ in.-lb.}$$

Maximum positive bending moment due to concentrated loads (Art. 127):

$$M_p = 84{,}000 \times 12 = 1{,}008{,}000 \text{ in.-lb.}$$

Total maximum positive bending moment at center line:

$$M_p = 1{,}008{,}000 + 660{,}000 = 1{,}668{,}000 \text{ in.-lb.}$$

For maximum negative bending the end moment M_n (Fig. 86) equals $\frac{wl^2}{11}$ for the distributed load plus 105,000 ft.-lb. for the two concentrated loads (Art. 127). The

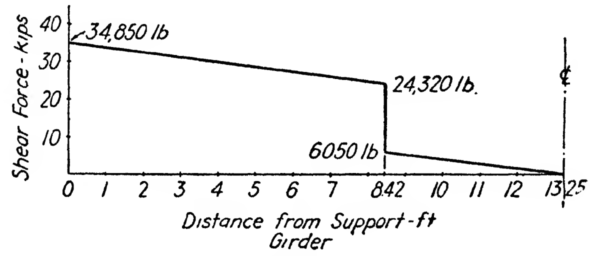


FIG. 87

value 105,000 ft.-lb. is the fixed-end moment for two concentrated loads placed as shown. The total value of the end moment M_n for maximum negative bending is

$$M_n = -\frac{1250 \times (26.5)^2}{11} - 105{,}000 = -185{,}000 \text{ ft.-lb.}$$

$$M_n = -2{,}220{,}000 \text{ in.-lb.}$$

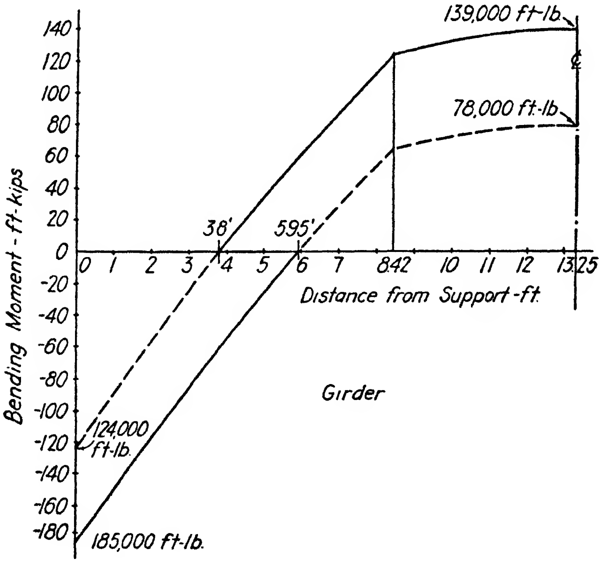


FIG. 88

142. Tension Steel. Girder. *Positive Steel.* The Type III system of reinforcement has been adopted for the girder (Art. 60). The positive steel should be four, six, or eight bars, so that half of it can be bent up as truss bars. Assume a moment arm ratio $j = 0.92$ for positive bending (Fig. 73*b*), a depth to the steel $d = 30 - 4 = 26$ in., and $\frac{1}{2}$-in. round stirrup bars.

$$A_p = \frac{1{,}668{,}000}{20{,}000 \times 0.92 \times 26} = 3.48 \text{ sq. in.}$$

Bond is checked at the point of inflection, which is at 3.8 ft. = 46 in. by Figure 88. At this section the shear force $V = 30{,}100$ lb. Using special anchorage and an allowable stress $u = 150$ lb. per sq. in.,

$$\Sigma = \frac{V}{uojd} = \frac{30{,}100}{150 \times o \times 0.92 \times d} = \frac{218}{od}$$

Number	Size in.	Steel Depth d	Actual Area A_s	Computed Area A_s	Number of Bars for Bond	Number of Rows
4	1 sq.	27.00	4.00	3.35	2.1	1
5	1 rd.	27.00	3.93	3.35	2.6	1
6	$\frac{7}{8}$ rd.	26.12	3.61	3.47	3.1	2
8	$\frac{3}{4}$ rd.	26.25	3.54	3.45	3.6	2

Adopt tentatively the eight $\frac{3}{4}$-in. round bars, as they closely approach the computed areas and give plenty of bars to bend. This choice is subject to change if satisfactory provision cannot be made for compression steel at the support.

Negative Steel. The negative tension steel will be provided from the top rows of positive tension steel in the two adjacent spans. By bending up four $\frac{3}{4}$-in. round bars from each side there are available at the support eight $\frac{3}{4}$-in. round bars. Assume a moment arm ratio $j = 0.87$ and a height to the steel $d = 26.49$ in. (Fig. 89). The top row of $\frac{1}{2}$-in. square bars in the cross and intermediate beams is 2.00 in. down. The top row of $\frac{3}{4}$-in. girder steel must be below these bars.

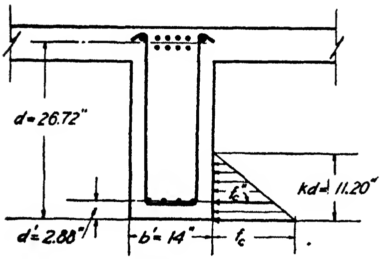


FIG. 89

$$A_n = \frac{185{,}000 \times 12}{20{,}000 \times 0.87 \times 26.49} = 4.82 \text{ sq. in.}$$

The eight $\frac{3}{4}$-in. bars supply 3.54 sq. in. area. There remains an excess of 1.28 sq. in., which is given by two 1-in. round bars. The 1-in. bars will not be bent down, but will run to the center line and carry the stirrups. The minimum spacings are satisfied with four $\frac{3}{4}$-in. rounds and two 1-in. rounds in the same row.

At support

$$u = \frac{34{,}850}{25.13 \times 0.87 \times 26.72} = 60 \text{ lb. per sq. in.}$$

143. Fiber Stress in Concrete. Girder. *Negative Bending.* The maximum negative bending moment $M_n = 2{,}220{,}000$ in.-lb. Without compression steel the rectangular beam at the support (Fig. 89) can safely carry a bending moment:

$$M_o = Kbd^2 = 157 \times 14 \times (26.72)^2 = 1{,}570{,}000 \text{ in.-lb.}$$

Therefore, compression steel must be used.

$$M = 2{,}220{,}000 \text{ in.-lb.}$$

$$M_c = 1{,}570{,}000 \text{ in.-lb.}$$

$$M'_s = 650{,}000 \text{ in.-lb.} = C'_s(d - d')$$

Assuming the compression steel to be $\frac{3}{4}$-in. bars in the bottom row, $d' = 2.88$ in.

$$C'_s = \frac{650{,}000}{26.72 - 2.88} = 27{,}200 \text{ lb.} = 2f'_sA'_s$$

If the maximum allowable stresses are $f_c = 900$ lb. per sq. in. and the actual steel stress $f_s = 18{,}700$ lb. per sq. in., the neutral axis ratio $k = 0.42$, and the neutral axis distance $kd = 0.42 \times 26.72 = 11.2$ in.

The fiber stress f'_s in the compression steel equals

$$f'_s = 15 \times 900 \frac{(11.20 - 2.88)}{11.20} = 10{,}000 \text{ lb. per sq. in.}$$

$$A'_s = \frac{C'_s}{2f'_s} = \frac{27{,}200}{2 \times 10{,}000} = 1.36 \text{ sq. in.}$$

This area is supplied by the four $\frac{3}{4}$-in. round positive tension steel in the bottom row of the girder. Since the actual area is 1.77 sq. in., the actual compression stress equals $f'_s = 2 \times 9050$ lb. per sq. in. and must be anchored 45 diameters, or 34 in., into the column.

Positive Bending. The designer knows that the stresses in the concrete are safe for positive bending.

However, if we desire to ascertain the actual stress, we solve first for the flange width b. This is computed as $b = 78$ in. The tension steel depth $d = 26.25$ in. Assuming that the neutral axis lies in the flange, we solve for the steel ratio $p = \frac{A_s}{bd}$ $= \frac{3.54}{78 \times 26.25} = 0.0017$. From Diagram 1 in the Appendix, the intersection of $p = 0.0017$ and $n = 15$ gives a neutral axis ratio $k = 0.20$ and a moment axis ratio $j = 0.93$. The neutral axis is a distance $kd = 0.20 \times 26.25 = 5.25$ in. below the top. This is below the slab, and the compression area is tee shaped. Assuming that the compression in the stem can be neglected, equation 73 (Art. 104) may be written:

$$M_c = \frac{f_cbt(jd)}{2kd}(2kd - t) = C_cf_cbd^2$$

where

$$C_c = \frac{tj}{2kd}\left(2k - \frac{t}{d}\right) = \frac{M_c}{f_cbd^2}$$

also $p = 0.0017$ and $np = 15 \times 0.0017 = 0.026$.

$$\frac{t}{d} = \frac{4}{26.25} = 0.152$$

From Diagram 10 in the Appendix, for $\frac{t}{d} = 0.152$ and $np = 0.026$, the value of the moment arm $j = 0.94$. The assumption of $j = 0.92$ is safe, but we shall now use $j = 0.94$. In the same diagram, for $\frac{t}{d} = 0.152$ and $np = 0.026$ the term $C_c = 0.09$.

$$\text{Maximum fiber stress } f_c = \frac{M}{C_c bd^2} = \frac{1{,}668{,}000}{0.09 \times 78 \times (26.25)^2} = 340 \text{ lb. per sq. in.}$$

Maximum allowable stress $f_c = 0.45f'_c = 900$ lb. per sq. in.

144. Checking by Tee-Beam Equations. The check of fiber stress given immediately above by plots is the usual commercial method. Cases may arise for which there are no plots and we should be able to use the fundamental equations. Let us check the maximum fiber stress for positive bending by the equations of Article 104.

$$\text{Neutral axis } kd = \frac{2ndA_s + bt^2}{2nA_s + 2bt} = \frac{2 \times 15 \times 26.25 \times 3.54 + 78 \times (4)^2}{2 \times 15 \times 3.54 + 2 \times 78 \times 4}$$

$$kd = \frac{2790 + 1250}{106 + 624} = \frac{4040}{730} = 5.52 \text{ in.}$$

This is the correct kd; the previous computation of 5.25 in. in Article 143 was made by the rectangular-beam theory and merely indicates that the result was outside the limit. The neutral axis is below the slab, and the compressive area is tee shaped.

$$z = \frac{t(3kd - 2t)}{3(2kd - t)} = \frac{4(3 \times 5.52 - 2 \times 4)}{3(2 \times 5.52 - 4)} = \frac{4 \times 8.56}{3 \times 7.04} = 1.62 \text{ in.}$$

$$jd = d - z = 26.25 - 1.62 = 24.63 \text{ in.}$$

also

$$j = \frac{jd}{d} = \frac{24.63}{26.25} = 0.94$$

$$f_s = \frac{M}{A_s jd} = \frac{1{,}668{,}000}{3.54 \times 24.63} = 19{,}100 \text{ lb. per sq. in.}$$

$$f_c = \frac{f_s}{n}\left(\frac{kd}{d - kd}\right) = \frac{19{,}100}{15}\left(\frac{5.52}{26.25 - 5.52}\right) = 340 \text{ lb. per sq. in.}$$

These equations are convenient for checking a design rather than for determining sizes of sections and areas of steel.

145. Placing Steel. Girder. *Positive Steel.* The positive steel consists of eight $\frac{3}{4}$-in. round bars in two rows. We must fulfill the following requirements while it is used as tension steel.

1. Eight $\frac{3}{4}$-in. rounds at the center line for bending moment.
2. Four bars at the point of inflection 46 in. from column face for bond.
3. Four bars running into the column and hooked, for anchorage and compression steel.

Run the lower row of four bars 34 in. into the column to anchor for the compression stress $2f'_s = 18{,}100$ lb. per sq. in. (Art. 143). This will care for bond, anchorage, and compression steel.

The bars in the upper row can be bent up as soon as the bending moment allows. From Figure 88 the positive bending moment diagram permits

$$\text{2 bent:} \quad M_6 = 20{,}000 \times 6 \times 0.442 \times 0.94 \times 26.54 = 1{,}320{,}000 \text{ in.-lb.}$$

$$\text{4 bent:} \quad M_4 = 20{,}000 \times 4 \times 0.442 \times 0.94 \times 27.13 = 900{,}000 \text{ in.-lb.}$$

$$\text{When } M_6 = 110{,}000 \text{ ft.-lb. bend 2 bars at 7.8 ft.} = 94 \text{ in.}$$

$$\text{When } M_4 = 75{,}000 \text{ ft.-lb. bend 2 more at 6.5 ft.} = 78 \text{ in.}$$

These bends cannot be made nearer the center line, and the values are maxima when bends for diagonal tension are considered.

Negative Tension Steel. The negative tension steel consists of eight $\frac{3}{4}$-in. round bars and two 1-in. round bars. The 1-in. bars will be run 3 in. beyond the center line in each direction.

Bond. The point of inflection for maximum negative bending occurs at a section 5.95 ft. = 71 in. from the column (Fig. 88). The total perimeters required at this section to satisfy bond requirements are

$$\Sigma o = \frac{V}{ujd} = \frac{27{,}400}{150 \times 0.87 \times 27.13} = 7.75 \text{ in.}$$

The steel height $d = 27.13$ assumes that all the bars considered are in the top row. The two 1-in. bars supply 6.28 in., so at least one $\frac{3}{4}$-in. bar must run to this section to satisfy bond. Use two $\frac{3}{4}$-in. bars for symmetry; they will be run a few inches or so into the "region of compression" and a standard 180° hook will be added. The other six $\frac{3}{4}$-in. bars can be bent down or cut off before they reach the point of inflection. As far as bond is concerned, six bars can be bent down at any part of the span, as the remainder is effective up to a shear force of 39,000 lb.

Bending Moment. Three-quarter-inch bars can be bent down in pairs as follows.

$$\text{2 bent:} \quad M_8 = 20{,}000 \times 4.22 \times 0.87 \times 27.00 = 1{,}980{,}000 \text{ in.-lb.}$$

$$\text{4 bent:} \quad M_6 = 20{,}000 \times 3.34 \times 0.87 \times 26.90 = 1{,}570{,}000 \text{ in.-lb.}$$

$$\text{6 bent:} \quad M_4 = 20{,}000 \times 2.45 \times 0.87 \times 27.13 = 1{,}157{,}000 \text{ in.-lb.}$$

$$\text{When } M_8 = 165{,}000 \text{ ft.-lb. bend 2 bars at 0.83 ft.} = 10 \text{ in.}$$

$$\text{When } M_6 = 130{,}700 \text{ ft.-lb. bend 2 more at 1.62 ft.} = 20 \text{ in.}$$

$$\text{When } M_4 = 96{,}300 \text{ ft.-lb. bend 2 more at 2.65 ft.} = 32 \text{ in.}$$

This last pair is bent down but is not made continuous with the positive steel.

The summary of the bending moment, bond, and anchorage data for the bent bars is shown in Figure 90. One pair of truss bars will be bent up to go into the

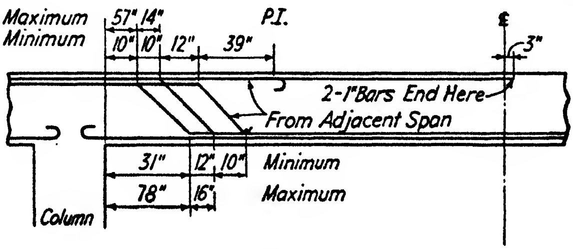


FIG. 90

upper row over the column and run to the point of inflection; the other pair will go into the lower row and bend down in the adjacent spans. We are now able to compute the diagonal tension steel, knowing the limitations for the bent bars due to their triple action as positive, negative, and diagonal tension steel.

146. Diagonal Tension Steel. Girder. The corrected maximum shear stress equals

$$v = \frac{V}{bjd} = \frac{34{,}850}{14 \times 0.87 \times 26.72} = 108 \text{ lb. per sq. in.}$$

The complete shear stress diagram is shown in Figure 91.

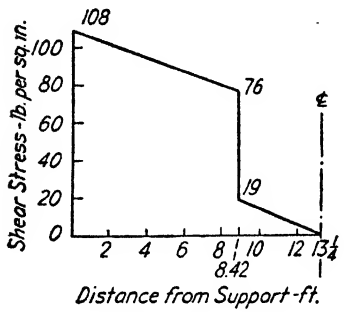


FIG. 91

Maximum Spacing. We have assumed two-rod $\frac{1}{2}$-in. round stirrups. Their maximum longitudinal spacing equals $\frac{d}{2} = \frac{26.72}{2} = 13.4$ in.

We are using $\frac{3}{4}$-in. round bars as bent bars. Their maximum longitudinal spacing equals $d = 26.7$ in.

Minimum Spacing. Assume that diagonal tension is computed for maximum positive bending with the tensile cracks occurring at the bottom of the girder. The first pair of bent bars cannot be nearer than 31 in. to a crack starting at the support. This is greater than the maximum longitudinal spacing of 26 in. We must first use stirrups.

Stirrups. The minimum spacing for stirrups is determined at the support.

$$108 = 60 + \frac{2 \times 0.196 \times 20{,}000}{14 \times s \times 1} = 60 + \frac{560}{s}$$

Minimum $s = \frac{560}{48} = 11.7$ in. This is too close to the maximum spacing of 13 in. If the size is changed to two-rod $\frac{3}{8}$-in. round stirrups the closest spacing equals

$$v = 60 + \frac{314}{s} \qquad \text{or} \qquad s = 6.6 \text{ in.}$$

This size permits some expansion as the shear stress drops, and we shall adopt the $\frac{3}{8}$-in. stirrups.

Spacing in.	Total Shear v	Distance from Column Face in.
6	. .	. .
8	99	20
10	91	46
12	86	62
	60	101

Bent Bars. It is desirable to use bent bars at their maximum spacing so that the extra cost of bending the bar may be offset by the number of stirrups eliminated. Also, it is desirable to make the bends as near the support as possible, because the stirrups are closely spaced in regions of high shear force.

A pair of $\frac{3}{4}$-in. bars bent at 45° can be used at their maximum spacing of 26 in. if the shear stress does not exceed

$$v = 60 + \frac{2 \times 0.44 \times 20{,}000}{14 \times 26 \times 0.707} = 129 \text{ lb. per sq. in.}$$

The shear stress does not exceed $v = 108$ lb. per sq. in., so the bent bars can be used at their maximum spacing anywhere in the span. By A.C.I. Article 706*a* stirrups must be spaced not more than

$$16 \text{ bar diameters} = 12 \text{ in.}$$

$$48 \text{ tie diameters} = 18 \text{ in.}$$

wherever compression steel is needed. The negative moment exceeds $M_c = 1{,}570{,}000$ in.-lb. for a distance of about 18 in. from the column face.

Placing Steel. The first bend can reach the bottom of the beam at 31 in. (Fig. 90). At a maximum longitudinal spacing of 26 in. this pair of bent bars can reinforce all sections between 5 in. and 31 in. However, use three stirrups spaced at 6 in. to reinforce from the column face as long as compression steel is needed. Then the first bent pair can be placed at 18 + 26 = 44 in., which is between the bending limits of 31 in. and 78 in. The second bent pair is spaced at its maximum of 26 in. and bends up at 70 in., which is also between the bending moment limits of 43 in. and 94 in. The third pair of negative bars can be bent down from the top at 32 in. out. Let them continue to the bottom and be hooked there. They can reach the bottom at 53 in.; but, if used as diagonal tension reinforcement, they should reach the bottom at 70 + 26 = 96 in. The edge of the intermediate beam is at 97 in., so no additional web steel is needed. The stirrup spacings and bent bars are shown in Figure 93.

147. Transverse Steel in Slab. Girder. The slab steel runs parallel to the girder. Therefore, there is no rigid connection between the slab and girder except that given by the prongs of the few stirrups, yet we are assuming that the slab acts as the flange of the girder section. In mid-span the slab will tend to deflect more than the girder, and cracks may appear at or near the junction of slab and girder stem. To prevent the formation of such cracks and to tie the slab and girder together it is customary to place steel in the top of the slab running transversely over the girder. A.C.I. Article 705*c* specifies that this steel shall be designed to carry the load on the portion of the slab assumed as the girder flange. This load will act on a cantilever beam (Fig. 92). The maximum bending moment occurs at the edge of the beam stem and for a cantilever 1 ft. wide equals

$$M = \frac{wl^2}{2} = 180 \times \frac{32}{12} \times \frac{32}{2} = 7680 \text{ in.-lb.}$$

This transverse steel goes in the top of the slab but must be under the slab steel in order to preserve the fireproofing clearance. The transverse steel will be immediately below the main slab steel and can be placed to dodge the occasional girder stirrup which also hooks over at this level. The spacing of the transverse steel shall not exceed five times the slab thickness, nor exceed 18 in. Assume $\frac{3}{8}$-in. round bars for trial. The steel height $d = 4 - \frac{3}{4} - \frac{3}{8} - \frac{3}{16} = 2.69$ in.

$$K = \frac{M}{bd^2} = \frac{7680}{12 \times (2.69)^2} = 88$$

From Diagram 2, in the Appendix, steel ratio $p = 0.005$. From Diagram 1 the value of the moment arm ratio $j = 0.89$.

$$\text{Tension steel area } A_s = \frac{7680}{20{,}000 \times 0.89 \times 2.69} = 0.16 \text{ sq. in.}$$

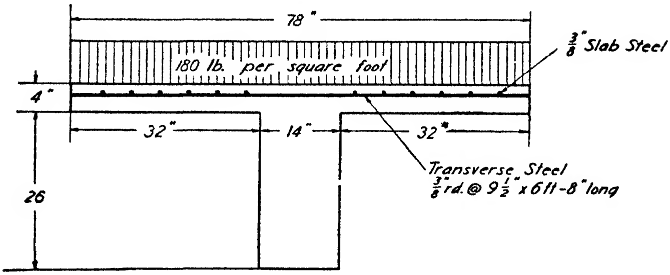


FIG. 92

Use $\frac{3}{8}$-in. round bars spaced at $8\frac{1}{2}$ in. This spacing is within the maximum allowable of $5t = 20$ in. and within the other maximum of 18 in.

148. Final Sketch. Girder. Figure 93 shows the complete steel sketch for this span. The design of this floor system has been given in considerable detail because tee-beam designs constitute so great a part of concrete design. The designer would simplify much of the procedure by means of tables and plots.

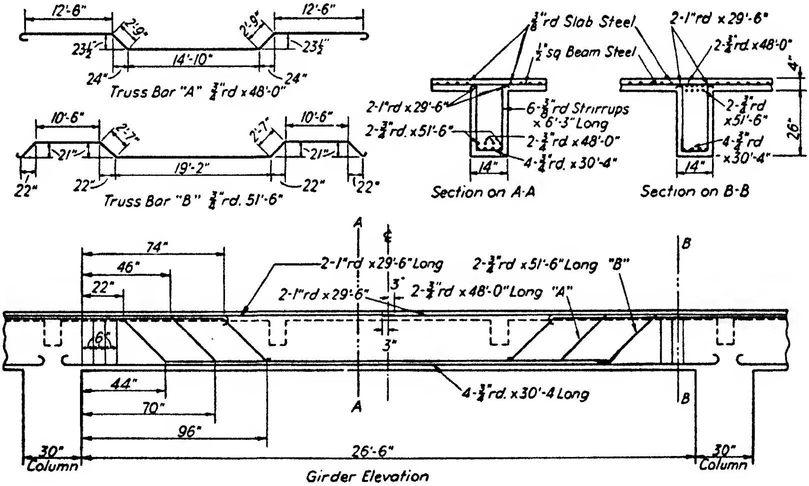


FIG. 93

The truss bars A and B are long and have many bends plus hooks. Type III reinforcement using truss bars is not well adapted to the present requirements for special anchorage. If ordinary anchorage had been used the hooks could be omitted and the final bend down for anchorage in the adjacent spans for bar B. Both A

and B bars would then resemble those of Figure 31 and be shorter and easier to fabricate. When special anchorage is employed the U bars of Type II reinforcement fit more easily into the requirements.

PLASTIC THEORY

ILLUSTRATIVE PROBLEM 18

149. Tee-Beam Size by Plastic Theory. Let us determine for the beam and girder floor system of Problem 17 the sizes of tee-beam sections to use for the intermediate beam and the girder.

Intermediate Beam. The known or assumed data are

Thickness of slab = 4 in.

Beam stem = 8 in. by 16 in.

Load from slab = $180 \times 9\frac{2}{3}$ = 1740 lb. per ft.

Stem weight = 135 lb. per ft.

The load diagram is shown in Figure 68 and the shear force and bending moment diagrams are given in Figure 69. These shears and moments will be used without change for the plastic theory, on the assumption that they were obtained from the ultimate values by dividing by a suitable factor of safety.

Maximum positive moment = 19,200 ft.-lb.

Maximum negative moment = 27,900 ft.-lb.

Negative Bending at Support (*Fig. 59b*). This section is a rectangular beam. Assuming no compression steel and balanced design, the size by equation 20 is

$$bd^2 = \frac{3M}{f'_c} = \frac{3 \times 27{,}900 \times 12}{800} = 1253 \text{ cu. in.}$$

The allowable stress f_c has been taken as $0.4f'_c$. If $b = 8$ in., $d = 12.5$ in., $h =$ 15 in.

Positive Bending (*Fig. 59a*). If $h = 15$ in., $d = 12.25$ in. approximately. Mr. Whitney recommends that the beam flange $b = 8t + b' = 40$ in. It is probable that the compression area will lie wholly in the flange. Assuming an under-reinforced rectangular beam, by equation 19,

$$a = 12.25\left[1 - \sqrt{1 - \frac{2.35 \times 19{,}200 \times 12}{800 \times 40 \times (12.25)^2}}\right] = 0.71 \text{ in.}$$

$$c = d - \frac{a}{2} = 11.90 \text{ in.}$$

$$A_p = \frac{M}{f_y c} = \frac{19{,}200 \times 12}{20{,}000 \times 11.90} = 0.97 \text{ sq. in.}$$

If one 1-in. square bar is used, the bond stress is too high, so use two $\frac{7}{8}$-in. round bars. The negative steel equals

$$A_n = \frac{M}{0.732 f_y d} = \frac{27{,}900 \times 12}{0.732 \times 20{,}000 \times 12.5} = 1.83 \text{ sq. in.}$$

Use two 1-in. square bars, which are also satisfactory for bond. No compression steel is needed at the support. The comparison of the designs by the two theories gives

	Straight-Line	Plastic
Stem width, in.	8	8
Total depth, in.	17	15
Actual A_p, sq. in.	0.88	1.20
Actual A_n, sq. in.	1.50	2.00
Theoretical A'_s, sq. in.	0.44	None

The compression steel does not count in this comparison, as the two bars of positive steel will run into the support anyway and must be well anchored to justify "special anchorage."

Girder. A similar analysis of the girder gives

$$M_p = 139{,}000 \text{ ft.-lb.}$$

$$M_n = 185{,}000 \text{ ft.-lb.}$$

At the support

$$bd^2 = \frac{3 \times 185{,}000 \times 12}{800} = 8320 \text{ cu. in.}$$

If $b = 14$ in., $d = 24.4$ in. and $h = 29$ in.

$$A_n = \frac{185{,}000 \times 12}{0.732 \times 20{,}000 \times 25.00} = 6.06 \text{ in.}$$

Use eight 1-in. round bars, which are also satisfactory for bond.

For positive bending, assuming $d = 25.26$ in. (two rows),

$$a = 25.26 \left[1 - \sqrt{1 - \frac{2.35 \times 139{,}000 \times 12}{800 \times 46 \times (25.26)^2}} \right] = 2.20 \text{ in.}$$

$$c = 25.26 - 1.10 = 24.16 \text{ in.}$$

$$A_p = \frac{139{,}000 \times 12}{20{,}000 \times 24.16} = 3.45 \text{ sq. in.}$$

Use $\frac{7}{8}$-in. round bars. The comparison of results gives

	Straight-Line	Plastic
Stem width, in.	14	14
Total depth, in.	30	28
Actual A_p, sq. in.	3.54	3.61
Actual A_n, sq. in.	5.11	6.28
Theoretical A'_s, sq. in.	1.36	None

In general, plastic design gives a somewhat smaller section with no compression steel at the support. However, the negative tension steel is considerably increased and the positive tension steel is slightly greater. These comparisons are made on the assumption that bond, anchorage, and diagonal tension are satisfied.

RIBBED OR JOIST FLOORS

150. Ribbed or Joist Floors. The ribbed floor consists of a thin slab, $1\frac{1}{2}$ in. to 3 in. thick, supported by small joists spaced not more than 30 in. in the clear. This type of floor construction is used in the endeavor to save weight and is especially adapted for office buildings, schools, etc., where the live loads are light. Since the slab and joists are poured as a unit the design again deals with a tee-shaped section. The space between the joists may be filled with light-weight tile to give a plane undersurface for plaster (Fig. 94*a*), or the joists may taper slightly so that

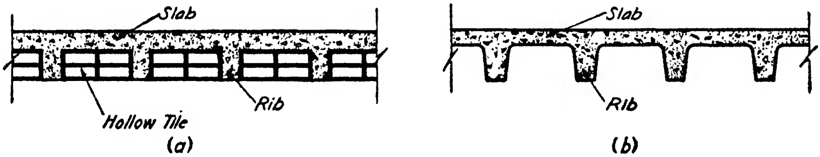


FIG. 94

the forms of sheet metal may be withdrawn, leaving this space open (Fig. 94*b*). If tile is used the clear distance between joists is usually 12 in.; if the metal forms are used longer distances up to 30 in. can be employed.

Concrete or clay tiles are used as fillers. Before the floor system is designed it is desirable to learn what depths of tile are available in the locality where they are to be purchased. The weights of concrete tile with 12-in. by 12-in. area for varying heights are

Height in.	Weight * lb.	Height in.	Weight * lb.
4	16	8	30
5	20	9	33
6	22	10	35
7	27	12	40

* From *Concrete Engineers Handbook*, Hool and Johnson.

A.C.I. Article 708 requires that the ribs shall be considered straight, not less than 4 in. wide, nor deeper than 3 times the width. The clear spacing of ribs shall not exceed 30 in. Where removable forms are used, the slab thickness shall not be less than $\frac{1}{12}$ the clear span between ribs, nor less than 2 in. If tile are used, the slab thickness shall not be less than $\frac{1}{12}$ the clear span between ribs, nor less than $1\frac{1}{2}$ in. If the tile are so placed that the tile joints in alternate rows do not occur at the same cross section of the rib, the tile webs in contact with the rib may be used in the region of negative bending for computations involving shear and bending moment. Shrinkage reinforcement must be provided as in ordinary slabs.

ILLUSTRATIVE PROBLEM 19

151. Design of Ribbed Floor. Using the floor dimensions of Figure 61, design a tile ribbed floor to span between the girders, omitting the intermediate and cross beams. Adopt a 2000-lb.[3] concrete and ordinary anchorage. The tile system is used for this illustration because it parallels the design for removable forms and the tile computations must be made in addition.

Assume the girder to be 12 in. wide; then the clear span for the continuous ribbed floor will be 13 ft. 6 in. Assume the minimum slab thickness $t = 1\frac{1}{2}$ in. and the minimum rib thickness of $b' = 4$ in. Each tee-beam joist carries a strip of floor 16 in. wide (Fig. 95).

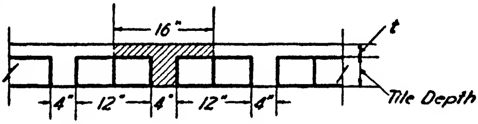


FIG. 95

152. Slab. The thin slab spans continuously across the joists for many spans. Assuming that they are fixed at the ends and disregarding any support the tile may give, the negative fixed-end moment equals $M_n = \dfrac{wl^2}{12}$. The load equals

$$
\begin{aligned}
\text{Live load} &= 130 \text{ lb. per sq. ft.}\\
1\tfrac{1}{2}\text{-in. slab} &= \underline{\ 19\ }\\
w &= 149 \text{ lb. per sq. ft.}
\end{aligned}
$$

$$M_n = \frac{wl^2}{12} = \frac{149 \times (1)^2 \times 12}{12} = 149 \text{ in.-lb.}$$

The slab is of plain concrete, except for shrinkage steel, and the maximum tensile stress equals

$$f = \frac{My}{I} = \frac{149 \times 6}{12 \times (1.5)^2} = 33 \text{ lb. per sq. in.}$$

The allowable tensile stress is given by the "shear equivalent" of $v = 0.02f'_c = 40$ lb. per sq. in. (Art. 61). The $1\frac{1}{2}$-in. slab is satisfactory.

153. Tee-Beam Rib. The dimensions of this small beam should be fixed before any steel is computed. The narrow compression area at the support (Fig. 96*b*)

[3] See footnote of Article 32.

makes that section the critical one. Assuming an 8-in. tile, the load on each beam equals

$$\begin{aligned}
&\text{Live and slab loads} = 149 \times \tfrac{16}{12} &&= 199 \text{ lb. per ft. length}\\
&\text{Beam stem} = \frac{4 \times 8}{144} \times 150 &&= 33\\
&\text{8-in. tile} &&= \underline{30}\\
&w &&= 262 \text{ lb. per ft. length}
\end{aligned}$$

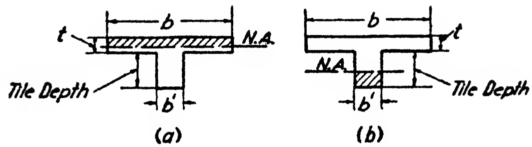


FIG. 96

154. Shear. **Joist.** The maximum shear force $V = \dfrac{262 \times 13.5}{2} = 1770$ lb. The tile walls adjacent to the joist are assumed to be $\frac{3}{4}$ in. thick. The shear stress equals

$$v = \frac{1770}{5.5 \times 0.87 \times 8.50} = 44 \text{ lb. per sq. in.}$$

This stress is safe if special anchorage is used. No diagonal tension steel is needed nor is it desired.

155. Fiber Stress at Support. The maximum negative moment equals

$$M_n = \frac{wl^2}{11} = 262 \times (13.5)^2 \times \frac{12}{11} = 52{,}000 \text{ in.-lb.}$$

The compression area is rectangular at the support (Fig. 96*b*). Counting the adjacent tile walls, the rectangular beam constant equals

$$K = \frac{M}{bd^2} = \frac{52{,}000}{5.5 \times (8.50)^2} = 131$$

From Diagrams 1 and 2 for $K = 131$ and $f_s = 20{,}000$,

$$f_c = 800 \text{ lb. per sq. in.} \qquad \text{and} \qquad j = 0.875. \quad \text{Safe.}$$

156. Positive Bending Moment. (Figure 96*a*.) The maximum positive moment $M_p = \dfrac{wl^2}{16} = 35{,}700$ in.-lb. Assuming the steel to be $\frac{1}{2}$-in. bars with $1\frac{1}{2}$-in. fireproofing and the neutral axis to be located in the flange.

$$K = \frac{M}{bd^2} = \frac{35{,}700}{16 \times (7.75)^2} = 37$$

From Diagrams 1 and 2 for $K = 37$ and $f_s = 20{,}000$,

$$f_c = 360 \text{ lb. per sq. in.} \qquad j = 0.93 \qquad \text{and } k = 0.22$$

The neutral axis is at a distance $kd = 0.22 \times 7.75 = 1.71$ in. from the top. This is below the flange and the compression area is tee shaped. The positive steel equals

$$A_p = \frac{M}{f_s jd} = \frac{35{,}700}{20{,}000 \times 0.93 \times 7.75} = 0.25 \text{ sq. in.}$$

Use one $\frac{1}{2}$-in. square bar.

By Diagram 10 for $\frac{t}{d} = \frac{1.5}{7.75} = 0.194$ and $np = \frac{15 \times 0.25}{16 \times 7.75} = 0.03$,

$$C_c = 0.10 \quad \text{and} \quad j = 0.93$$

$$f_c = \frac{M}{C_c bd^2} = \frac{35{,}700}{0.10 \times 16 \times (7.75)^2} = 370 \text{ lb. per sq. in.} \quad \text{Safe.}$$

It will be noticed that the results for f_c and j by the false assumption of the neutral axis in the flange are correct in this case because the neutral axis is only a short distance below the flange.

The positive steel is checked for bond at the point of inflection. This occurs at $0.15l = 24$ in. from the support. At this section $V = 1240$ lb. The number of bars required is

$$\Sigma = \frac{V}{uojd} = \frac{1240}{150 \times 2.00 \times 0.93 \times 7.75} = 0.6 \text{ bars.} \quad \text{Safe.}$$

157. Negative Tension Steel.

$$A_s = \frac{52{,}000}{20{,}000 \times 0.875 \times 8.50} = 0.35 \text{ sq. in.}$$

Use two $\frac{1}{2}$-in. round bars.

At the support

$$u = \frac{V}{\Sigma ojd} = \frac{1770}{2 \times 1.57 \times 0.875 \times 8.50} = 76 \text{ lb. per sq. in.} \quad \text{Safe.}$$

These bars will be carried out to the point of inflection and anchored beyond that section by hooks. The point of inflection occurs at $0.24l = 39$ in. The steel arrangement is shown in Figure 97.

Shrinkage Steel. This steel can best be supplied by wire mesh, since there is no steel in the thin slab and it is advisable to have shrinkage reinforcement in both directions. Using the slab thickness of $t = 1.5$ in., the steel area in both directions equals

$$A_s = 0.0025 \times 12 \times 1.5 = 0.045 \text{ sq. in. per sq. ft. of floor surface}$$

158. Supporting Beam. *Rectangular Beam.* The beam supporting the joist floor system will have the section shown in Figure 97 when opposite a row of tiles. If the

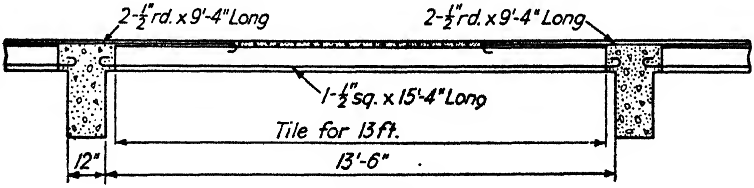


FIG. 97

tiles run for 13.5 ft., it will be necessary to introduce a half-tile into each row to give the necessary length and to assure that the tile joints are not opposite each other in two adjacent rows. Each slab rib brings to the supporting beam in each 16 in. interval a load of

$$\begin{aligned}
\text{Live and dead loads} &= 262 \times 13.5 = 3540 \text{ lb.}\\
\text{Beam} &= 12 \times 24 \times \tfrac{150}{144} \times \tfrac{16}{12} = \underline{400 \text{ lb.}}\\
\text{Total} &= 3940 \text{ lb.}
\end{aligned}$$

The load per foot of length $w = 3940 \times \frac{12}{16} = 2950$ lb. per ft. Disregarding the thin $1\frac{1}{2}$-in. slab, the beam section is rectangular. The maximum numerical moment is the negative.

$$M_n = \frac{wl^2}{11} = 2950 \times (26.5)^2 \times \frac{12}{11} = 2{,}260{,}000 \text{ in.-lb.}$$

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{2{,}260{,}000}{157 \times b}} = \frac{120}{\sqrt{b}}$$

A beam with a width of 12 in. will be about 38 in. deep, and a better section would be 16 in. by 33 in. In order to use tile or half-tile between beams 16 in. wide, the row of tiles could only be 13 ft. long. The beam would then have a flange of solid slab extending out 1 in. each side of the beam stem.

Tee Beam. A lighter design can be made with a tile row of 13 ft., if the beam stem is made 12 in. wide with the solid slab extending out 3 in. on either side. The beam section for positive bending is tee shaped with a flange 18 in. wide and 9.5 in. deep. Let us assume the stem to be 12 in. by 21 in. Each joist unit brings in a load for each 16-in. length of the beam equal to

$$\begin{aligned}
\text{Live load and } 1\tfrac{1}{2}\text{-in. slab} &= 149 \times \tfrac{16}{12} \times 14.5 = 2890 \text{ lb.}\\
\text{Joist stem} &= \frac{4 \times 8}{144} \times 150 \times 13 = 435\\
\text{Tile} &= 30 \times 13 = 390\\
\text{Solid slab} &= \frac{9.5 \times 16 \times 6}{(12)^3} \times 150 = \underline{80}\\
&\qquad 3795 \text{ lb.}
\end{aligned}$$

The load per foot length of the beam equals

$$\begin{aligned}
3795 \times \tfrac{12}{16} &= 2840 \text{ lb. per ft.}\\
\text{Beam weight} = \frac{12 \times 29}{144} \times 150 &= \underline{360}\\
&\ 3200 \text{ lb. per ft.}
\end{aligned}$$

$$\text{Maximum positive moment} = \frac{wl^2}{16} = 3200 \times (26.5)^2 \times \frac{12}{16} = 1{,}690{,}000 \text{ in.-lb.}$$

Assuming the neutral axis to be below the 9.5-in. flange and $d = 27.5$ in.,

$$A_p = \frac{1{,}690{,}000}{20{,}000 \times 0.88 \times 27.5} = 3.50 \text{ sq. in.}$$

$$np = \frac{15 \times 3.50}{18 \times 27.5} = 0.106 \qquad \text{and} \qquad \frac{t}{d} = \frac{9.5}{27.5} = 0.35$$

From Diagram 10, $f_c = 810$ lb. per sq. in., $j = 0.88$. This concrete stress is safe. Use three $1\frac{1}{8}$-in. square bars in one row with $d = 27.56$ in.

Negative Moment and Shear. Maximum negative moment $= \dfrac{wl^2}{11} = 2{,}460{,}000$ in.-lb. Use four $1\frac{1}{8}$-in. square bars in one row with $d = 28.06$ in. The concrete, without compression steel, can safely carry a moment of

$$M_c = Kbd^2 = 157 \times 12 \times (28.06)^2 = 1{,}480{,}000 \text{ in.-lb.}$$

The compression steel moment $M'_s = 980{,}000$ in.-lb. and a compression steel area $A'_s = 1.95$ sq. in. is required.

The maximum shear stress equals

$$v = \frac{V}{bjd} = \frac{3200 \times 13.25}{12 \times 0.87 \times 28.06} = 145 \text{ lb. per sq. in.}$$

If special anchorage is used, this is safe.

Weight of floor. The weight of a joist unit 16 in. wide by 13.5 ft. long = 1,245 lb.

Weight of floor panel 29 ft. by 13.5 ft.	= 27,000 lb.
Weight of beam, 12 in. by 30.5 in., 26.5 ft. long	= 10,100 lb.
Weight of floor system, 29 ft. by 14.5 ft.	= 37,100 lb.

159. Comparison with Solid One-Way Slab. If this same panel, 29 ft. by 14.5 ft., is designed for a slab spanning from girder to girder, the slab load equals

Live load	= 130 lb. per sq. ft.
6-in. slab	= 75
Total on slab	= 205 lb. per sq. ft.

$$\text{The maximum numerical moment} = M_n = \frac{wl^2}{11} = 205 \times (13.5)^2 \times \frac{12}{11} = 40{,}800 \text{ in.-lb.}$$

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{40{,}800}{157 \times 12}} = 4.65 \text{ in.}$$

Assuming $\frac{1}{2}$-in. steel, $h = 4.65 + 0.75 + 0.50 = 5.90$ in. Use 6-in. slab.

Beam. A tee beam, 12 in. wide by 31 in. deep, with a stem 12 in. by 25 in., will carry this slab and the live load safely.

Weight of floor, 6 in. deep, 29 ft. by 13.5 ft.	= 29,400 lb.
Weight of beam, 12 in. by 31 in., 26.5 ft. long	= 10,300 lb.
Weight of floor system, 29 ft. by 14.5 ft.	= 39,700 lb.

For the loads and dimensions used in this problem little saving in weight results from the use of the joist floor system. If the metal forms are used and no tile, so that the joists are much farther apart, there is often a saving. If spans are less and loads are lighter, the solid slab approaches its minimum of 4 in. thick and fireproofing clearances make such a slab inefficient. The somewhat deeper joist floor will then often compare favorably in weight.

The joist floor system can also be used with two-way steel and joists. The division of loads is made in the same manner as that for two-way solid slabs (Chapter 10).

CHAPTER 8

DEFLECTION AND TORSION

160. Deflection and Flow. The term *deflection* will be used to denote the displacement of the neutral layer of a beam which occurs within a short time after the application of the load. The usual beam formulae give deflections. The term *flow* will be used to represent the additional deformations caused by shrinkage and by the yielding under long-continued loading.

Neither deflection nor flow is extreme for beams of short span. Beams of medium span designed for fiber stress are usually safe for deflection. Beams of long span must be checked for both deflection and flow, as the safe load will be determined by the necessity of having reasonable values for these deformations.

161. Deflection of Beams. The theory of deflection for beams is based on the usual limitations and assumptions of the general beam theory. A reinforced concrete beam does not fulfill all these requirements. It is not of homogeneous and isotropic material, nor does it have uniform cross sections as required by the limitations. The assumption that the fiber stresses are uniformly varying is true for the steel and at least approximately for the concrete in compression. If we consider the concrete as also helping to carry the tensile stresses, the tensile stresses in the concrete will be so near the ultimate that a constant proportionality between the stress and strain no longer holds (Fig. 3*a*). As long as the concrete still carries some tension, the load-deflection diagram will be curved as is the line OA in the typical deflection diagram of Figure 98. After the concrete cracks in tension the concrete carries the compression stresses and the steel the tensile stresses in a straight-line relation AB. When the concrete reaches the higher compression stress and the steel passes its elastic limit the load-deflection diagram is no longer straight, but continually greater deflections are recorded for the same increment of load, as is shown by BC. The difficulty of fitting a rational deflection theory to both the OA and AB portions of

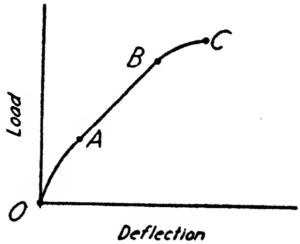


FIG. 98

the load-deflection curve which shall allow for the changes produced by concretes of different strengths and also by the great variation of possible steel arrangements is apparent. The variations are so great that the theory must be modified by test results to give reasonable accuracy.

The usual equations for the maximum deflection v of beams take the form

$$v_{\text{max.}} = C_1 \frac{Wl^3}{EI} \tag{92}$$

where W = total load on the span
l = span
E = modulus of elasticity of the material
I = moment of inertia of the section
C_1 = a coefficient (C_1 varies with the fixity of the support and with the type of loading whether it be concentrated, distributed, etc.)

If this relation is adopted as the basis of reinforced concrete deflection computations, it is necessary to decide what we shall use for the modulus of elasticity E and for the representative, or average, moment of inertia I for the span.

162. Modulus of Elasticity. The usual beam theory assumes that the modulus of elasticity of the homogeneous material is the same in tension as in compression. This is not true for the reinforced concrete beam.

On the compression side of the beam the concrete stresses may be assumed proportional to the strains. There will be a constant value E_c of the modulus of elasticity in compression for all sections whether lightly or heavily loaded.

On the tension side the concrete will have cracked at the sections of maximum bending moment even for light loads. Other sections will still be able to take some tension wherever these tensile stresses are low. When we design for fiber stress we consider the section of maximum bending and very properly do not consider the concrete as taking any tension. On the other hand, when we design for diagonal tension at sections of small bending moment we allow for the tension in the concrete. Similarly, dealing with deflection, and considering the action of the beam as a whole, we should allow for the tension in the concrete for a great part of the span. At a section with small bending moment the maximum tensile stresses may vary from zero to A (Fig. 99). In such a case all particles have the same modulus of elasticity E_t, which is the slope of the line OA. This value E_t approximates the modulus of elasticity E_c in compression. At a section more heavily loaded the tensile stresses may vary from zero to B. The particles with stresses varying from zero to C have a common modulus which is the slope of the line OC. Each particle beyond C has a different modulus (secant modulus), such as the slope at particle B. The whole section will act as though the tensile stresses obeyed some unknown average modulus.

Some other section may have tensile stresses varying from zero to F with a different average modulus. It is impossible, therefore, to compute a modulus of elasticity E_t for concrete in tension which shall be a good average for sections lightly or heavily stressed in tension and sections which have already cracked. In addition, on the tension side there are varying amounts of steel whose modulus of elasticity E_s is a constant.

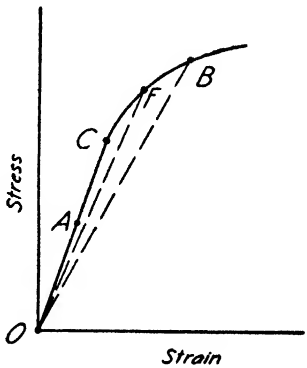


FIG. 99

We cannot determine the correct average modulus for all particles in a beam. As we shall use the transformed section for our computations, it seems advisable to adopt the modulus of elasticity E_c of the concrete in compression for use in the deflection formulae and let any error due to the fact that it may not be the correct average be taken care of by an empirical value n' of the ratio $\frac{E_s}{E_c} = n'$. This term n' will also allow for the variation of steel areas and position at different sections and for the variation of moment of inertia at different parts of the span. It becomes an empirical constant that justifies our simplifications and assumptions by giving the correct deflection. The analysis of available deflection tests suggests that the value of n' be taken as 8, 10, or 12, no matter what the mix may be. A value of $n' = 8$ appears to be a good average for all types of beams. The deflection figured when the term n' is used as a constant will not give exact results but will indicate whether the deflection is $\frac{1}{8}$ in., $\frac{1}{4}$ in., 1 in., or more. The designer can then judge whether this approximate deflection is excessive or not.[1]

163. Moment of Inertia. There has been previous discussion of the fact that the moment of inertia of the cross sections of reinforced concrete beams varies in different parts of the span. At the center sections with positive bending the cross section has the appearance of Figure 100*a* with the equivalent transformed areas of Figure 100*b* or 100*c*, depending on the position of the neutral axis. At the support with nega-

[1] At the end of 4 or 5 years plastic flow makes it advisable to use values of n' of 40 or 50 to estimate the total deflection due to loads, shrinkage, and flow.

tive bending the section is shown in Figure 101*a* with tension and compression steel. The equivalent transformed area is given in Figure 101*b*. The positive and negative transformed areas are quite different, and their moments of inertia are different. Near the points of inflection,

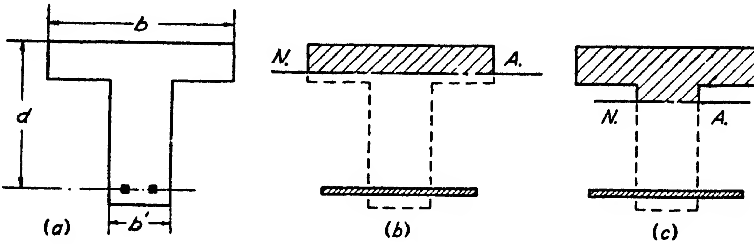


Fig. 100

while the steel is bending up or down, the transformed section will continually vary. It is difficult, therefore, to obtain by theory or by test an average moment of inertia which can be easily computed. The following discussion gives some justification for using in computations the moment of inertia of the sections in positive bending.

Maximum deflection is obtained by the live-load arrangement which gives the least restraint at the support or, in other words, gives the maximum positive bending in the span. The continuous beam will then

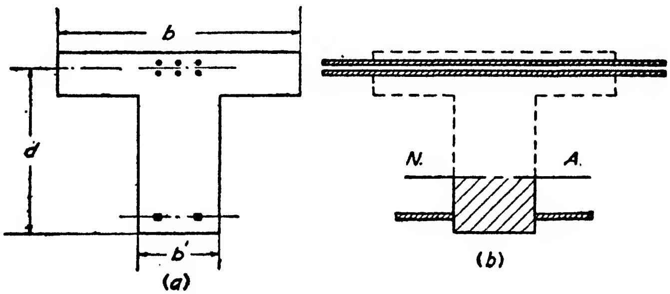


Fig. 101

approach the case for greatest deflection, which is that of a beam supported at the ends without restraint. The discussion of slope-deflection solutions of continuous frames in Chapter 14 shows that the deflection can be found graphically from the $\frac{M}{EI}$ diagram. The moment of this diagram about a section gives the deflection at that section. If we consider a beam of constant moment of inertia I supported at the ends with a uniformly distributed load, the $\frac{M}{EI}$ diagram can be plotted as in Figure 102a. The moment of the portion of the diagram to the left of the center line gives (Fig. 102b):

$$v_{\max.} = \frac{wl^3}{24EI}\left(\frac{l}{2}\right) - \frac{wl^3}{24EI}\left(\frac{3l}{16}\right) = \frac{wl^3}{24EI}\left(\frac{5l}{16}\right) = \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384}\frac{Wl^3}{EI}$$

The same result is obtained if we take the $\frac{M}{EI}$ diagram in two parts divided by the section at $\frac{l}{4}$ of the span (Fig. 102c). The maximum

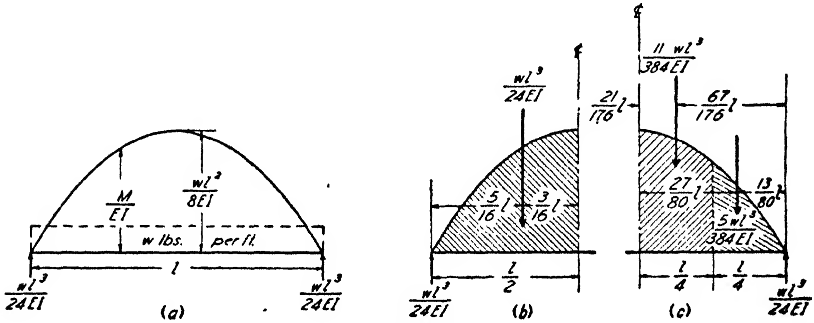


FIG. 102

deflection at the center section is the moment of two couples about the center line:

$$v_{\max.} = \frac{11}{384}\frac{wl^3}{EI}\left(\frac{67}{176}l\right) + \frac{5}{384}\frac{wl^3}{EI}\left(\frac{13}{80}l\right) = \frac{(67+13)wl^4}{384\times 16\ EI} = \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384}\frac{Wl^3}{EI}$$

This second analysis shows that $\frac{67}{80} = 0.84$ of the maximum deflection is due to the $\frac{M}{EI}$ loading in the middle half of the beam.

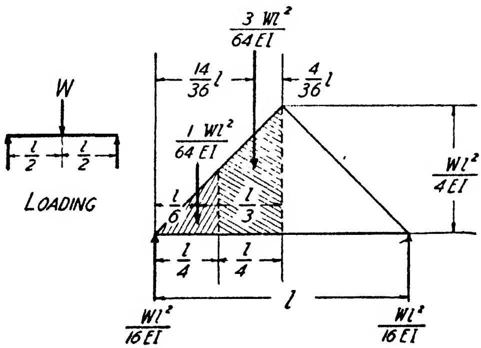
A similar analysis for the supported beam with a central concentrated load W gives a deflection (Fig. 103):

$$v_{\max.} = \frac{Wl^2}{64EI} \times \frac{l}{6} + \frac{3Wl^2}{64EI} \times \frac{14}{36} l = \frac{(2 + 14)Wl^3}{64 \times 12EI} = \frac{Wl^3}{48EI}$$

In this case $\frac{14}{16} = 0.875$ of the maximum deflection is due to the $\frac{M}{EI}$ diagram in the middle half of the beam.

Analyses of the cases for concentrated loads of W due to two or three intermediate beams show that the loading of the center half causes more than 80 per cent of the maximum deflection.

Therefore, since the center half of a reinforced concrete beam has a constant moment of inertia, and since more than 80 per cent of the maximum deflection is due to the $\frac{M}{EI}$ loading of this part of the beam, it seems reasonable to adopt the moment of inertia of the sections in positive bending for use in computations for maximum deflection.



$\frac{M}{EI}$ DIAGRAM

FIG. 103

This is admittedly not the average for the whole span, but the correction will be made from test data by the adoption of the suitable constant n' in the relation $E_s = n'E_c$.

164. Maximum Deflection. The maximum deflection of reinforced concrete beams will be computed by using equation 92:

$$v_{\text{max.}} = C_1 \frac{Wl^3}{E_c I_p} \tag{93}$$

where E_c = modulus of elasticity of the concrete in compression

I_p = moment of inertia of the transformed section for sections in positive bending. (The tension in the concrete is to be included between the neutral axis and the center of gravity of the tension steel.)

The values of the coefficient C_1 for the usual loadings are given in Table 2 in the Appendix.

165. Deflection of Rectangular Beams, Tension Steel Only. A general expression for the moment of inertia I_p for sections in positive bending can be deduced. The section, transformed to equivalent concrete, will consist of a concrete rectangle bd and equivalent concrete substituted for the steel in the amount $A_c = n'A_s = n'pbd$ (Fig. 104). As the tension in the concrete is also considered, the neutral axis will not be at a distance kd from the top given by the formulae of Chapter 2.

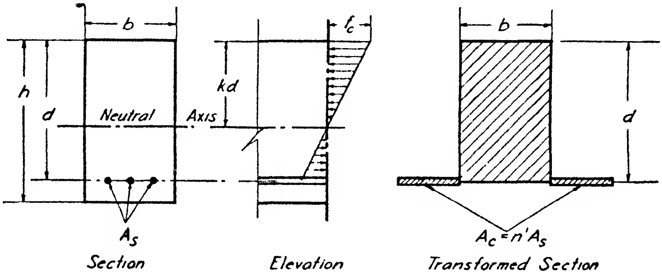


FIG. 104

The neutral axis is at the center of gravity of the transformed section and can be found by taking moments of the transformed area about the upper edge.

$$bd \times \frac{d}{2} + n'pbd \times d = (bd + n'pbd)kd$$

Dividing by bd^2 and assembling k terms,

$$k = \frac{1 + 2n'p}{2(1 + n'p)} \tag{94}$$

The moment of inertia of the concrete projections, substituted for the steel, about their center of gravity is so small that it will be neglected. The total moment of inertia about the neutral axis equals

$$I_p = \frac{b(kd)^3}{3} + \frac{b(d - kd)^3}{3} + n'pbd(d - kd)^2$$

$$I_p = \frac{bd^3}{3}[k^3 + (1 - k)^3 + 3n'p(1 - k)^2]$$

$$= \frac{bd^3}{3}[3(1 + n'p)k^2 - 3(1 + 2n'p)k + (1 + 3n'p)]$$

Substituting the value of k from equation 94 gives

$$I_p = \frac{(1 + 4n'p)}{(1 + n'p)}\frac{bd^3}{12} \tag{95}$$

ILLUSTRATIVE PROBLEM 20

166. Deflection of a Slab. Compute the deflection of the slab designed in Problem 1 (Chapter 3). The data deal with an interior span of a continuous slab. The greatest deflection will occur for the live-load arrangement that gives a maximum positive bending moment $M_p = \frac{wl^2}{16}$ and, hence, a minimum negative moment $M_n = -\frac{wl^2}{16}$.

Clear span $= 13$ ft. 6 in.

Slab thickness $t = 6$ in.

Depth to steel $d = 4.25$ in. for positive bending

$$\text{Steel ratio } p = \frac{A_s}{bd} = \frac{0.39}{12 \times 4.25} = 0.0077$$

Adopt a value of the constant $n' = 8$.

$$n'p = 8 \times 0.0077 = 0.0616$$

$$I_p = \frac{(1 + 4n'p)}{(1 + n'p)} \frac{bd^3}{12} = \frac{1.246 \times 12 \times (4.25)^3}{1.062 \times 12} = 90 \text{ (in.)}^4$$

The modulus of elasticity $E_c = \frac{E_s}{n'} = \frac{30{,}000{,}000}{8} = 3{,}750{,}000$ lb. per sq. in. The coefficient C_1 by Table 2, for a uniformly distributed load and an end restraint of $M_n = -\frac{wl^2}{16}$, equals $C_1 = 0.0052$.

$$v_{\max.} = \frac{C_1 W l^3}{E_c I_p} = \frac{0.0052(205 \times 13.5)(13.5 \times 12)^3}{3{,}750{,}000 \times 90}$$

$$\text{Maximum deflection } v_{\max.} = 0.18 \text{ in.}$$

If the maximum allowable deflection is taken as $\frac{l}{400} = \frac{13.5 \times 12}{400} = 0.41$ in., the slab is safe for deflection, even if $v_{\max.} = 0.18$ in. is somewhat approximate.

ILLUSTRATIVE PROBLEM 21

167. Deflection of a Rectangular Beam. Compute the deflection of the rectangular beam of Problem 8 (Chapter 5). The data deal with the interior span of a continuous beam. The greatest deflection occurs for the live-load arrangement that gives a maximum positive moment $M_p = \frac{wl^2}{16}$ with an accompanying negative value $M_n = -\frac{wl^2}{16}$. By Table 2 the coefficient $C_1 = 0.0052$.

Clear span $l = 26$ ft. 6 in.

Width of beam $b = 16$ in.

$$\text{Depth to steel } d = 29.12 \text{ in. for positive bending}$$

$$\text{Steel ratio } p = \frac{A_s}{bd} = \frac{3.00}{16 \times 29.12} = 0.00643$$

Adopting a value of the constant $n' = 8$,

$$n'p = 8 \times 0.00643 = 0.0515$$

$$I_p = \frac{1.206 \times 16 \times (29.12)^3}{1.05 \times 12} = 37{,}600 \text{ (in.)}^4$$

$$\text{Modulus of elasticity } E_c = \frac{E_s}{n'} = \frac{30{,}000{,}000}{8} = 3{,}750{,}000 \text{ lb. per sq. in.}$$

$$v_{\text{max.}} = \frac{C_1 W l^3}{E_c I_p} = \frac{0.0052(2705 \times 26.5)(26.5 \times 12)^3}{3{,}750{,}000 \times 37{,}600} = 0.085 \text{ in.}$$

If the maximum allowable deflection is taken as $\frac{l}{400} = 0.80$ in., the beam is safe for deflection.

168. Deflection of a Rectangular Beam with Compression Steel. The expression for the moment of inertia I_p of a rectangular beam with compression steel can be obtained by the use of the transformed section

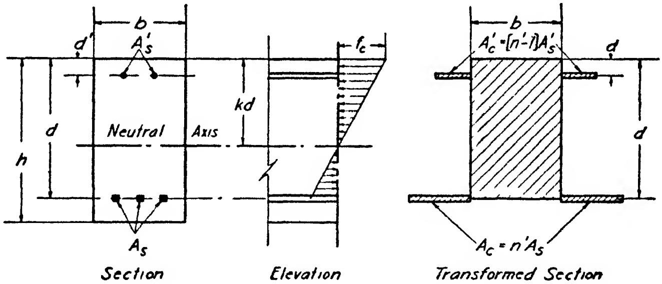


FIG. 105

(Fig. 105). This concrete area consists of a rectangle bd. The compression steel is pulled out and the holes filled with an equal area of concrete. The rest of the equivalent concrete for the compression steel, having an area $(n' - 1)p'bd$, is placed as a projection, or fin, at a distance d' from the upper edge. The equivalent concrete for the tension steel is placed as a fin, or projection, with an area of $n'pbd$ at a distance d from the extreme compression edge. To be strictly accurate, some of the equivalent concrete should be used to fill the semicircular holes left in the rectangle bd when the tension steel was removed. This precision is not necessary and has been neglected.

Including as before the tension in the concrete, the neutral axis is found by taking moments about the upper edge:

$$bd \times \frac{d}{2} + n'pbd \times d + (n' - 1)p'bd \times d' = [bd + n'pbd + (n' - 1)p'bd]kd$$

Dividing by bd^2 gives

$$\frac{1}{2} + n'p + (n' - 1)p'\frac{d'}{d} = [1 + n'p + (n' - 1)p']k$$

$$k = \frac{1 + 2n'p + 2(n' - 1)p'\dfrac{d'}{d}}{2[1 + n'p + (n' - 1)p']} \tag{96}$$

Neglecting again the moment of inertia of the projections about their own centers of gravity, we obtain for the moment of inertia about the neutral axis

$$I_p = \frac{b(kd)^3}{3} + \frac{b(d - kd)^3}{3} + n'pbd(d - kd)^2 + (n' - 1)p'bd(kd - d')^2$$

Expanding these terms does not simplify the expression, so we shall leave the moment of inertia as

$$I_p = \frac{bd^3}{3}\left[k^3 + (1 - k)^3 + 3n'p(1 - k)^2 + 3(n' - 1)p'\left(k - \frac{d'}{d}\right)^2\right] \tag{97}$$

ILLUSTRATIVE PROBLEM 22

169. Deflection of a Rectangular Beam with Compression Steel. The beam of Problem 8 actually has two $1\frac{1}{4}$-in. square bars on the compression side of the sections near the center of the span. If this steel is counted as part of the reinforcement, the beam will be somewhat stiffer and the computed deflection should be less. Using the data of Problem 21 with the addition of the compression steel,

$$\text{Compression steel ratio } p' = \frac{3.12}{16 \times 29.12} = 0.0067$$

$$(n' - 1)p' = 0.047$$

$$\text{Depth to compression steel } d' = 2.50 \text{ in.}$$

$$\text{The ratio } \frac{d'}{d} = \frac{2.50}{29.12} = 0.086$$

By equation 96,

$$k = \frac{1 + 2 \times 0.0515 + 2 \times 0.047 \times 0.086}{2(1 + 0.0515 + 0.047)} = 0.506$$

By equation 97, the moment of inertia equals

$$I_p = \frac{16 \times (29.12)^3}{3} [(0.506)^3 + (0.494)^3 + 3 \times 0.0515 \times (0.494)^2 + 3 \times 0.047 \times (0.420)^2]$$

$$I_p = \frac{16 \times 24{,}700 \times 0.313}{3} = 41{,}180 \text{ (in.)}^4$$

$$v_{\text{max.}} = \frac{0.0052(2705 \times 26.5)(26.5 \times 12)^3}{3{,}750{,}000 \times 41{,}180} = 0.078 \text{ in.}$$

This is a reduction of about 9 per cent in deflection due to the inclusion of the steel present on the compression side of the section.

170. Deflection of Tee Beams. The moment of inertia I_p for positive bending for a tee-shaped cross section can also be determined by the transformed section. Since the tension in the concrete is included, the transformed area has the shape shown in Figure 106.

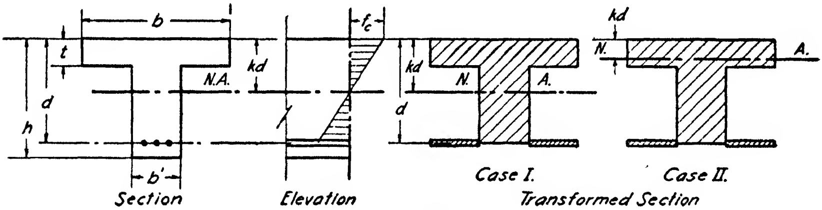


FIG. 106

The expression for the neutral axis ratio k can be found by taking moments of the areas about the upper edge.

$$bt \times \frac{t}{2} + b'(d - t) \times \left(t + \frac{d - t}{2}\right) + n'pbd \times d = [bt + b'(d - t) + n'pbd]kd$$

Multiply by $\dfrac{2}{bd^2}$:

$$\left(\frac{t}{d}\right)^2 + \frac{b'}{b}\left[1 - \left(\frac{t}{d}\right)^2\right] + 2n'p = 2k\left[\frac{t}{d} + \frac{b'}{b}\left(1 - \frac{t}{d}\right) + n'p\right]$$

$$k = \frac{\left(\frac{t}{d}\right)^2 + \frac{b'}{b}\left[1 - \left(\frac{t}{d}\right)^2\right] + 2n'p}{2\left[\frac{t}{d} + \frac{b'}{b}\left(1 - \frac{t}{d}\right) + n'p\right]} \qquad (98)$$

The moment of inertia about the neutral axis equals

$$I_p = \frac{b(kd)^3}{3} - \frac{(b - b')(kd - t)^3}{3} + \frac{b'(d - kd)^3}{3} + n'pbd(d - kd)^2$$

$$I_p = \frac{bd^3}{3}\left[k^3 - \left(1 - \frac{b'}{b}\right)\left(k - \frac{t}{d}\right)^3 + \frac{b'}{b}(1 - k)^3 + 3n'p(1 - k)^2\right] \qquad (99)$$

These equations can be used if the neutral axis is in the stem or the flange.

A.C.I. Article 705 permits the use of a flange width $b = (16t + b')$ at the section of maximum bending moment, providing the span and spacing do not limit this width. However, the flange width for deflection should be representative of the whole beam span. Conservative designers tend to limit the flange width to $b = (12t + b')$, or even $b = (8t + b')$.

ILLUSTRATIVE PROBLEM 23

171. Deflection of a Tee Beam. Compute the deflection of the intermediate beam in Problem 17 (Chapter 7).

Intermediate Beam. This is an interior span of a beam supported by girders. The loading of Figure 75 (Art. 130) necessitates a special computation for the coefficient C_1.

From Figure 75, for $x = 0$ to $x = 1.83$ ft.,

$$M = 9140x - \frac{110x^2}{2} + M_n \qquad (100)$$

$$EI\frac{dv}{dx} = \frac{9140x^2}{2} - \frac{110x^3}{6} + M_nx + c_1 \qquad (101)$$

$$EIv = \frac{9140x^3}{6} - \frac{110x^4}{24} + \frac{M_nx^2}{2} + c_1x + c_2 \qquad (102)$$

For $x = 1.83$ ft. to $x = 6.67$ ft. (center line),

$$M = 9140x - \frac{110x^2}{2} - \frac{1740}{2}(x - 1.83)^2 + M_n \qquad (103)$$

$$EI\frac{dv}{dx} = \frac{9140x^2}{2} - \frac{110x^3}{6} - \frac{1740}{6}(x - 1.83)^3 + M_nx + c_3 \qquad (104)$$

$$EIv = \frac{9140x^3}{6} - \frac{110x^4}{24} - \frac{1740}{24}(x - 1.83)^4 + \frac{M_nx^2}{2} + c_3x + c_4 \qquad (105)$$

To evaluate the constants of integration:

When $x = 1.83$ equation 101 = equation 104 and $c_1 = c_3$

When $x = 6.67$ $\quad \frac{dv}{dx} = 0 \quad$ and $c_3 = -164{,}940 - 6.67M_n$

When $x = 0$ $\quad v = 0 \quad$ and $c_2 = 0$

When $x = 1.83$ equation 102 = equation 105 and $c_4 = 0$

The maximum deflection occurs when $x = 6.67$ ft. (equation 105):

$$EIv_{\text{max.}} = -697{,}000 - 22.22M_n$$

For maximum positive bending there is a corresponding minimum negative moment of

$$M_n = -19{,}700 \text{ ft.-lb. (Fig. 77)}$$

and

$$v_{\text{max.}} = -\frac{259{,}200}{EI} = -\frac{C_1Wl^3}{EI} = -\frac{C_1 \times 18{,}280 \times (13.33)^3}{EI}$$

$$C_1 = 0.00598$$

Iti s also known that

Clear span	= 13 ft. 4 in.
Thickness of slab t	= 4 in.
Flange width b	= 40 in. This happens to correspond to $b = (b' + 8t)$, which is conservative (see Art. 170).
Stem width b'	= 8 in.
Depth to tension steel d	= 14.37 in.
Tension steel ratio p	$= \frac{A_s}{bd} = \frac{0.88}{40 \times 14.37} = 0.00153$

Adopting a value of the constant $n' = 8$,

$$n'p = 0.00153 \times 8 = 0.0122$$

Also $\frac{t}{d} = \frac{4}{14.37} = 0.278$ and $\frac{b'}{b} = \frac{8}{40} = 0.20.$

$$k = \frac{(0.278)^2 + 0.20[1 - (0.278)^2] + 2 \times 0.0122}{2[0.278 + 0.20(1 - 0.278) + 0.0122]} = 0.329$$

$$I_p = \frac{40(14.37)^3}{3}[(0.329)^3 - (1 - 0.20)(0.329 - 0.278)^3 + 0.20(1 - 0.329)^3 + 3 \times 0.0122 \times (1 - 0.329)^2]$$

$$I_p = 39{,}580 \times 0.1124 = 4450 \text{ (in.)}^4$$

Modulus of elasticity $E_c = \frac{E_s}{n'} = 3{,}750{,}000$ lb. per sq. in.

$$v_{\text{max.}} = \frac{C_1Wl^3}{E_cI_p} = \frac{0.00598 \times 18{,}280 \times (13.33 \times 12)^3}{3{,}750{,}000 \times 4450} = 0.027 \text{ in.}$$

If the maximum allowable deflection is taken as $\frac{l}{400} = \frac{13.33 \times 12}{400} = 0.40$ in., the actual deflection is a safe value.

The deflection of the cross beam will be less than this amount as its span is less and its ends are more rigidly restrained.

TORSIONAL STRESSES IN BEAMS

Members in bending may also be subjected to torsional couples. Wall beams and beams framing about openings receive their loads wholly from one side of the beam axis with tendencies to rotate as well as bend. Concentrated loads may also be applied to produce a considerable torsional moment.

The discussion of moment distribution in Chapter 14 shows that the torsional stiffness of the usual reinforced concrete beam section is very small compared with the bending stiffness, so that a negligible amount of torsional moment will be transmitted to a given beam span by its action as part of a continuous frame. In general, then, it is sufficient to investigate torsional stresses due to the loads brought directly to the member.

172. Torsional Stresses in Plain Concrete. *Circular Sections.* A member of circular section subjected to a torsional couple T will have shear stresses on the cross section whose magnitudes are

$$v = \frac{2Tr}{\pi R^4} \tag{106}$$

where R = outside radius of the section
r = distance from center of section to the particle considered.

The maximum shear stress occurs on the particles on the outside perimeter and equals

$$v_{\text{max.}} = \frac{2T}{\pi R^3} \tag{107}$$

If this member is loaded to destruction, failure does not occur on the cross section but on a plane inclined 45° with the cross section. This failure is due to excessive tensile stresses. By the theory of elasticity the maximum tensile stress t on a given particle equals

$$t = \frac{f}{2} + \frac{1}{2}\sqrt{f^2 + 4v^2} \tag{108}$$

where f = fiber stress on the cross-sectional plane through the particle.

This tensile stress occurs on a plane making an angle with the cross section, where

$$\tan 2\alpha = \frac{2v}{f} \tag{109}$$

If only the effect of the torsional couple is considered, there is no fiber stress and the maximum tensile stress equals

$$t = v \tag{110}$$

Then, $\tan 2\alpha = \dfrac{2v}{0} = \infty$ and $\alpha = 45°$. For plain concrete, the tensile stress t will determine the torsional moment carried at failure.

Square Section. A plain concrete member with a square cross section will have maximum shear stresses on the particles in the cross section which are on the outer perimeter at the ends of the axes of symmetry. This maximum shear stress equals

$$v_{\text{max.}} = \frac{4.8T}{b^3} \tag{111}$$

where b = side dimension of the square sections.

If the maximum stresses on the circular and square sections are equated,

$$b^3 = \frac{4.8\pi}{2} R^3 = 7.54R^3$$

$$b = 1.96R = 0.98D$$

where D = diameter of the cylinder.

The lower shear stresses elsewhere in the square section are not easily computed. As far as stresses are concerned the square section can be closely approximated by using the inscribed circle for computations.

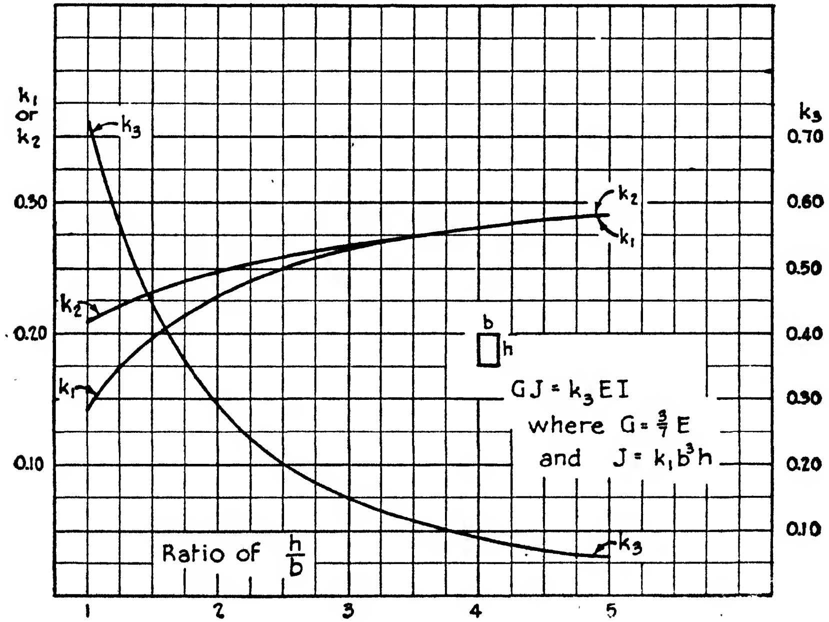


FIG. 107

Rectangular Section. The maximum shear stress due to torsion for a rectangular section occurs at the outside particle on the long side h at the end of the short axis of symmetry. Its magnitude [2] is

$$v_{\text{max.}} = \frac{T}{k_2 b^2 h} \tag{112}$$

where b = short side of rectangle
h = long side of rectangle
k_2 = numerical term depending on ratio $\frac{h}{b}$ (see Fig. 107).

For a square section, $k_2 = 0.208$, $b = h$, and equation 112 simplifies to equation 111.

173. Torsion Reinforcement. *Circular Section.* The most efficient torsional section is circular. Though this is not a common structural shape the derivation of the amount of torsional reinforcement will first be made for this shape.

It has been shown that the maximum tensile stress on a 45° plane equals the maximum shear stress due to torsion on the cross section. If the torsional moment produces tensile stresses greater than the allowable tension of the concrete used, torsional reinforcement must be used. The theory [3] makes use of the same procedure employed for the design of web steel to resist diagonal tension stresses in a beam (Art. 61). The portion of the section where the concrete tensile stresses are less than the allowable is considered safe and the torsion reinforcement is designed to take the excess tension above the allowable for the portion of the section near the perimeter.

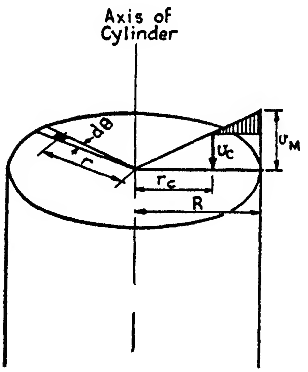


FIG. 108

In Figure 108 the shear stress is plotted on a radius. On each particle the principal tensile stress on a 45° plane equals the shear stress on the cross section. Let

v_m = maximum shear stress at the perimeter, R in. out
v = shear stress on a particle r in. out
v_c = shear stress on the particle r_c in. out, whose principal tensile stress t_c equals the allowable tension in the concrete

[2] *Theory of Elasticity,* Timoshenko.
[3] "Experiments with Concrete in Torsion," Paul Andersen, *Trans. A.S.C.E.,* 1935, p. 949.

If we consider as a rigid body a particle whose area in the cross section is $r\,dr\,d\theta$ (Fig. 109), and bounded by the two principal planes at 45°, this body is in equilibrium under the following forces:

$vr\,dr\,d\theta$ on the cross section
$tr\cos 45°\,dr\,d\theta$ on the principal tensile plane
$cr\cos 45°\,dr\,d\theta$ on the principal compression plane

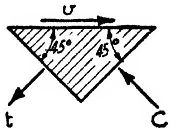


FIG. 109

also

$v = t = c$ in magnitude

Wherever t exceeds the allowable tensile stress t_c, the excess tensile force on the principal tensile plane equals

$$0.707(t - t_c)r\,dr\,d\theta = 0.707(v - v_c)r\,dr\,d\theta$$

The total excess tensile force equals

$$0.707\int_0^{2\pi}\int_{r_c}^{R}(v - v_c)r\,dr\,d\theta$$

This excess tensile force can be resolved into two components, one parallel to the cross section and perpendicular to the radius, the other parallel to the axis of the cylinder. Each component is equal to the total excess tension times the sine or cosine of 45°. The axial component has no moment about the axis of the cylinder. The cross-sectional component has a moment about this axis of

$$(0.707)^2\int_0^{2\pi}\int_{r_c}^{R}(v - v_c)r^2\,dr\,d\theta$$

By Figure 108, $\dfrac{v_m}{v} = \dfrac{R}{r}$, or $v = v_m\dfrac{r}{R}$.

If the reinforcing steel is placed at 45° with the cross section and acts perpendicular to the principal tensile plane at a distance r_s from the cylinder axis, its moment about the axis equals

$$f_sA_s\cos 45°\,r_s$$

where A_s = the area of the reinforcement perpendicular to its axis. Equating these moments,

$$0.707f_sA_sr_s = (0.707)^2\int_0^{2\pi}\int_{r_c}^{R}\left(\frac{v_mr - v_mr_c}{R}\right)r^2\,dr\,d\theta$$

$$f_sA_sr_s = \frac{0.707 \times 2\pi v_m}{R}\left[\frac{R^4 - r^4_c}{4} - \frac{r_c}{3}(R^3 - r^3_c)\right]$$

$$f_sA_sr_s = \frac{1.414\pi v_m}{12R}[3R^4 - 4r_cR^3 + r^4_c]$$

or, substituting $r_c = \dfrac{v_c}{v_m} R = \dfrac{t_c}{v_m} R$,

$$f_s A_s r_s = 0.370 \left(\frac{R}{v_m}\right)^3 [3v^4_m - 4v^3_m t_c + t^4_c]$$

Let the reinforcement be supplied in the form of 45° spirals whose longitudinal pitch equals p. Then

$$A_s = \frac{2\pi r_s}{p} a_s$$

where a_s = cross-sectional area of one wire.

$$\frac{a_s}{p} = \frac{1}{17 f_s r_s^2} \left(\frac{R}{v_m}\right)^3 [3v^4_m - 4v^3_m t_c + t_c^4] \tag{113}$$

Equation 113 can be used to select a spiral size to give a reasonable pitch. Figure 110 shows a member of circular section subjected to a

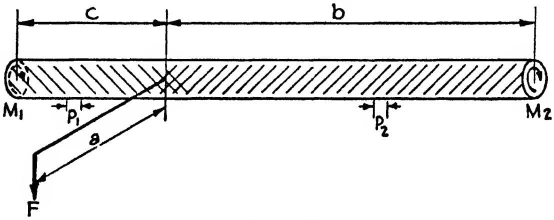


FIG. 110

torsional moment $T = Fa$. The supporting couples must be of such a magnitude that they produce the same angle of twist at the loaded section. Therefore,

$$\frac{M_1}{M_2} = \frac{b}{c} \tag{114}$$

The spiral reinforcement can be figured for each supporting couple and is placed as shown in order to be perpendicular to possible tension cracks in the concrete.

174. Allowable Tensile Stresses. Diagonal tension can be computed for a member in flexure by equation 108. The principal tensile stress will be somewhat greater than the shear stress on the cross section and makes a variable angle with the cross section (equation 109). Diagonal tension in beams is only of importance when the angle α approaches 45°, that is, when the fiber stress is comparatively small (Art. 61). Diagonal tension reinforcement for beams is satisfied by means of shear stress computations and the building codes give allowable shear stresses v_m and

v_c which insure safe diagonal tension stresses. These allowable shear values should be less than the permissible tension in concrete.

For members in torsion the principal tension equals the shear stress (equation 110). Therefore the allowable values of the shear stresses v_m and v_c can be taken somewhat greater than the allowable values for beams given in building codes. Code values are certainly conservative ones to use.

175. Torsion Reinforcement in a Square Section. It has been shown, in Article 172, that the maximum torsional shear stress on a square section is closely equal to the stress on the inscribed circular section. It is rather improbable that a square helix would be used as reinforcement, as that would probably involve welding short bars at an angle of 45° to the longitudinal steel. If such reinforcement is used, allowance must be made for the variation in position of the bars (dimension r_s) when taking the moment of this reinforcement about the central axis for the derivation of equation 113. It is much easier both for design and construction to use spiral reinforcement.

Rectangular Section. Examination of equation 112 shows that, as long as a rectangular cross-sectional area is kept constant, the maximum shear stress increases as the ratio of depth h to width b increases. The square section is the logical shape to use if torsional moments are large enough to require torsion reinforcement. It is also difficult to supply torsional reinforcement for a section that is markedly rectangular.

ILLUSTRATIVE PROBLEM 24

176. Design of Torsion Reinforcement. Figure 111 shows a beam subjected to a load of 50,000 lb. which is 10 in. distant from the center line of the member. The

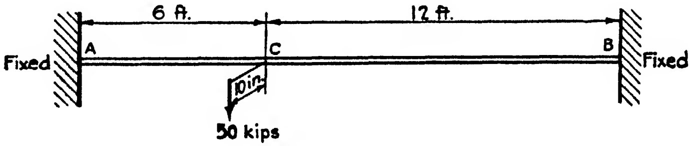


FIG. 111

ends are assumed to be fixed. The load can be transferred to the center line; this will give a downward load at C and a torsional moment of 500,000 in.-lb.

Bending. Considering first the effect of the load at C, the fixed-end bending moments equal (Table I, Art. 341)

$$M_A = \frac{50{,}000 \times (12)^2 \times 6 \times 12}{(18)^2} = 1{,}600{,}000 \text{ in.-lb.}$$

$$M_B = \frac{50{,}000 \times 12 \times (6)^2 \times 12}{(18)^2} = 800{,}000 \text{ in.-lb.}$$

The supporting forces are $V_A = 37{,}000$ lb. and $V_B = 13{,}000$ lb. Let us assume the beam to be 22 in. square. The fixed-end moments due to the beam weight of 500 lb. per ft. equal

$$M_A = M_B = \frac{wl^2}{12} = \frac{500 \times (18)^2 \times 12}{12} = 162{,}000 \text{ in.-lb.}$$

Additional supporting forces $= V_A = V_B = 4500$ lb.

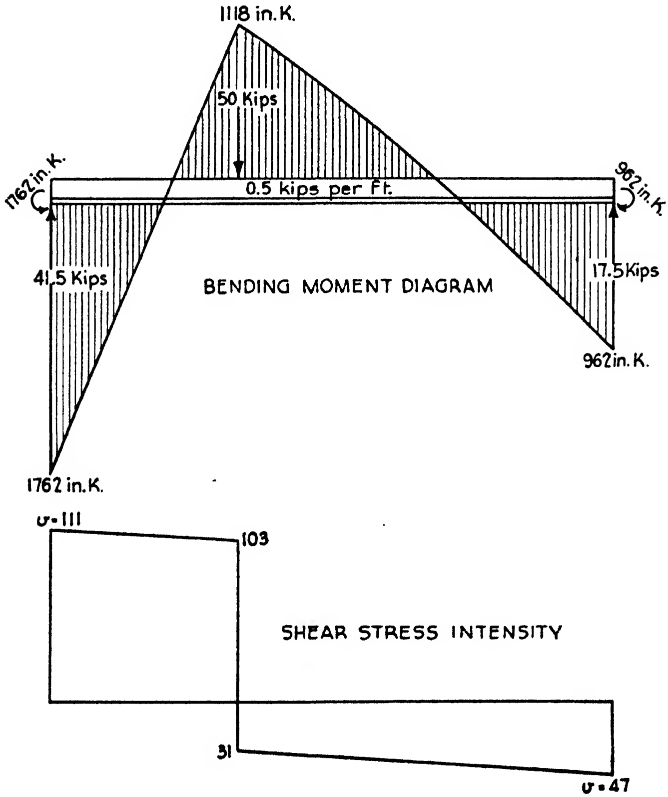


FIG. 112

The combined bending moment diagram is given in Figure 112. Assuming a concrete strength $f'_c = 3000$ lb. per sq. in., with $f_c = 1350$ lb. per sq. in. and $f_s = 20{,}000$ lb. per sq. in., the depth to the steel equals

$$\text{Minimum } d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{1{,}762{,}000}{123 \times 22}} = 18.5 \text{ in.}$$

Total depth $h = 18.5 + 1.5 + 0.5 + 0.44 = 20.94$ **in. Use** $h = 22$ **in.**

$$\text{Maximum shear stress } v_A = \frac{V}{bjd} = \frac{41{,}500}{22 \times 0.87 \times 19.56} = 111 \text{ lb. per sq. in.}$$

$$\text{Positive steel } A_p = \frac{1{,}118{,}000}{20{,}000 \times 0.87 \times 19.56} = 3.38 \text{ sq. in.}$$

Use six $\frac{7}{8}$-in. round bars. Run two into each support.

Negative steel.

$A_A = 5.29$ sq. in. Use nine $\frac{7}{8}$-in. round bars

$A_B = 2.88$ sq. in. Use five $\frac{7}{8}$-in. round bars

Torsion. The torsional couple of 500,000 in.-lb. is supported by end torsional moments (equation 114).

$$\frac{T_A}{T_B} = \frac{12}{6} \quad \text{or} \quad T_A = 333{,}000 \text{ in.-lb.} \quad \text{and} \quad T_B = 167{,}000 \text{ in.-lb.}$$

The maximum torsional shear stresses at each support equal

$$v_A = \frac{4.8T}{(b)^3} = \frac{4.8 \times 333{,}000}{(22)^3} = 150 \text{ lb. per sq. in.}$$

$$v_B = 75 \text{ lb. per sq. in.}$$

In each section, at one end of the horizontal axis of symmetry the shear stresses due to bending and torsion act in the same direction. The resultant vertical shear stress on this particle equals

$$v_A = 111 + 150 = 261 \text{ lb. per sq. in.}$$

$$v_B = 47 + 75 = 122 \text{ lb. per sq. in.}$$

When special anchorage is used, the maximum allowable shear stress $v = 0.12 \times 3000 = 360$ lb. per sq. in. The maximum allowable shear stress, used as a measure of tensile stress, $v_c = 0.03 \times 3000 = 90$ lb. per sq. in. At section A let us divide this shear equivalent between bending and torsion computations in proportion to their maximum stresses.

For bending, $v_c = \frac{111}{261} \times 90 = 38$ lb. per sq. in.

For torsion, $v_c = \frac{150}{261} \times 90 = 52$ lb. per sq. in.

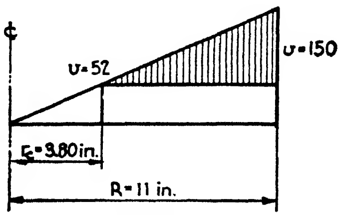


FIG. 113

For torsion reinforcement (Fig. 113), $R = 11$ in., $r_c = 3.80$ in., and $r_s = 8.1$ in. By equation 113,

$$\frac{a_s}{p} = \frac{1}{17 \times 20{,}000 \times (8.1)^2}\left(\frac{11}{150}\right)^3 [3(150)^4 - 4 \times (150)^3 \times 52 + (52)^4] = 0.0146$$

With $\frac{1}{4}$-in. round spirals the pitch figures 3.4 in. One spiral unit makes a complete turn in $2\pi 8.1 = 51$ in. Use 17 spiral units to give a pitch of 3 in. from A to C. Between B and C the pitch for $\frac{1}{4}$-in. spirals figures about 31 in. but five spiral units will be used to give a pitch of 10.2 in. This is less than the maximum spacing of half the depth, or 11 in.

Stirrups. Using $\frac{1}{2}$-in. round bars, stirrups will be spaced at 4 in. from A to C and at 10-in. spacing from C to B. The sketch of the reinforcement is given in Figure 114.

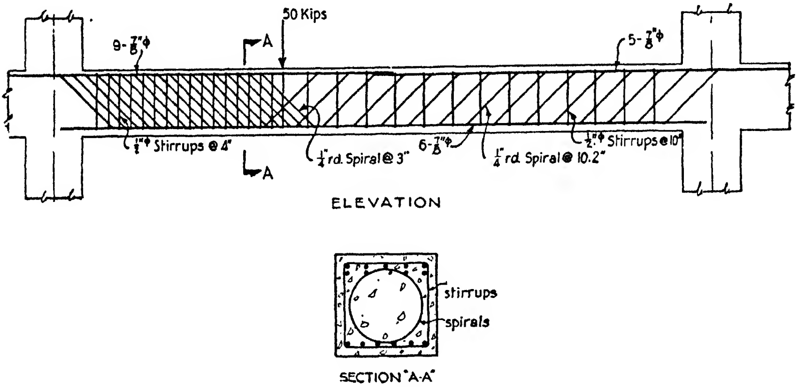


FIG. 114

ILLUSTRATIVE PROBLEM 25

177. Wall Girder. A common example of direct torsional loading of beams is that of spandrel beams at the outside wall of a structure, or beams framing around an opening. These beams receive a load from one side only of their axis. A concrete floor system provides such beams with a flange on one side only, which is not a symmetrical section. However, such beams are designed as an angle section for bending computations. It has been common office practice to neglect the torsional moments and stresses but these may often be of considerable magnitude, especially as the section is not the most economical for resisting torsion. In this problem allowance will be made for torsional stresses. The computations must be regarded as approximate because the presence of the slab is a restraint of unknown amount against twisting of the beam. For this reason, the section will be regarded as rectangular for torsional computations and the computed stresses are probably in excess of actual values.

Design the wall girder parallel to the interior girder EH of Figure 61. In addition to the floor loads this girder carries a curtain wall weighing 80 lb. per ft. of length but the steel window frame is supported by the columns. Let us assume that the design for bending results in a girder stem 12 in. by 22 in. (26 in. total depth) and that the maximum shear stress due to bending equals 80 lb. per sq. in. at the face of the support. It is desired to compute the additional shear stresses due to torsion.

The intermediate beams that are supported by this girder were designed for a minimum negative moment at this exterior support of $M_n = -\dfrac{wl^2}{24}$. This moment was assumed in order that a reasonable amount of negative tension steel may be placed in the top of the cross beam at its support. We have no assurance that the wall girder supplies the restraint to develop this moment. It will now be advantageous to estimate the restraint, at least approximately. The procedure will be that discussed in Article 374.

Wall Girder RS (Fig. 115). If Poisson's number $m = 6$, $G = \frac{3}{7}E$, while $J = k_1 b^3 h$. The wall girder has dimensions $b = 12$ in. and $h = 26$ in., so $k_1 = 0.236$ from Figure 107. Using center line dimensions, $RM = MP = PS = 116$ in. and

$$K_{MR} = K_{MP} = K_{PS} = K_T = \frac{3}{7} E \times \frac{0.236 \times (12)^3 \times 26}{116} = 39E$$

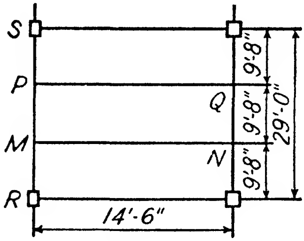


FIG. 115

Beams MN and PQ. The intermediate beam stems are 8 in. by 13 in. with a 4-in. flange 40 in. wide. Using the gross section the moment of inertia about its center of gravity $I = 6240$ (in.)^4. The bending stiffness equals

$$K_{MN} = K_{PQ} = K_B = \frac{4EI}{l} = \frac{4 \times 6240E}{14.5 \times 12} = 143E$$

The fixed end moment at M or P equals 24,800 ft. lb. (Art. 130). Assuming fixity at S, R, N, and Q, the method of moment distribution (Art. 361 *et seq.*) can be used to determine the moments in the wall girder as joints M and P are released.

MOMENT DISTRIBUTION AT JOINTS M AND P

Joint	R	M			P			S
Member	RM	MR	MN	MP	PM	PQ	PS	SP
Carry-over factor		-1	$+0.5$	-1	-1	$+0.5$	-1	
Stiffness K		$39E$	$143E$	$39E$	$39E$	$143E$	$39E$	
Ratio $\frac{K}{\Sigma K}$	0	0.177	0.646	0.177	0.177	0.646	0.177	0
F.E.M. (ft. k.)			-24.8			-24.8		
Distribute		$+4.4$	$+16.0$	$+4.4$	$+4.4$	$+16.0$	$+4.4$	
Carry over	-4.4			-4.4	-4.4			-4.4
Distribute	0	$+0.7$	$+3.0$	$+0.7$	$+0.7$	$+3.0$	$+0.7$	0
Carry over	-0.7							-0.7
Final Moment	-5.1	$+5.1$	-5.8	$+0.7$	$+0.7$	-5.8	$+5.1$	-5.1

It is sufficiently accurate to assume the wall girder ends to be fixed if the girder is continuous and frames into a sizeable wall column, but a more accurate solution could be made by applying moment distribution to a bent consisting of the continuous intermediate beam right across the building with the far ends of upper and lower columns assumed fixed (Art. 355).

Using the results given above, the end moment in the beams MN and PQ are 0.234 the fixed end moment, which is much less than $\frac{wl^2}{24} = 0.5$ F.E.M. used to compute the negative tension steel at M and P.

The maximum torsional shear stress at half depth of the wall girder at the support R, or S, can be figured by equation 112, where $k_2 = 0.252$ is obtained from Figure 107 for the ratio $\frac{h}{b} = 2.17$.

$$v_T = \frac{5100 \times 12}{0.252 \times (12)^2 \times 26} = 65 \text{ lb. per sq. in.}$$

This shear stress acts in the same direction as the shear due to bending on the inside face of the section. The total shear at half depth equals $v = 65 + 80 = 145$ lb. per sq. in. This shear is a measure of diagonal tension and reinforcement can be supplied by stirrups without regard to possible bends of the tension bars.

ILLUSTRATIVE PROBLEM 26

178. Wall Beam. Design the wall beam PQ of Figure 61. This beam acts as end support for the slab whose one-way steel runs into this beam. The beam will be considerably deeper than the slab and it is certainly conservative to assume a moment of $-\frac{wl^2}{24}$ in the slab at this support. Using center-to-center dimensions for the slab and referring to slab loads (Art. 129) the end moment equals

$$M_n = \frac{wl^2}{24} = \frac{180 \times (9.67)^2 \times 12}{24} = 8420 \text{ in.-lb. per ft.}$$

The torsional moment at the support of the beam PQ will be taken as the summation of these slab moments for the half span, namely,

$$M_T = 8420 \times 6.58 = 55{,}500 \text{ in.-lb.}$$

Following the procedure of Problem 25, the section of beam PQ will be determined by the usual procedure for bending loads but the diagonal tension reinforcement can be increased by adding the shear stress v_T due to torsion to the shear stress v_B due to bending. In the past most designers have neglected to make torsional computations, in view of the difficulty of ascertaining true torsional moments. They have made the section slightly larger than needed for bending computations. The fact that few cracks in wall beams or girders can be assigned to overstress due to torsion indicates that the approximate estimates made here are conservative.

CHAPTER 9

SHRINKAGE, FLOW, AND PRESTRESS

The beam theory presented in previous chapters for working-stress straight-line assumptions deals only with the elastic strains produced instantaneously by application of the live or dead load. In several places mention has been made that other strains of considerable magnitude are produced in concrete by temperature variation and by time effects known generally as *shrinkage* and *plastic flow.*

Since 1916 these additional strains have been increasingly investigated. There is not yet a complete agreement in definition of these terms nor in explanation of the observed strains. For the purposes of the discussion in this chapter the following definitions of these terms will be used.

Shrinkage of concrete is the volume change due to drying out of the cement gel between the aggregate particles. There may also be some shrinkage of the aggregate itself. The unit shrinkage strain is determined by tests of plain or reinforced concrete on unloaded specimens.

Plastic flow or *creep* will be regarded as the additional volume changes of loaded specimens obtained by subtracting from the total strain (time $= t$) the elastic strain due to load application (time $= 0$) and the shrinkage strain as defined above. The flow strain may be caused by the sustained load or by non-uniform shrinkage due to unequal drying out of thick concrete members; but for convenience all these interrelated effects will be denoted plastic flow. A concise explanation of plastic flow or creep is that given by Lorman:[1]

> It has been suggested that creep of concrete may involve all three of the following types of yielding: (*a*) crystalline flow (in a crystalline mass, slippage along planes within the crystals); (*b*) seepage (due to applied pressure, flow of adsorbed water from the cement gel); and (*c*) viscous flow (movement of particles, as in the flow of asphalt). A portion of the creep possibly may be due to crystalline or viscous flow; nevertheless, it is believed that the major portion is caused by seepage, which would appear to be the most acceptable explanation of creep. The hydration of portland cement results in the formation of an amorphous or gelatinous mass, ordinarily termed "gel," which serves to connect the aggregate particles. Water may exist in the concrete mass in three principal

[1] "The Theory of Concrete Creep," by W. R. Lorman, *Proc. A.S.T.M.*, 1940, p. 1082.

forms: (*a*) chemically-combined water (in chemical combination with the cement), (*b*) adsorbed water (adsorbed by the cement gel), and (*c*) free water (water within the microscopic pores or spaces between the gel particles). According to Lynam, chemically-combined and free water play no direct part in volume changes. Thus, except for the effect of hydration, gain or loss of adsorbed water from the gel appears to be the basis of volume changes resulting from ambient moisture variations or from sustained pressure. The gel may be considered as having microscopic pores; with the removal of water the pore spaces collapse and the gel shrinks, while upon the addition of moisture the pore spaces adsorb water and the gel expands. This process is dependent upon frictional resistance to flow of water along the capillary channels which permeate the mass of concrete. Other things being equal, the total frictional resistance is governed by the moisture gradient. The steeper the moisture gradient, the easier the flow of water through the capillary channels. Volume change of the gel may, on the other hand, be dependent upon seepage caused by applied external pressure. Subjecting the concrete to an external load, the adsorbed water is expelled from the gel. The rate of expulsion of moisture in this instance is a function of the applied load and of the friction in the capillary channels. The greater the applied load, the steeper the pressure gradient with consequent increase in rate of moisture expulsion. By the foregoing hypothesis shrinkage or swelling due to loss or gain of moisture and creep due to seepage are interrelated phenomena. Despite this relation the two are considered separately.

SHRINKAGE

179. Shrinkage. The drying out of concrete varies with time and with the exposure of the member. Structures in regions of low humidity, such as Arizona, or members located in a heated building will attain maximum shrinkages. Members exposed to high humidities will have little shrinkage, whereas those under water may even expand. Concrete that is alternately wet and dried will have alternate expansion and contraction of volume but the net result at the end of each cycle is usually a residual shrinkage.

Lean mixes have less shrinkage than rich. Mixes of high water-cement ratio shrink more than dry mixes if exposed in air, but they expand less under water. Large members will dry out more rapidly at their surfaces and non-uniform shrinkage may produce unequal strains and warping of the member.

If a member drys out at constant temperature and humidity the shrinkage strains increase rapidly at once and then at a decreasing rate for several years. The variation of strain against time gives a plot similar in shape to that of plastic flow strains in Figure 118.

180. Shrinkage of Symmetrically Reinforced Members. A plain concrete specimen will shrink as it dries out but there will be no stress in the member due to shrinkage. Steel does not shrink as the concrete dries out, so its restraint tends to reduce the volume change of reinforced concrete. After a certain time the concrete will have tensile stresses and the steel will be in compression.

Assuming perfect bond between the steel and concrete these stresses can be computed for members, such as columns, whose steel is symmetrically placed in the cross section. It is understood that only the effect of shrinkage is considered.

If the shrinkage strain is s during the time $t = 0$ to $t = t_1$, a plain concrete member of length l will shorten sl. The action on the reinforced member can be visualized as a shortening of the concrete due to shrinkage plus a partial pull back by the steel. The final length of the steel and concrete must be the same (Fig. 116). Equating the strains,

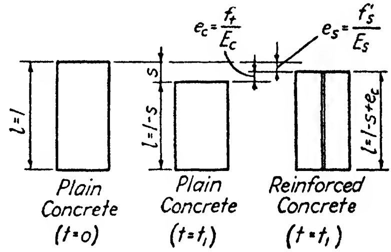


FIG. 116

$$\frac{f'_s}{E_s} = s - \frac{f_t}{E_c} \tag{115}$$

where f'_s = compressive stress in the steel
f_t = tensile stress in the concrete.

The stresses at a given section must be in equilibrium.

Equating the compressive force on the steel to the tension in the concrete

$$f'_s pA = f_t(1 - p)A \tag{116}$$

Solving for f'_s in equation 116 and substituting in equation 115,

$$f_t = \frac{np}{1 + (n - 1)p} sE_c \tag{117}$$

Similarly,

$$f'_s = \frac{1 - p}{1 + (n - 1)p} sE_s \tag{118}$$

The above discussion is representative of constant conditions for nearly the full length of the member. Near the ends the stress f'_s in the steel must be transferred back to the concrete before the bar ends. The pressure on the end of the bar plus the friction between concrete and the steel surfaces will both be included in the bond resistance. By equation 28 the length of imbedment of the bars, a distance in which the shrinkage equations do not apply, is

$$l = \frac{f'_s}{4u} a \tag{119}$$

The shrinkage discussion applies also to columns with spiral steel reinforcement, since tests show that the spiral steel is not stressed until well beyond working loads. The shrinkage of symmetrically reinforced members whose cross sections vary in thickness, or those of curved center line, is discussed in Article 391.

ILLUSTRATIVE PROBLEM 27

181. Shrinkage Stresses in a Column. Let us assume the designs of the rodded column of Problem 38 (Chapter 11) for a concrete strength of $f'_c = 3000$ lb. per sq. in. Assume it to be in a heated building and to have a shrinkage strain $s = 0.0004$ at the end of two years.

Plain Concrete. If this column were of plain concrete and its ends were fully restrained by the floor system above and below, the tensile stress in the concrete due to shrinkage would equal

$$f_t = sE_c = 0.0004 \times 3{,}000{,}000 = 1200 \text{ lb. per sq. in.}$$

This concrete would probably crack in tension at stresses around 300 lb. per sq. in. The compressive stresses due to the dead load would not be great enough to reduce the tensile stress from 1200 to 300 lb. per sq. in.; so this column should have tension cracks due to shrinkage some time before the two-year age. In practice, however, the other columns on this floor also shrink and the upper floor moves downward; in other words, its ends are not fully restrained.

Reinforced Section. Low Percentage of Steel. The design of this column for a low percentage of steel gives

Square section $t = 21$ in. and steel ratio $p = 0.0109$

By equations 117 and 118,

$$f_t = \frac{10 \times 0.0109}{(1 + 9 \times 0.0109)} \times 0.0004 \times 3{,}000{,}000 = 119 \text{ lb. per sq. in. (tension)}$$

$$f'_s = \frac{(1 - 0.0109)}{(1 + 9 \times 0.0109)} \times 0.0004 \times 30{,}000{,}000 = 10{,}800 \text{ lb. per sq. in. (compression)}$$

The concrete stress of 119 lb. per sq. in. will be reduced by the stresses due to the live and dead loads and by plastic flow; so there should be no shrinkage cracks. The steel stresses due to the loads will be materially increased by the shrinkage stresses.

Reinforced Section. High Percentage of Steel. The design for a high percentage of steel gives

Square section $t = 17$ in. and steel ratio $p = 0.0351$

$$f_t = \frac{10 \times 0.0351}{1.316} \times 1200 = 320 \text{ lb. per sq. in. (tension)}$$

$$f'_s = \frac{(1 - 0.0351)}{1.316} \times 12{,}000 = 8800 \text{ lb. per sq. in. (compression)}$$

The greater steel area more effectively reduces the actual concrete deformation, giving a higher tensile stress in the concrete. This value of 320 lb. per sq. in. will be reduced by compression stresses due to live and dead loads and by the effect of plastic flow.

182. Shrinkage of Beams—Concrete Taking Some Tension. If the steel is not symmetrically placed about the center line of the cross section, shrinkage of the concrete adds to the tensile stresses in the concrete due to bending and causes compression stresses in the steel. These shrinkage compression stresses decrease the tensile stresses in the steel due to bending. As the effect of shrinkage becomes operative many more sections in a beam are cracked as a result of excessive tensile stresses. On the compression side there is no steel to prevent the complete shrinkage deformation from occurring, so the deflection of the beam will be greater because of this shrinkage warping.

Stresses Due to Shrinkage. Assume that the concrete is uncracked for a distance of qd from the compression side (Fig. 117). The steel pulls

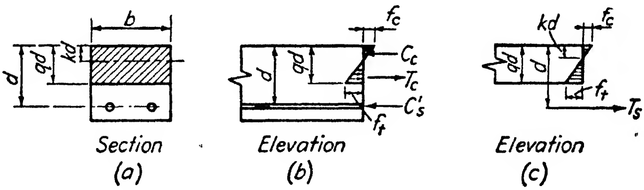


FIG. 117

on the concrete, tending to reduce the full shrinkage deformation in the concrete nearby. The concrete area $b(qd)$ is affected as though a tensile force T_s were applied at the center of gravity of the steel (Fig. 117c). The maximum tensile stress f_t and compression stress f_c in the concrete equal

$$f_t = \frac{T_s}{b(qd)} + \frac{6T_s(d - \frac{1}{2}qd)}{b(qd)^2} = \frac{2T_s}{bqd} \times \frac{3 - q}{q} \tag{120}$$

$$f_c = \frac{T_s}{b(qd)} - \frac{6T_s(d - \frac{1}{2}qd)}{6(qd)^2} = -\frac{2T_s}{bqd} \times \frac{3 - 2q}{q} \tag{121}$$

The neutral axis ratio k equals

$$k = \frac{(3 - 2q)}{3(2 - q)} q \tag{122}$$

Returning now to the full section of concrete and steel (Fig. 117*b*), the strain in the concrete at the level of the steel, were it uncracked, would be equal to that of the steel. By equation 115,

$$\frac{f'_s}{E_s} = s - \frac{f''_t}{E_c} \tag{123}$$

where f''_t = tensile stress at level of tensile steel, if uncracked. Since a plane section remains plane, the strains are proportional to their distances from the neutral axis, so

$$\frac{f''_t}{f_c} = \frac{d - kd}{kd} = \frac{1 - k}{k}$$

$$f''_t = \frac{1 - k}{k} f_c$$

Substituting into equation 123,

$$f'_s = sE_s - \frac{(1 - k)}{k} nf_c$$

The fact that the concrete has cracked means merely that the ultimate value of f_t has been exceeded, and the concrete cracks in order that the section may remain plane.

It is also true that the total compression force on the section equals the total tensile force (Fig. 117*b*).

$$f'_sA_s + \frac{f_c}{2} b(kd) = \frac{f_t}{2} b(qd - kd)$$

Substituting,

$$\frac{f_t}{f_c} = \frac{q - k}{k} \qquad \text{and} \qquad A_s = pbd$$

$$f'_sp + \frac{f_c}{2} k = \frac{f_c}{2} \frac{(q - k)^2}{k}$$

$$f_c = \frac{2kp}{(q - 2k)q} f'_s \tag{124}$$

$$f_t = \frac{2(q - k)p}{(q - 2k)q} f'_s \tag{125}$$

$$f'_s = \frac{sE_s}{1 + \dfrac{2np(1 - k)}{q(q - 2k)}} \tag{126}$$

The non-uniform shrinkage of the concrete will cause the beam to bend. There will be constant stress conditions at all sections due to shrinkage only and the deflection can be computed for a uniform bending moment M.

$$\frac{e}{y} = \frac{d^2v}{dx^2} = \frac{M}{EI}$$

where e = strain in concrete at a distance y from the neutral axis; then

$$v_{\text{max.}} = \frac{Ml^2}{8EI} = \frac{e}{y} \times \frac{l^2}{8}$$

or

$$v_{\text{max.}} = \frac{f_c l^2}{8E_c(kd)} \tag{127}$$

No Cracks. For the special cases of no cracking, $q = 1$, and equations 120 and 121 reduce to

$$f_c = \frac{2T_s}{bd}$$

$$f_t = \frac{4T_s}{bd} = 2f_c$$

The neutral axis ratio $k = \frac{1}{3}$ and equations 124, 125, 126, and 127 become

$$f_c = 2pf'_s \tag{128}$$

$$f_t = 4pf'_s \tag{129}$$

$$f'_s = \frac{sE_s}{1 + 4np} \tag{130}$$

$$v_{\text{max.}} = \frac{3f_c l^2}{8E_c d} \tag{131}$$

183. Stresses Due to Loads—Concrete Taking Some Tension. Assuming that the combined effect of shrinkage and loads leave the concrete uncracked for the distance qd from the top of the section (Fig. 117a), the center of gravity of the transformed area is a distance kd from the top.

$$\frac{b(qd)^2}{2} + npbd^2 = [b(qd) + npbd]kd$$

or

$$k = \frac{q^2 + 2np}{2(q + np)} \tag{132}$$

The moment of inertia about the neutral axis equals

$$I = \frac{b(kd)^3}{3} + \frac{b(qd - kd)^3}{3} + npbd(d - kd)^2$$

$$I = \frac{bd^3}{3}[k^3 + (q - k)^3 + 3np(1 - k)^2] \tag{133}$$

The maximum fiber stresses in the concrete equal

$$\text{Compression} \quad f_c = \frac{M(kd)}{I} \tag{134}$$

$$\text{Tension} \quad f_t = \frac{Md}{I}(q - k) \tag{135}$$

These stresses vary from section to section with the bending moment and the height of the cracks will vary from section to section. For this reason, since some loads are present on the beam during the drying-out period, accurate shrinkage computations are difficult to make for members in bending.

No Cracks. If no cracks occur above the tension steel, $q = 1$, and the equations simplify. The results are the same as those of the deflection derivation in Article 165.

$$k = \frac{1 + 2np}{2(1 + np)} \tag{136}$$

$$I = \frac{(1 + 4np)}{(1 + np)} \frac{bd^3}{12} \tag{137}$$

ILLUSTRATIVE PROBLEM 28

184. Shrinkage Stresses in a Slab. Assume that the slab of Problem 1 (Chapter 3) has partially dried out before the live load is applied. The shrinkage strain will be taken as $s = 0.0002$. The essential data from Problem 1 and Figure 15 give

Depth t	$= 6$ in.
Maximum positive moment	$= 28{,}000$ in.-lb.
Positive steel area A_p	$= 0.39$ sq. in. per ft. of width
d	$= 4.25$ in.
Maximum negative moment	$= 40{,}800$ in.-lb.
Negative steel area A_N	$= 0.52$ sq. in. per ft. of width
d	$= 5.00$ in.

Live load	= 130 lb. per sq. ft.
Dead load	= 75 lb. per sq. ft.
f'_c	= 2000 lb. per sq. in.
Ultimate tensile stress f_t	= 200 lb. per sq. in.

No Cracks

	Positive Bending	Negative Bending
Shrinkage Stresses		
Steel ratio p	0.00765	0.00865
Compression stress in steel f'_s	4120	3950
Maximum compression stress in concrete f_c	63	68
Maximum tensile stress in concrete f_t	126	136
Maximum Stresses Due to Dead Load		
Neutral axis ratio k	0.551	0.557
Moment of inertia I	100	168
Maximum moment (dead load) M	10,250	14,900
kd	2.34	2.79
$d - kd$	1.91	2.21
Maximum compressive stress in concrete f_c	240	247
Maximum tensile stress in concrete f_t	196	196
Maximum tensile stress in steel $f_s = nf_t$	2940	2940
Maximum Stresses—Shrinkage plus Dead Load		
Maximum compressive stress in concrete f_c	303	315
Maximum tensile stress in concrete f_t	322	332
Compressive stress in steel f'_s	1188	1010

It is apparent that the assumption of "no cracks" is definitely in error, if the effect of plastic flow is neglected. During positive bending the maximum moment that can be carried without cracks will produce a tensile stress in the concrete due to the dead load of $f_t = 200 - 126 = 74$ lb. per sq. in. The moment equals

$$M = \frac{fI}{d(1 - k)} = \frac{74 \times 100}{1.91} = 3870 \text{ in.-lb.}$$

The concrete will be cracked for about 7.5 ft. in the center of the 13.5-ft. span. A similar computation for the negative moments shows that the concrete will be cracked for about 2 ft. from each support. Only about 2 ft. of the total span will be uncracked. More complete drying out plus the addition of the live load would tend to crack the slab throughout were it not for plastic flow.

At the section of maximum positive bending at the center, by the process of trial and error, a maximum tensile stress of 200 lb. per sq. in. in the concrete is found to be at a distance $qd = 2.08$ in. from the top. When the value of $q = 0.49$ is used, the stresses due to shrinkage and dead load and the resultant stresses are

$$f_t = 122 + 77 = 199 \text{ lb. per sq. in.}$$

$$f_c = 98 + 300 = 398 \text{ lb. per sq. in.}$$

$$f_s = -756 + 7050 = 6294 \text{ lb. per sq. in. (tension)}$$

Any other section will have a different bending moment due to dead load and a different distance qd to the limiting tensile stress $f_t = 200$ lb. per sq. in. Therefore it is impracticable to determine the combined shrinkage and load effect for a member in bending. This holds true also for the additional deflection due to shrinkage. The modification of these results by consideration of plastic flow will be investigated in Problems 32 to 34.

PLASTIC FLOW

185. Plastic Flow Coefficient. Plastic flow has just been defined as the strain due to a sustained load. In test data it is obtained by measurement of the total strain of the loaded specimen at some definite time t_1 after loading. From this total strain the elastic strain ($t = 0$) due to the application of the load is subtracted as well as the shrinkage strain ($t = t_1$) of a duplicate reinforced specimen which has remained unloaded.

The plastic flow strain c, thus obtained, is caused by sustained application of the load (dead always and quiescent live loads) plus whatever shrinkage effect due to non-uniform drying out occurs, but should not include temporary adjustments due to temperature changes. As used in the following derivations the plastic flow strain c is the unit strain due to a unit stress at some definite time t_1 after application of the quiescent loads. Various investigations evaluate the actual plastic strain e_p for axially loaded members in different terms. Some of the results are

Straub $\quad e_p = C_1 f^p{}_c t^q$

Thomas $\quad e_p = C_2 f_c[1 - e^{-A\{(t+a_2)^x - a^x{}_2\}}]$

Shank $\quad e_p = C_3 \sqrt[a_3]{t}$

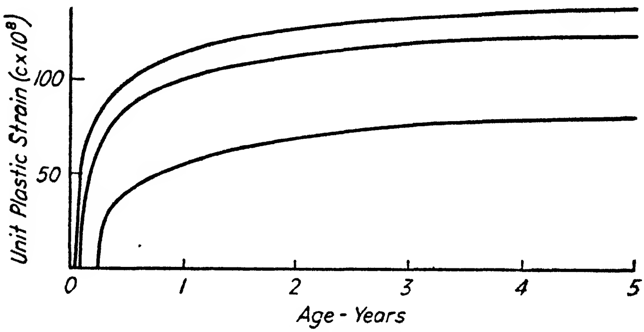
Lorman $\quad e_p = \left(\dfrac{mt}{C_4 + t}\right) f_c$

where C_1, C_2, C_3, C_4 = constants determined by test
f_c = uniform normal stress intensity in the concrete
t = time, in days
p = exponent determined by test for varying stresses
a_3 or q = exponent determined by test for varying ages
e = Naperian base
A = area of cross section
a_2 = constant of viscous creep
x = exponent determined by test
m = final value of unit strain c after several years of sustained load.

For working loads it is generally agreed that the strain e_p varies directly with the stress, so that the actual strain at any time t_1 equals

$e_p = f_c c$, or the unit strain c may be taken as varying only with the time. Professors Maney and Lagaard have recently stated that all plastic flow strains for working loads are due to non-uniform shrinkage since the member dries out more rapidly at its exposed surfaces. The result is a warping of the member and a readjustment of stress distributions. Whatever the explanation may be the plastic flow equations record the strain changes observed in the test specimens.

The plastic flow unit strain c varies with the age after the load application. The strains increase rapidly at first, but the rate of increase decreases as time goes on until the rate of change approaches zero at the end of 4 or 5 years. Figure 118 shows the general form of the plastic



Plastic Flow Increase with Age

FIG. 118

flow strain curve in terms of time. A similar curve is recorded for shrinkage strain s versus time. At the end of 4 or 5 years the specimen responds elastically if there are no moisture changes in the concrete. Figure 118 also shows that the greater the age of the concrete when loaded the less is the total plastic flow and the more rapid the rate of increase of strain immediately after loading. Three or four months later the increase of strain is the same for all ages of loading. Since plastic flow tends to relieve the concrete particles that are highly stressed it is advantageous to remove the forms at an early age in order to obtain the maximum readjustment due to flow. These flow readjustments also occur if there is movement of the supports.

Tests of concrete specimens subjected to torsion show a similar increase of the angle of twist and the shearing strains due to a sustained twisting moment. The angle of twist versus time relation gives a curve similar to that of Figure 118.

186. Stress Changes Due to Plastic Flow under Sustained Load. *Axial Loads.* Let us assume a concrete member symmetrically reinforced and loaded with an axial load N. By equation 186 (Art. 231) the elastic stress f_o in the concrete at the instant of loading equals

$$f_o = \frac{N}{A[1 + (n - 1)p]} \tag{138}$$

After loading, as the concrete creeps, the total force at the section remains unchanged, so the increase in stress in the steel Δf_s and the decrease of stress in the concrete Δf_c must be related.

$$\Delta f_s A_s = -\Delta f_c A_c$$

$$\Delta f_s = -\Delta f_c \frac{A_c}{A_s} = -\Delta f_c \frac{1 - p}{p} \tag{139}$$

At some time t_1 after loading the strain in the concrete will have increased. If the unit elastic strain equals e and the additional plastic strain at this time t_1 equals c, the total strain due to the stress f_c now equals

$$\text{Unit strain} \times \text{stress} = \text{total strain} = f_c e + \Sigma f_c \Delta c$$

In this equation the product $f_c e$ is obtained by using the actual stress f_c at the time t_1. The summation of the plastic strains must be made by using corresponding values of f_c as increments of Δc are taken. In a short increase of time Δt beyond t_1 the deformation in the concrete will equal the elastic strain due to a change of stress Δf_c and the yield due to the stress f_c $(t = t_1)$, or

$$\frac{\Delta f_c}{E_c} + f_c \Delta c = \frac{\Delta f_s}{E_s} \tag{140}$$

Usually Δf_c will have a negative value as the concrete stress decreases when the steel stress increases.

$$f_c \Delta c = \frac{\Delta f_s}{E_s} - \frac{\Delta f_c}{E_c}$$

Substituting the value of Δf_s from equation 139,

$$f_c \Delta c = -\frac{\Delta f_c}{E_s} \frac{1 - p}{p} - \frac{\Delta f_c}{E_c} = -\frac{\Delta f_c}{E_c}\left[\frac{1 - p + np}{np}\right]$$

$$\Delta f_c = -f_c \Delta c E_c \left[\frac{np}{1 + (n - 1)p}\right] = -\frac{f_c \Delta c}{b}$$

where

$$b = \frac{1}{E_c}\left[\frac{1 + (n - 1)p}{np}\right]$$

Passing to the limit,

$$\frac{d(f_c)}{f_c} = -\frac{d(c)}{b}$$

Integrating between limits of $t = 0$ to $t = t_1$, or f_o to f_c and $c = 0$ to $c = c$,

$$\ln f_c - \ln f_o = -\frac{c}{b}$$

$$\frac{f_c}{f_o} = e^{-\frac{c}{b}} \quad \text{or} \quad f_c = \frac{f_o}{e^{\frac{c}{b}}} \tag{141}$$

Equation 141 gives the concrete stress f_c at any time t_1 after loading, providing the elastic stress f_o is computed and the plastic flow unit strain c for the time t_1 is known for the concrete used.

A similar derivation will give the changes in stress in the steel. Let f_i be the initial elastic stress and f_s the final stress at time t_1.

$$f_i = nf_o = \frac{nN}{A[1 + (n - 1)p]} \tag{142}$$

The change in stress in the concrete $f^{\Delta}{}_c$ equals

$$f^{\Delta}{}_c = f_o - f_c = f_o\left(1 - \frac{1}{e^{\frac{c}{b}}}\right)$$

Let the change in stress in the steel be $f^{\Delta}{}_s$. From equation 139, for numerical equality,

$$f^{\Delta}{}_s = f^{\Delta}{}_c \frac{1 - p}{p} = f_o\left(1 - \frac{1}{e^{\frac{c}{b}}}\right)\left(\frac{1 - p}{p}\right)$$

$$f^{\Delta}{}_s = f_i\left(\frac{e^{\frac{c}{b}} - 1}{e^{\frac{c}{b}}}\right)\left(\frac{1 - p}{np}\right) \tag{143}$$

Note that equation 141 gives the final concrete stress f_c whereas equation 143 gives the change of stress $f^{\Delta}{}_s$ in the steel.

187. Plastic Flow Stress Changes with Length Constant. A simple illustration of stress changes under constant length is that of the concrete cylinder loaded with a compression force in a testing machine. When the load is applied there is an elastic deformation. If the testing heads remain unchanged in position the load on the scale beam decreases as flow takes place.

At the instant of application of the initial axial load the member undergoes a unit elastic strain e. If the length of a reinforced concrete member remains unchanged thereafter owing to the restraint of its supports, the steel remains unchanged in length and its stress will be constant. The load on the concrete will cause unit flow strain c and the total strain $(e + c)$ at any time t_1 must equal the initial elastic strain e. The concrete stress and the total load will decrease. The relation between the initial elastic stress f_o and the final stress f_c can be found. The change in elastic strain is equal and opposite to the plastic strain.

$$\frac{d(f_c)}{E_c} = -f_c d(c)$$

Integrating

$$\ln f_c - \ln f_o = -E_c c$$

$$f_c = \frac{f_o}{e^{E_c c}} \tag{144}$$

For members in bending the tendency is to decrease the initial maximum stress f_o on particles near the surface of the compression side, and the stress variation from neutral axis to outer fiber is no longer uniformly varying. This justifies such discussions as the plastic theory of beams. Mr. Charles S. Whitney comments, "This indicates another valuable property of concrete which appears to make it advisable to place it under dead load at as early an age as possible. The plastic flow causes an adjustment of stress, which reduces the maximum, making structures stronger under live load. This flow in the material also tends to compensate irregularities in the stiffness of the material itself, accounting no doubt for some of the increase which has been noted in the strength of concrete subjected to sustained loads."

ILLUSTRATIVE PROBLEM 29

188. Plastic Flow Stresses for an Axially Loaded Column. Let us consider the effect of plastic flow for the same columns of Problem 27. Assume that the unit plastic flow strain $c = 100 \times 10^{-8}$ for the concrete used at the end of 2 years of a sustained load of 150,000 lb. This includes the dead load and sustained live load. The variable live load is also 150,000 lb.

Low Percentage of Steel. The column designed for a concrete strength of $f'_c = 3000$ lb. per sq. in. has a section 21 in. square and a steel ratio $p = 0.0109$. By equation 138,

$$f_o = \frac{150{,}000}{(21)^2(1.098)} = 310 \text{ lb. per sq. in.} \quad \text{(compression)}$$

$$b = \frac{1}{3{,}000{,}000}\left[\frac{1.098}{0.109}\right] = 3.355 \times 10^{-6}$$

$$\frac{c}{b} = \frac{100 \times 10^{-8}}{3.355 \times 10^{-6}} = 0.298 \qquad \text{and} \qquad e^{\frac{c}{b}} = 1.347$$

By equation 141,

$$f_c = \frac{310}{1.347} = 230 \text{ lb. per sq. in.} \quad \text{(compression)}$$

The initial steel stress $f_i = nf_o = 3100$ lb. per sq. in. (compression)

By equation 143 the change of stress in the steel equals,

$$f^{\Delta}_s = 3100\left(\frac{1.347 - 1}{1.347}\right)\left(\frac{0.989}{0.109}\right) = 7250 \text{ lb. per sq. in. (increase)}$$

The summary of combined effect of loads, shrinkage, and flow at the end of 2 years is

	Concrete lb. per sq. in.	Steel lb. per sq. in.
Elastic stress (sustained load)	310 C	3,100 C
Shrinkage	119 T	10,800 C
Plastic flow	230 C *	7,250 C *
Resultant stress (sustained load)	111 C	21,150 C
Elastic stress (live load)	310 C	3,100 C
Resultant stress (fully loaded)	421 C	24,250 C

* The concrete stress of 230 replaces initial stress of 310; steel stress of 7250 is an addition to the initial stress of 3100. Computations of stresses due to loads only would give $f_c = 620$ lb. per sq. in. and $f_s = 6200$ lb. per sq. in. The resultant effect of shrinkage and flow is to reduce the concrete stress 32 per cent and increase the steel stress 290 per cent. In the past, when columns were designed for loads alone, the designs were safe because the allowable concrete stress was held low enough so that the allowable steel stress did not exceed 10,000 lb. per sq. in. The shrinkage and flow readjustments did not raise the resultant steel stress to the yield point.

High Percentage of Steel. This column was 17 in. square with a steel ratio $p = 0.0351$. Solving as before,

$$f_o = \frac{150{,}000}{(17)^2(1.316)} = 395 \text{ lb. per sq. in.}$$

$$b = 1.25 \times 10^{-6} \qquad \frac{c}{b} = 0.800 \qquad \text{and} \qquad e^{\frac{c}{b}} = 2.226$$

$$f_c = \frac{395}{2.226} = 178 \text{ lb. per sq. in.}$$

Steel $f_i = 3950$ lb. per sq. in.

$$f^{\Delta}_s = 3950\left(\frac{2.226 - 1}{2.226}\right)\left(\frac{0.965}{0.351}\right) = 5960 \text{ lb. per sq. in.}$$

Summary.

	Concrete lb. per sq. in.	Steel lb. per sq. in.
Elastic stress (sustained load)	[395 C]	3,950 C
Shrinkage	320 T	8,800 C
Plastic flow	178 C	5,960 C
Sustained load stresses (2 yr.)	142 T	18,710 C
Elastic stress (live load)	395 C	3,950 C
Resultant stress (fully loaded)	253 C	22,660 C

In this case the effect of shrinkage and flow compared with elastic stresses is to reduce the concrete stress 68 per cent and increase the steel stress 187 per cent. The shrinkage and flow readjustments have greatest influence on the steel stresses when low steel ratios are used.

189. Approximation of Shrinkage-Flow Readjustments. The calculations in the previous article preserve equilibrium at the section. For example, the column with the low percentage of steel carries at time $t = 0$ a sustained load of

$$N = A[f_c(1 - p) + f_s p] = 441[310 \times 0.989 + 3100 \times 0.0109]$$
$$= 150{,}000 \text{ lb.}$$

After 2 years the load sustained is

$$N = 441[111 \times 0.989 + 21{,}150 \times 0.0109] = 150{,}000 \text{ lb.}$$

The elastic stresses were computed with the assumption that the steel stress $f_i = nf_o$. The final stresses can also be computed for dead plus live loads by equation 138 if a value of n is used equal to

$$n = \frac{f_s}{f_c} = \frac{24{,}250}{421} = 57.7$$

$$N = 421 \times 441[1 + 56.7 \times 0.0109] = 300{,}000 \text{ lb.}$$

Similarly the dead plus live load stresses for the high percentage of steel can be obtained by equation 138 by using a value

$$n = \frac{f_s}{f_c} = \frac{22{,}660}{253} = 89.8$$

$$N = 253 \times 289[1 + 88.8 \times 0.0351] \doteq 300{,}000 \text{ lb.}$$

The designer can approximate the final stresses by using values for n of 60 to 90 for the particular concrete strength, shrinkage, flow strains, and dead-live ratio adopted for this problem. It would be logical to use a greater value of n, say 150 to 200, applied to the sustained load only, and the usual elastic ratio $n = 10$ to the short time live loads, except that occasionally (as with the high percentage of steel) the sustained stresses in the concrete result in tensile values.

190. Plastic Flow in Beams. It is stated in Article 187 that the result of shrinkage and flow in members in bending is to produce a nonlinear stress variation in the concrete on the compression side. An assumption of a parabolic variation is frequently made. The neutral axis is lowered and this gives a greater compression area. Computations for stresses or deflection which purport to allow for shrinkage and flow are of doubtful value; consequently the designer usually approximates for these values by using a greatly increased value of n. As Professor Shank says, "In any practical problem uncertainties about the crack penetration, the real elastic properties of the concrete, and the plastic flow expression are great enough so that it is hardly practical to use any theory more complex than the modular ratio change method." [2] For illustration of plastic flow in beams see Problems 32 to 34 under prestressed concrete. In these cases the tensile stresses are kept at low values and cracks do not affect the results for working loads.[3]

TEMPERATURE

191. Temperature Stresses. Changes in the temperature will produce strains in a reinforced concrete member. If the member is part of a continuous frame these strains will be accompanied by stresses in the steel and concrete. At early ages the temperature stresses in the concrete are modified by plastic flow. For this reason the actual temperature during deposition of the concrete is not important and it is sufficiently accurate to compute temperature changes above and below the average of the temperature range. After 4 or 5 years the rate of plastic flow approaches zero and the concrete acts as an elastic material but with a modulus of elasticity considerably higher than it possesses at early ages. Temperature computations made at ages greater than 4 years should not rely on readjustment by plastic flow. For concrete exposed outdoors the temperature range internally in thick sections can

[2] "The Plastic Flow of Concrete," Ohio State University, *Eng. Exp. Sta. Bulletin 91*, p. 50.

[3] An interesting analysis of plain concrete slabs whose surfaces are exposed to drying out (non-uniform shrinkage) and direct stresses or bending is given by Mr. Gerald Pickett, in *Jour. A.C.I.*, Feb., 1942, p. 333. For this simple case Mr. Pickett's analysis supports Prof. Maney's statement that plastic flow for working strains is due to non-uniform shrinkage.

be taken to be about 60 per cent of the yearly range; thin sections will approach the air temperatures.

The coefficient of expansion of concrete varies with the mix and materials used. Laboratory and field tests give values of the coefficient ϵ varying from 4×10^{-6} to 7×10^{-6} per degree Fahrenheit.[4] In designs it has been customary to assume the coefficient to be equal to that of steel and to have a value $\epsilon = 5.5 \times 10^{-6}$. If the coefficients are not equal, stress will be caused in the steel by variations of temperature above or below that at the time the concrete hardened. To evaluate the stresses due to different coefficients, neglect plastic flow and note that strain equals the coefficient ϵ times temperature change T. The difference in concrete and steel strains will be $(\epsilon_s - \epsilon_c)T$. If the temperature drops and ϵ_s is greater than ϵ_c the steel will be placed in tension and the concrete in compression, so

$$(\epsilon_s - \epsilon_c)T = \frac{f_c}{E_c} + \frac{f_s}{E_s} \tag{145}$$

The section must be in equilibrium, so

$$f_cA_c = f_sA_s \qquad \text{or} \qquad f_c(1 - p) = f_sp \tag{146}$$

Substituting values of f_s or f_c from equation 146 in equation 145,

$$f_s = (\epsilon_s - \epsilon_c)TE_s\frac{1 - p}{1 + (n - 1)p} \tag{147}$$

$$f_c = (\epsilon_s - \epsilon_c)TE_c\frac{np}{1 + (n - 1)p} \tag{148}$$

If the temperature rises, the steel will be in compression and the concrete in tension.

192. Temperature Stresses in Concrete Chimneys. An extreme example of reinforced concrete exposed to temperature changes is the concrete chimney. The unlined chimney may be subjected to great differences in temperature on opposite faces of a comparatively thin shell. The derivation given here is taken from the Tentative Standard of the American Concrete Institute (505-36T) for Design and Construction of Reinforced Concrete Chimneys.

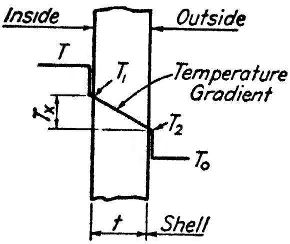


FIG. 119

Figure 119 shows an elevation of the chimney shell with a representation of the

[4] "Plain and Reinforced Concrete Arches," Chas. S. Whitney, *Jour. A.C.I.*, March 1932, p. 479.

temperature gradient from flue gas to outside air. Let

T = maximum temperature of flue gas, degrees Fahrenheit
T_1 = temperature of concrete at inside surface of shell
T_2 = temperature of concrete at outside surface
T_o = minimum temperature of outside air
u = coefficient of overall heat transmission from the gas inside the chimney to the air outside, British thermal units per square foot of surface per hour per degree temperature difference
K_1 = coefficient of absorption and convection at inside concrete surface, British thermal units per square foot per hour per degree
K_2 = coefficient of radiation and convection at outside concrete surface, British thermal units per square foot per hour per degree
C = coefficient of thermal conductivity of concrete, British thermal units per square foot of surface per inch thickness per degree

The amount of heat received at the inside surface equals the heat conducted through the shell and the heat discharged to the air, or

$$K_1(T - T_1) = \frac{C}{t}(T_1 - T_2) = K_2(T_2 - T_o) = u(T - T_o) \quad (149)$$

also

$$T - T_o = (T - T_1) + (T_1 - T_2) + (T_2 - T_o) \quad (150)$$

From equation 149,

$$T - T_1 = \frac{u(T - T_o)}{K_1}, \quad T_1 - T_2 = \frac{t}{C}u(T - T_o),$$

$$T_2 - T_o = \frac{u(T - T_o)}{K_2}$$

Substituting in equation 150,

$$u = \frac{1}{\dfrac{1}{K_1} + \dfrac{t}{C} + \dfrac{1}{K_2}} \quad (151)$$

Let

$$T_x = T_1 - T_2 = \frac{t}{C}u(T - T_o) = \frac{t(T - T_o)}{C\left(\dfrac{1}{K_1} + \dfrac{t}{C} + \dfrac{1}{K_2}\right)}$$

Let $\dfrac{C}{K_1} + \dfrac{C}{K_2} = K_o$, a constant determined by experiment. For the unlined chimney

$$T_x = \frac{(T - T_o)t}{K_o + t} \quad (152)$$

If the chimney is lined, the value of K_o will also include the surface coefficients, thickness, and conductivity of the lining. It is suggested in the absence of test data that $K_o = 12$ for unlined chimneys and $K_o = 30$ for lined chimneys where the lining is at least 4 in. thick with an air space of at least 2 in. between the lining and the concrete shell.

The analysis for stresses due to temperature differences includes the assumption that the combined effect of dead load, wind, and temperature may crack the concrete particles which have tensile stresses. Therefore, the concrete is assumed to be cracked on the tension side. The coefficient of expansion ϵ of concrete and steel are assumed to be the same. The temperature gradient at any particular level is assumed to be the same around the circumference; the section remains circular and horizontal.

Vertical Steel. The inner hotter particles tend to expand more than the outer ones. If the section remains horizontal, the outer particles must expand more than the temperature dictates and the inner ones less. The inner stresses will be compression, the outer stresses tension. Some particles in between, on a definite circumference, will expand an amount agreeing with the temperature change T_3 (Fig. 120). These particles will have zero temperature stress. Let us assume that these particles are located a distance kt from the inner surface (note that this distance is kt, and not kd). Let the distance to the steel be zt. The particles on the inner surface have compressive strains due to restraint of $\epsilon(T_1 - T_3)$ and a stress

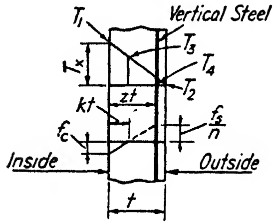


FIG. 120

$$f_c = \epsilon(T_1 - T_3)E_c$$

But

$$\frac{T_1 - T_3}{T_x} = \frac{kt}{t} \qquad \text{or} \qquad T_1 - T_3 = kT_x$$

Then

$$f_c = \epsilon T_x k E_c \text{ (compression)} \tag{153}$$

Similarly the steel is given a forced elongation corresponding to the temperature difference $(T_3 - T_4)$. The steel stress equals

$$f_s = \epsilon(T_3 - T_4)E_s = \epsilon T_x(z - k)E_s \tag{154}$$

The forces on the section must be in equilibrium. Using an average circumference,

$$\frac{f_c}{2} kt(2\pi r) = f_s pt(2\pi r)$$

Substituting the values of f_c and f_s from equations 153 and 154,

$$k = \sqrt{np(np + 2z)} - np \tag{155}$$

These equations give stresses due to temperature and are uniform around the circumference.

Circumferential Steel. The derivation for stresses on a cross section is given above. If one takes a vertical section, say 1 ft. high, the same equation may be used to compute compressive stresses on the inner concrete and tensile stresses in the circumferential steel. Of course, in this case the distance zt locates this circumferential steel and the steel ratio p is computed by using the area of this steel.

PRESTRESS

One of the disabilities which reinforced concrete members in bending suffer is the loss of the concrete area on the tension side in computations at sections near the maximum bending moment. This loss is due to the assumption that the concrete has cracked to the neutral axis. Members that must be watertight have the additional handicap that such cracks reduce the thickness of impermeable concrete. The prestressing of some or all of the steel is one method of reducing or eliminating tensile stresses.

Methods of prestressing steel for reinforced concrete pipes, tanks, and beams were initiated by M. Freyssinet and a variety of devices for prestressing have been introduced by others. Most of these methods are patented. The following discussion will cover members in direct tension, such as pipes and tanks, and members in bending.

193. Prestressed Pipes. M. Freyssinet developed a reinforced concrete pipe of great strength. The concrete mix and water content were determined with great care. The concrete is deposited into a form containing the circular steel reinforcement. With the outer form immovable the inner form is expanded. The compressed concrete becomes denser and some excess water is drained off. Then the outer form expands while the inner form follows it to maintain the pressure on the concrete. The circular steel must also increase in diameter and a predetermined tensile stress is produced in this steel. The concrete now hardens at a temperature of 180° F. The result is a concrete whose compressive strength may be as high as 7000 to 8000 lb. per sq. in. After hardening the inner form is collapsed and the pull in the steel causes the pipe to decrease slightly in diameter. The result is that the empty pipe has initial compression stresses in the concrete and tension in the steel. Let

f_p = tensile stress in the steel due to expansion of inner form
f_r = tensile stress in the steel on release of inner form
f_c = compressive stress in the concrete

$$p = \text{steel ratio} = \frac{A_s}{t \times 1}$$

Figure 121*a* shows the half section of concrete pipe, one unit long, acted upon by the pressure of the contracting steel. The final force in

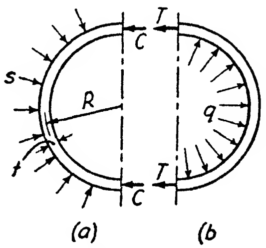


FIG. 121

the steel amounts to $f_rA_s = f_rp(t \times 1)$. The force in the concrete will equal $C = f_cA_c = f_c(1 - p)(t \times 1)$. These two forces must be equal, so

$$f_c = \frac{p}{1 - p} f_r$$

Also, the concrete and steel strains must be equal:

$$\frac{f_p - f_r}{E_s} = \frac{f_c}{E_c} \quad \text{or} \quad f_c = \frac{f_p - f_r}{n}$$

Solving these two equations,

$$f_r = \frac{1 - p}{1 + (n - 1)p} f_p \tag{156}$$

When the pipe goes into service an internal pressure q (Fig. 121*b*) will produce a tension T per unit length of shell. This is resisted by both steel and concrete, if the concrete is not overstressed in tension.

$$T = qR = f_t(t \times 1)[1 + (n - 1)p] \tag{157}$$

also

$$f_s = nf_t \tag{158}$$

The effect of prestressing and internal pressure can be combined to give resultant stresses.

ILLUSTRATIVE PROBLEM 30

194. Design of Prestressed Pipes. Determine the effect of prestressing a pipe of 30 in. internal diameter with shell 1.5 in. thick. Use $f'_c = 5000$ lb. per sq. in. and reinforcing steel of No. 5 American Steel and Wire Gage spaced at 1.5 in. center to center of bars. This steel is to be prestressed to $f_p = 80{,}000$ lb. per sq. in.

Number 5 wire has a diameter of 0.207 in. and area of 0.0336 sq. in.; $p = 0.0149$.

By equation 156,

$$f_r = \frac{80{,}000 \times 0.985}{1.075} = 73{,}350 \text{ lb. per sq. in.}$$

$$f_c = \frac{f_p - f_r}{n} = \frac{6650}{6} = 1113 \text{ lb. per sq. in.} \quad \text{(compression)}$$

Determine the maximum internal pressure to reduce the concrete prestress to zero.

$$15.00q = 1113 \times 1.5 \times 1.075 \qquad \text{or} \qquad q = 120 \text{ lb. per sq. in.}$$

If it is deemed permissible to stress the concrete to a resultant tensile stress of 400 lb. per in., the internal pressure $q = 163$ lb. per sq. in.

Shrinkage and Flow. The curing of this pipe should result in a rather complete drying out. This shrinkage will cause the pipe to contract more; there will be a tendency to produce tension in the concrete and compression in the steel. Let us assume a shrinkage strain $s = 0.0004$. By equations 117 and 118 there will be a shrinkage tensile stress in the concrete of

$$f_t = \frac{6 \times 0.0149}{1.075} \times 0.0004 \times 5{,}000{,}000 = 167 \text{ lb. per sq. in.}$$

$$\text{Steel stress } f'_s = \frac{0.9851}{1.075} \times 0.0004 \times 30{,}000{,}000 = 11{,}000 \text{ lb. per sq. in.}$$

The process of curing is not lengthy but some time may elapse before the pipe is placed in service. Let us assume that a flow strain $c = 40 \times 10^{-8}$ represents the deformations ascribed to flow with due allowance for the fact that shrinkage is reducing the prestresses in both steel and concrete. By the flow equations for sustained load,

$$b = \frac{1.075}{6 \times 0.0149} \times \frac{1}{5{,}000{,}000} = 2.39 \times 10^{-6}$$

By equation 141,

$$f_c = \frac{f_o}{e^{\frac{c}{b}}} = \frac{1113}{1.182} = 942 \text{ lb. per sq. in.}$$

The forces at the section are in equilibrium, or

$$942 \times 1.5 \times 0.9851 = f_s \times 1.5 \times 0.0149$$

$$f_s = 62{,}200 \text{ lb. per sq. in.}$$

If we combine shrinkage and flow effects, the stresses as the pipe is put in service appear to be

$$f_c = 942 - 167 = 775 \text{ lb. per sq. in.} \quad \text{(compression)}$$

$$f_s = 62{,}200 - 11{,}000 = 51{,}100 \text{ lb. per sq. in.} \quad \text{(tension)}$$

If the pipe transports water, the greater part of the shrinkage will be eliminated by a subsequent expansion. Plastic flow will be non-existent if the resultant concrete stresses are close to zero. If the pipe stands empty for a long time, flow may be resumed but will depend on whether the pipe is above ground or whether it receives additional sustained compressive stresses due to the weight of fill above it.

In any case the effect of shrinkage and flow before the pipe is in service is to reduce the effect of prestressing and therefore reduce the allowable internal pressure that produces tensile stresses in the concrete.

195. Prestressed Tanks. The walls of a water or oil tank of cylindrical shape can be prestressed as a means of crack prevention. For tanks of considerable diameter, say 50 ft. to 150 ft., steel has been used as large as $1\frac{1}{8}$-in. rounds with rolled threads on upset ends perhaps 30 to 35 ft. long. These bars are joined by turnbuckles and form bands or continuous helices around the tank wall. The steel is placed on the outside of the poured wall and wherever the turnbuckles occur vertical slots or recesses are made in the wall so the turnbuckles may be rotated for tightening. After prestressing the steel is covered with a layer of concrete or gunite. This extra layer, placed after prestressing, adds concrete whose stress is theoretically zero before shrinkage, flow, or loading takes place.

By tightening the turnbuckles the length of steel is reduced and the concrete is prestressed in compression, while the steel has tensile stresses. The prestress analysis is similar to that of the concrete pipe except that the prestressed bar starts at zero stress and ends with the maximum stress f_p. With the usual lever bar for turning the turnbuckle, prestresses of 20,000 to 30,000 lb. per sq. in. can be obtained. If the diameter of the tank is very great compared to the wall thickness, an average radius R may be used. The steel is placed outside the wall so the concrete area equals $(t_1 \times 1)$ and the steel ratio is $\dfrac{A_s}{t_1 \times 1}$. Equating the forces at a section one unit high,

$$f_cA_c = f_pA_s \qquad \text{or} \qquad f_c = pf_p \tag{159}$$

When the tank is filled the tensile stresses in steel and concrete are given by equations 157 and 158, where the thickness t is the sum of the wall thickness t_1 plus gunite cover t_2. As in Problem 30 it is possible to adjust the steel area A_s and prestress f_p so that the concrete can take a reasonable tension due to the internal pressure. The gunite cover does not have the advantage of prestressing but it can well be proportioned with a greater tensile strength and the action of plastic flow will tend to relieve the greater resultant tensile stress.

A low prestress, say 30,000 lb. per sq. in., suffers a large percentage of reduction due to shrinkage and flow in the concrete. Therefore, recent tanks have been prestressed by wrapping wire around the wall with prestresses as high as $f_p = 150{,}000$ lb. per sq. in.

ILLUSTRATIVE PROBLEM 31

196. Prestressed Tank Wall. Let us check the prestressing effect on a tank, 90 ft. in diameter, which is to be filled with water. Assume that the preliminary design gives an 8-in. wall at a position 20 ft. below the water level; use $f'_c = 3000$ lb. per sq. in. The wall is prestressed with $\frac{7}{8}$-in. round bars spaced at $3\frac{1}{2}$ in. at this depth. The prestress equals $f_p = 25{,}000$ lb. per sq. in. After prestressing 3 in. of gunite is placed to cover the steel bars.

Prestress. Since the steel is placed outside the wall the steel ratio for a vertical section 1 ft. high equals

$$p = \frac{2.06}{8 \times 12} = 0.0215$$

By equation 159,

$$f_c = pf_p = 0.0215 \times 25{,}000 = 537 \text{ lb. per sq. in.}$$

The section is now in equilibrium, for $25{,}000 \times 2.06 = 51{,}500 = 537 \times 96$. The gunite covering is now placed and has no stress when it has hardened.

Shrinkage and Flow. If some time elapses before the tank is filled, shrinkage and flow will modify these stresses. Let us assume that the measured deformation corresponds to a flow strain $c = 30 \times 10^{-8}$, and the cross section remains parallel to its original position. The concrete in the wall will decrease in circumference because of flow but will tend to increase in circumference because of the reduction of compressive strength. Reversing the left terms of equation 140, the concrete strain equals

$$\frac{537 + f_c}{2} \times \frac{30}{10^8} - \frac{537 - f_c}{3 \times 10^6} \tag{160}$$

where f_c is the final stress and the average stress is used for Δf_c.

The gunite will receive a compressive stress f_g due to this net shortening. Shrinkage of the gunite will cause it to shorten and flow due to the compressive stress will cause it to shorten. Assuming the net change also to equal $c = 30 \times 10^{-8}$, the gunite strain equals

$$\frac{f_g}{2} \times \frac{30}{10^8} + \frac{f_g}{3 \times 10^6} \tag{161}$$

Equating these strains,

$$\frac{537 + f_c}{2} \times \frac{30}{10^8} - \frac{537 - f_c}{3 \times 10^6} = \frac{\Delta f_s}{30 \times 10^6} = \frac{f_g}{2} \times \frac{30}{10^8} + \frac{f_g}{3 \times 10^6} \tag{162}$$

For equilibrium, after flow has occurred,

$$96f_c + 33.94f_g = (25{,}000 - \Delta f_s)2.06 \tag{163}$$

The values of Δf_s and f_g can be expressed in terms of f_c by equation 162 and substituted in equation 163. The results give

$$f_c = 404 \text{ lb. per sq. in.}$$

$$f_g = 200 \text{ lb. per sq. in.} \quad \text{(compression)}$$

$$\Delta f_s = 2900 \text{ lb. per sq. in. (reduction) or}$$

$$f_s = 22{,}100 \text{ lb. per sq. in.}$$

These results are approximate as equation 140 should be a summation of small increments of Δc with corresponding values of Δf_c, Δf_g, and Δf_s, but this approximation indicates a reduction in the steel stress of about 12 per cent, which is checked by test values.

Effect of Water Pressure. Let us consider the portion of the tank wall between 19 ft. and 20 ft. below the water level. The average water pressure equals $19.5 \times 62.5 = 1220$ lb. per sq. ft. The steel ratio for the full thickness of concrete and gunite equals $p = \dfrac{2.06}{11 \times 12} = 0.0156$. By equation 157,

$$1220 \times 45 = 54{,}900 \text{ lb.} = f_t(11 \times 12)[1 + 9 \times 0.0156]$$

$$f_t = 365 \text{ lb. per sq. in.} \quad \text{(tension)}$$

$$\text{Resultant stress (concrete)} f_c = 404 - 365 = 39 \text{ lb. per sq. in.} \quad \text{(compression)}$$

$$\text{Resultant stress (gunite)} f_g = -200 + 365 = 165 \text{ lb. per sq. in.} \quad \text{(tension)}$$

Then

$$-39 \times 96 + 165 \times (36 - 2.06) + f_s \times 2.06 = 54{,}900$$

$$f_s = 25{,}750 \text{ lb. per sq. in.} \quad \text{(tension)}$$

Also,

$$f_s = 22{,}100 + 3650 = 25{,}750 \text{ lb. per sq. in.}$$

These are the computed stresses immediately after the water pressure is applied. Note that the steel stress only slightly exceeds the original prestress and that the gunite must be able to withstand safely a tensile stress of 165 lb. per sq. in.

Shrinkage and Flow. The water-filled tank will absorb moisture and the greater part of the shrinkage will be overcome by the consequent expansion of the concrete. The plastic flow due to stresses can be handled similarly to the flow due to prestress. Since shrinkage strains disappear, let us assume that the flow effect will correspond to a flow strain of 40×10^{-8}. The tensile force on the gunite is greater than the compressive force on the concrete. Assume that the section remains parallel to its original position and that it expands. Then

$$\left(\frac{165 + f_g}{2}\right)\frac{40}{10^8} - \frac{(165 - f_g)}{3 \times 10^6} = \frac{\Delta f_s}{30 \times 10^6} = \left(\frac{39 + f_c}{2}\right)\frac{40}{10^8} + \frac{39 - f_c}{3 \times 10^6}$$

The force equation for equilibrium at the section becomes

$$-96f_c + 33.94f_g + 2.06(25{,}750 + \Delta f_s) = 54{,}900$$

Solving,

$$f_c = 19.6 \text{ lb. per sq. in.} \quad \text{(compression)}$$

$$f_g = 75.3 \text{ lb. per sq. in.} \quad \text{(tension)}$$

$$f_s = 26{,}295 \text{ lb. per sq. in.} \quad \text{(tension)}$$

This approximate evaluation of the effect of flow indicates that the tendency is to relieve the concrete and gunite stresses and yet the resultant steel stress is not far from the original prestress.

197. Prestress of Rectangular Beams. The prestressing of beams is advantageous because properly located prestressed steel will produce compressive stresses at the bottom of the beam which will offset the tensile stresses caused by the loads. The prestressing causes a deflection upward which will reduce the resultant deflection when the loads are applied. Up to the present time it has not been economical to prestress continuous beams. Many prestressed beams are precast and then moved into place; they serve as beams of a single span with supported ends. It is difficult to produce continuity for prestressed beams cast in position. This discussion of prestressed beams will be limited to those of rectangular section, but similar procedures can be applied to the I sections often used, or the tee section of the normal floor systems. Two cases will be considered. In the first case the prestressing steel is coated with grease or asphalt, or wrapped in oiled paper, so that there is no bond between the steel and the adjacent concrete. In the second case the steel is assumed to be bonded to the concrete.

198. Prestressed Rectangular Beam—No Bond. Such beams are constructed by pouring the beam with the unstressed steel in place. After the concrete has hardened long enough to withstand prestress, the steel is prestressed to the determined value of f_p by a pull on its ends. The steel can slip easily in reference to the concrete but the bearing plates at the ends, or the bearing cones of concrete at the ends, cause the concrete to be compressed at the level of the steel. In the following derivation the concrete area will be considered to be bh as it is intended to have the resultant concrete stresses after loading entirely compressive or of very low tensile values. The steel ratio will be based on the full area bh and equals $p = \dfrac{A_s}{bh}$. The neutral axis ratio k will be a percentage of the depth h, and the steel depth d will be expressed as zh (Fig. 122).

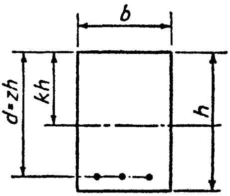


FIG. 122

Prestressing. Let the prestress intensity in the steel be denoted f_p and, for simplicity of equations, call the concrete area bh instead of $(bh - A_s)$. Upon completion of the prestressing the force in the steel equals $f_pA_s = C'_s$. This force will produce a pressure on the concrete, which acts at the center of gravity of the steel. Using the analysis developed for shrinkage (Art. 182), transfer this force to the center of gravity of the concrete area at $\frac{h}{2}$. On the bottom particles the concrete stress equals

$$f_c = \frac{C'_s}{bh} + \frac{6C'_s\left(zh - \frac{h}{2}\right)}{bh^2} = \frac{2C'_s}{bh}(3z - 1) = 2f_p p(3z - 1) \quad (164)$$

Similarly the top particles have a tensile stress of

$$f_t = 2f_p p(3z - 2) \quad (165)$$

Assuming a straight-line relation between concrete stress and strain, the neutral axis ratio equals

$$k = \frac{3z - 2}{6z - 3} \quad (166)$$

If we now consider the whole section, the steel has a tensile stress f_p, the lower part of the concrete is in compression, and the portion above the neutral axis is in tension. The forces at the section must be in equilibrium, or

$$\frac{f_t}{2} b(kh) + f_p pbh = \frac{f_c}{2} b(h - kh)$$

Substitution of values of f_t and f_c and k shows that this equation is an identity.

Stresses Due to Loads. The effect of application of loads will be considered separately from the prestress, but it will again be assumed that the combined effects produce tensile stresses in the concrete which are below the working tensile strength.

The loads on the beam will cause it to deflect and the unbonded steel will deflect with the concrete but may slip with relation to any concrete particle at the level of the steel. As the beam bends, the tensile steel will be lengthened, its ends being anchored, and it receives an additional tensile force T. This force T, acting on the end anchors, will load each concrete section with an equal compressive force $C'_s = T$ applied at the level of the steel. If M equals the moment at any section due to the external loads, the net moment M_c on the concrete area equals $M_c = M - Th(z - \frac{1}{2})$. By the methods used in deducing equations 164 and 165,

$$f_c = \frac{T}{bh} + \frac{6M_c}{bh^2} \quad \text{and} \quad f_t = \frac{T}{bh} - \frac{6M_c}{bh^2}$$

At the level of the tensile steel the stress f''_c in the concrete equals

$$f''_c = \frac{T}{bh} - \frac{12M_c h(z - \frac{1}{2})}{bh^3} = \frac{T}{bh} - \frac{12M_c}{bh^2}(z - \tfrac{1}{2})$$

When the value of M_c from above is substituted, the stress f''_c becomes

$$f''_c = \frac{T}{bh}[1 + 12(z - \tfrac{1}{2})^2] - \frac{12(z - \frac{1}{2})M}{bh^2}$$

The concrete strain at this level equals $e''_c = \dfrac{f''_c}{E_c}$ and is negative (tension). This strain varies from section to section as the external bending moment M varies. The total change of length at this level must equal the change in length Δl of the steel, or

$$-\frac{Tl}{A_s E_s} = \Delta l = \int_0^l e''_c\,dx = \frac{1}{E_c}\int_0 f''_c\,dx$$

$$= \frac{1}{E_c}\left\{\frac{Tl}{bh}[1 + 12(z - \tfrac{1}{2})^2] - \frac{12(z - \frac{1}{2})}{bh^2}\int_0^l M\,dx\right\} \qquad (167)$$

ILLUSTRATIVE PROBLEM 32

199. Design of Prestressed Beam—Steel Not Bonded. It is customary to use wires which have a high yield point as prestressed steel, so that a high prestress can be used. Thus plastic flow will not neutralize the prestressing.

Assume a beam section, 10 in. by 20 in. in section, supported at the ends of a 16-ft. span and carrying a uniformly distributed dead load of 520 lb. per ft. and a live load of 1040 lb. per ft. It is subjected to a maximum dead load moment of 200,000 in.-lb. and maximum live-plus-dead moment of 600,000 in.-lb. It is reinforced by ten No. 5 (American Steel and Wire Gage) wires prestressed to 150,000 lb. per sq. in. The center of gravity of this steel is located 3 in. from the bottom of the beam and each wire has a diameter of 0.207 in. (Fig. 123). The concrete strength $f'_c = 3000$ lb. per sq. in.

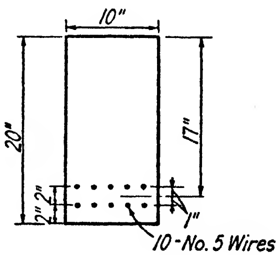


FIG. 123

Prestresses.

$$A_s = 10 \times 0.0336 \qquad \text{and} \qquad p = \frac{0.336}{10 \times 20} = \frac{1.68}{10^3}$$

By equation 164 and 165,

$$f_c = 2 \times 150{,}000 \times \frac{1.68}{10^3}(3 \times 0.85 - 1) = 782 \text{ lb. per sq. in.}$$

$$f_t = 2 \times 150{,}000 \times \frac{1.68}{10^3}(3 \times 0.85 - 2) = 277 \text{ lb. per sq. in.}$$

$$k = \frac{277}{277 + 782} = 0.262 \tag{168}$$

The stress diagram is shown in Figure 124a. The stresses are low enough so that a straight-line distribution is reasonable. The tensile stress is close to the ultimate value for 3000-lb. concrete.

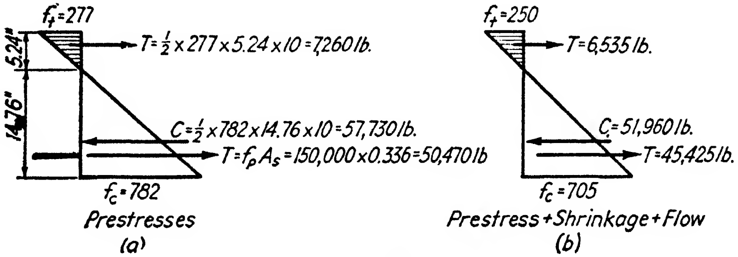


FIG. 124

Shrinkage and Flow. Shrinkage will cause the concrete to shorten, whereas flow due to stresses will cause the concrete in compression to shorten. The steel will also shorten and the prestress will be reduced. The section must remain in equilibrium under the reduced stresses. Assume that the test data indicate a 10 per cent reduction in steel stress, or a stress f_p = 135,000 lb. per sq. in. The compressive stress in the concrete becomes f_c = 705 lb. per sq. in. and the tensile stress f_t = 250 lb. per sq. in. The stress distribution is given in Fig. 124b.

Stresses Due to Dead Load. The maximum stresses due to the dead-load moment of 200,000 in.-lb. can be found. Using equation 167 and assuming the steel to be concentrated at its center of gravity,

$$-\frac{T \times 192}{0.336 \times 30 \times 10^6} = \frac{1}{3 \times 10^6}\left\{\frac{T \times 192}{200}[1 + 12 \times (0.35)^2] - \frac{12 \times (0.35)}{10 \times 400}\left(200{,}000 \times 192 \times \frac{2}{3}\right)\right\}$$

$$-57.2T = +2.37T - 26{,}880 \tag{169}$$

$$T = 451 \text{ lb.} \qquad \text{and} \qquad f_s = 1340 \text{ lb. per sq. in.}$$

$$\text{Maximum } M_c = 200{,}000 - 451 \times 20 \times 0.35 = 196{,}840 \text{ in.-lb.}$$

$$f_c = \frac{451}{200} + \frac{6 \times 196{,}840}{10 \times 400} = 297.5 \text{ lb. per sq. in.}$$

$$f_t = -293.0 \text{ lb. per sq. in.}$$

Combined Effect of Prestress and Dead Load. The combined stresses after application of the dead load are given below and plotted in Figure 125*a*.

Top particles f_c $= 298 - 250 = 48$ lb. per sq. in. (compression)

Bottom particles f_c $= 705 - 293 = 412$ lb. per sq. in. (compression)

Steel f_s $= 135{,}000 + 1340 = 136{,}340$ lb. per sq. in. (tension)

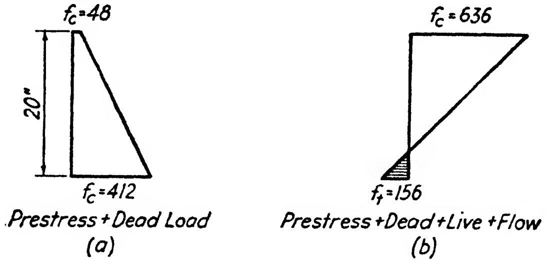


FIG. 125

Shrinkage and Flow—Dead Load. At the time the live load is applied assume that the additional shrinkage and flow amount to a flow strain $c = 30 \times 10^{-8}$. Tests indicate that a plane section remains plane after flow and the stresses are assumed to be uniformly varying for working loads. A solution can be made by using a greatly reduced modulus of elasticity. The elastic stress modulus $E = \frac{1}{e}$, or the elastic strain for a unit stress equals $e = \frac{1}{E}$. An equation in Article 11 suggests that the modulus of elasticity, after flow, may be defined as $R = \frac{1}{e + c}$. In this problem,

$$e = \frac{1}{3 \times 10^6} \qquad \text{and} \qquad c = \frac{3}{10^7}$$

Then

$$R = \frac{1}{e + c} = 1{,}580{,}000 \text{ lb. per sq. in.} \tag{170}$$

Using this value in equation 167,

$$T = 827 \text{ lb.} \qquad \text{and} \qquad f_s = 2460 \text{ lb. per sq. in.}$$

$$f_c = 295.4 \text{ lb. per sq. in.} \qquad \text{and} \qquad f_t = -287.2 \text{ lb. per sq. in.}$$

Flow reduces the concrete stresses only slightly, because there is such a small area of steel to prevent additional deflections.*

Combined Stresses before Live Load Is Applied.

Top particles f_c $= 295 - 250 = 45$ lb. per sq. in.

Bottom particles f_t $= 705 - 287 = 418$ lb. per sq. in.

Steel f_s $= 135{,}000 + 2460 = 137{,}460$ lb. per sq. in.

* An alternative is to apply a percentage reduction of the stresses of Figure 125*a*.

Live Load. The live load will be regarded as intermittently applied; therefore it will produce only elastic stresses. The additional live-load maximum moment equals 400,000 in.-lb. The stresses produced will be twice those of the dead load when it was initially applied. There is now less justification for superposing the results of separate stress computations, but the combined concrete stresses are low and do not vary much after the dead load is applied; therefore such a summation indicates the tendencies of stress variation.

Let us assume that the flow, after several years of continued dead-load application, amounts to $c = 100 \times 10^{-8}$. In this case $R = 750{,}000$ lb. per sq. in. Using equation 167, the stresses due to dead load become,

$$T = 1615 \text{ lb.} \qquad \text{and} \qquad f_s = 4810 \text{ lb. per sq. in.}$$

$$f_c = 291.2 \text{ lb. per sq. in.} \qquad \text{and} \qquad f_t = -275.0 \text{ lb. per sq. in.}$$

Top particles f_c $= 291 + 2 \times 297.6 - 250 = 636$ lb. per sq. in. (compression)

Bottom particles f_t $= 705 - 275 - 2 \times 293 = -156$ lb. per sq. in. (tension)

Steel f_s $= 135{,}000 + 4810 + 2 \times 1340 = 142{,}490$ lb. per sq. in. (tension)

The final stress distribution in the concrete is plotted in Figure 125*b*. The tensile stresses are moderate and should not produce cracks. The final stress in the steel does not exceed the original prestress. The low resultant compression stress may suggest that the section could be reduced in size, but a survey of such reduction should include allowable tensile stresses at the top due to prestress and at the bottom due to all factors. A reduction in section may also give difficulty in the introduction of the necessary area of steel. A discussion of diagonal tension stresses is given in Problem 33.

200. Prestressed Rectangular Beams—Steel Bonded to Concrete. This steel is installed in place and prestressed. The concrete is then poured; after it has hardened sufficiently to withstand prestress, the prestressing mechanism is released and the bearing plates, or bearing cones, press upon the concrete, causing it to oppose the shortening of the steel. The concrete is thereby prestressed.

In this case the original steel prestress f_p is reduced to some value f_r. The concrete at the level of the steel will receive a prestress in compression of f_{cs}. If the two materials are bonded, the strains at this level must be equal, or

$$\frac{f_p - f_r}{E_s} = \frac{f_{cs}}{E_c} \qquad \text{or} \qquad nf_{cs} = f_p - f_r \tag{171}$$

The force exerted by the steel on the concrete equals $C'_s = f_r A_s = f_r pbh$. Paralleling the derivation in Article 198,

$$\text{Bottom particles } f_c = 2f_r p(3z - 1) \tag{172}$$

$$\text{Top particles } f_t = 2f_r p(3z - 2) \tag{173}$$

also

$$k = \frac{3z - 2}{6z - 3} \tag{174}$$

$$f_{cs} = \left(\frac{z - k}{1 - k}\right) f_c \quad \text{and} \quad f_t = \left(\frac{k}{1 - k}\right) f_c \tag{175}$$

After prestressing, the section is in equilibrium, or

$$f_r pbh + \frac{f_t}{2} b(kh) = \frac{f_c}{2} b(h - kh)$$

$$2f_r p + f_t k = f_c(1 - k) \tag{176}$$

Stresses Due to Loads. The solution for stresses due to dead or live loads can be made by the use of the transformed area, if the resultant tensile stresses are assumed to be low enough so that the whole concrete section can be considered in the computations.

ILLUSTRATIVE PROBLEM 33

201. Design of Prestressed Beam with Bonded Steel. Assume the dimensions, loads, and stresses of Problem 32.

Prestresses. The center of gravity of the steel is 17 in. from the top. Therefore, since $z = 0.85$ and $k = 0.262$, $f_{cs} = 0.797f_c$ and $f_t = 0.355f_c$.

From equation 176,

$$2f_r \times \frac{1.68}{10^3} + 0.262f_t = 0.738f_c$$

or

$$f_c = 0.00531f_r \tag{177}$$

From equation 171,

$$10 \times 0.797f_c = 150{,}000 - f_r$$

Substituting the value of f_c in terms of f_r from equation 177,

$$f_r = 143{,}900 \text{ lb. per sq. in.}$$

Bottom particles $f_c = 2 \times 143{,}900 \times \dfrac{1.68}{10^3} (3 \times 0.85 - 1) = 749$ lb. per sq. in.

Top particles $f_t = 266$ lb. per sq. in.

These stresses differ only slightly from those obtained for non-bonded steel. The difference is due to the slightly lower steel stress f_r at the end of prestressing.

Shrinkage and Flow. Assume again that the steel stress is reduced 10 per cent to $0.9 \times 143{,}900 = 129{,}510$ lb. per sq. in. Then the maximum compressive stress in the concrete becomes $f_c = 674$ lb. per sq. in. and the maximum tensile stress $f_t = 239$ lb. per sq. in.

Dead Load. If the steel is bonded to the concrete, the effect of the dead load can be found by using the transformed area. The complete concrete area will be used in calculations because we expect the resultant stresses will not give tensile stresses large enough to crack the concrete. Take moments about the top of the section to determine the neutral axis position (Fig. 123), assuming the steel to be concentrated at its center of gravity:

$$\begin{array}{rlrl} 10 \times 20 = & 200 & \times 10 = & 2000 \\ 10 \times 0.336 = & \underline{\ 3.36} & \times 17 = & \underline{\ \ 57} \\ \Sigma A = & 203.4 & & 2057 = \Sigma M \end{array}$$

$$kh = \frac{2057}{203.4} = 10.11 \text{ in.} \qquad \text{and} \qquad k = 0.5055$$

The moment of inertia about the neutral axis equals

$$I = \frac{10 \times (10.11)^3}{3} + \frac{10 \times (9.89)^3}{3} + 10 \times 0.336 \times (6.89)^2 = 6835 \text{ (in.)}^4$$

$$\text{Maximum } f_c = \frac{200{,}000 \times 10.11}{6830} = 296 \text{ lb. per sq. in.}$$

$$\text{Maximum } f_t = 290 \text{ lb. per sq. in.}$$

$$f_s = 10 \times 290 \times \frac{6.89}{9.89} = 2015 \text{ lb. per sq. in.}$$

The remaining procedure parallels that of Problem 32. The results are tabulated below.

STRESSES IN PRESTRESSED BEAM

Due to		Line	Stresses (lb. per sq. in.)		
			Concrete		Steel
			Top	Bottom	150,000
Prestress		1	266 T	749 C	143,900
Shrinkage and flow (prestress)		2	239 T	674 C	129,510
Dead load		3	296 C	290 T	2,015
Combined—after dead load	2 + 3 =	4	57 C	384 C	131,525
Shrinkage and flow (dead)		5	293 C	281 T	3,700
Combined—before live load	2 + 5 =	6	54 C	393 C	133,210
Live load		7	592 C	580 T	4,030
Combined—after live load	6 + 7 =	8	646 C	187 T	137,240
Shrinkage and flow (dead)		9	287 C	263 T	7,207
Combined—after several years	2 + 7 + 9 =	10	640 C	169 T	140,747

The high value of the prestress may raise questions in regard to the bond stresses between steel and concrete. The high-strength wires used have a large perimeter-to-area ratio. There is a greater perimeter over which to distribute a given force. Also the drop from original prestress f_p to final prestress f_r tends to cause the wire to increase slightly in diameter and thus adds to the friction at its surface. Shrinkage and flow tend to offset the increase of steel stresses due to the loads and the resultant stresses do not exceed the original prestress f_p.

The prestresses are constant at all sections so there is no need to transfer varying steel forces to the concrete except at the ends of the wire. A properly designed bearing cone at the ends will safely transfer the steel stresses to the concrete. Subsequent bond stresses are due to the loads only. The usual bond relation can be restated as

$$u = \frac{V}{\Sigma ojd} = \frac{vbjd}{\Sigma ojd} = \frac{vb}{\Sigma o} \tag{178}$$

This shear stress v at the level of the steel is the maximum value for the usual reinforced concrete beam, as is shown in the distribution on section EE of Figure 16. However, the prestressed beam includes the concrete on the tension side in computations, and the shear stress varies as shown on section BB of Figure 16, if one disregards the effect on shear stresses of the small amount of steel. At the level of the steel in a prestressed beam the shear stress is very low as compared with section EE, and by equation 178 the bond stress is correspondingly low.

Diagonal tension reinforcement will usually be needed. Since the fiber stress distributions may resemble those of Figure 124 at all sections during prestress and continue to be much the same at sections where the bending moment due to loads is of low value, we do not have the usual condition for diagonal tensile reinforcement. Usually such reinforcement is needed at sections where the tensile fiber stresses are low and the shear stress is high. Therefore, it is probably better to proportion the diagonal tension reinforcement by the more exact equation 30 (Art. 61) than by the approximate equations in general use, such as equation 32, which consider shear stresses only. In European practice stirrups are often prestressed also to add compressive stresses on a horizontal plane. In that case the principal tensile stress can be computed by the equation

$$t = \frac{f + f_h}{2} + \frac{1}{2}\sqrt{(f - f_h)^2 + 4v^2} \tag{179}$$

where f_h = normal stress on horizontal plane. These normal stresses are negative, if compressive.

The maximum deflection of the beam upwards by the prestressing action can be computed by using equation 127 (Art. 182). This operates as an initial camber of the beam and subsequent deflections due to the loads and to flow should include this initial value.

Some of the patented systems of prestressing and some designers include ordinary reinforcing bars in the section in addition to the prestressed wires. The following problem will compute for comparison with Problem 32 the effect of prestressing such a beam.

ILLUSTRATIVE PROBLEM 34

202. Design of Prestressed Beam—Prestressed Steel Not Bonded. Using the data of Problem 32 add two 1-in. square reinforcing bars to the section (Fig. 126).

In this case the prestressing force in the wires acts upon the concrete and the reinforcing steel, whose steel ratio $p = 0.01$. The transformed area the prestress force acts upon equals

$$A = 200(1 + 9 \times 0.01) = 218 \text{ sq. in.}$$

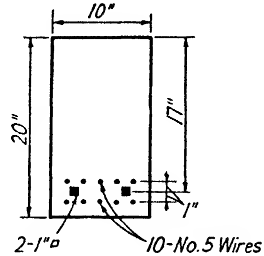


FIG. 126

The neutral axis ratio for concrete and steel bars equals $k = 0.529$ and the moment of inertia $I = 7477$ (in.)^4. The prestressing force $C'_s = 150{,}000 \times 0.336 = 50{,}400$ lb. Computations similar to those of equations 164 and 165 give

$$\text{Bottom particles } f_c = \frac{50{,}400}{218} + \frac{50{,}400(17 - 10.58) \times 9.42}{7477} = 639 \text{ lb. per sq. in.}$$

$$\text{Top particles } f_t = \frac{50{,}400}{218} - \frac{50{,}400(17 - 10.58) \times 10.58}{7477} = 227 \text{ lb. per sq. in.}$$

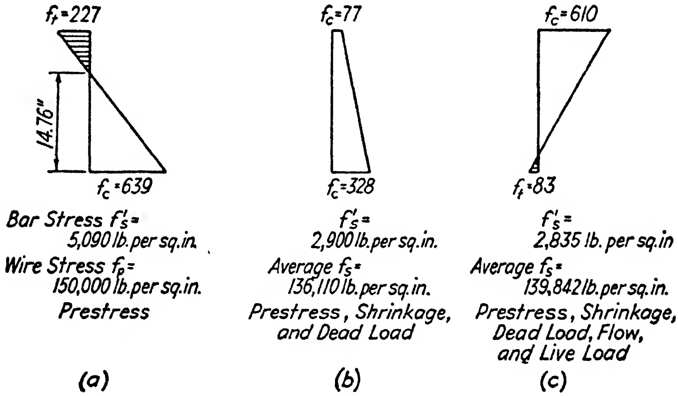


FIG. 127

These prestresses are shown in Figure 127*a* and they are somewhat less than those of Figure 124*a*. The compressive prestress in the reinforcing bars equals

$$f'_s = \frac{11.76}{14.76} \times 639 \times 10 = 5091 \text{ lb. per sq. in.}$$

Shrinkage and Flow. Assuming, as in Problem 32, that the result of shrinkage and flow is a 10 per cent reduction in the prestress in the wires, the reduced stresses in concrete and reinforcing bars become

$$f_c = 575 \text{ lb. per sq. in.}$$

$$f_t = 204 \text{ lb. per sq. in.}$$

$$f'_s = 4582 \text{ lb. per sq. in.}$$

Stresses Due to Dead Load. Upon the application of the loads, the concrete and bars act as a bonded unit, whereas the wires are not bonded. The derivation of Article 198 can be modified by substituting the transformed area of 218 sq. in. for the previous concrete area bh and the moment of inertia which equals 7477 (in.)^4 about the neutral axis 10.58 in. from the top of the section. Then

$$M_c = M - 6.42T$$

$$f_c = \frac{T}{218} + \frac{10.58M_c}{7477}$$

$$f_t = \frac{T}{218} - \frac{9.42M_c}{7477}$$

$$f''_c = \frac{T}{218} - \frac{6.42M_c}{7477} = \frac{1.01T}{10^2} - \frac{8.59}{10^4}M$$

$$-\frac{T \times 192}{0.336 \times 30 \times 10^6} = \frac{1}{3 \times 10^6}\left\{\frac{1.01 \times 192T}{10^2} - \frac{8.59}{10^4}\left(200{,}000 \times 192 \times \frac{2}{3}\right)\right\}$$

$$-57.2T = 1.94T - 22{,}000$$

$$T = 372 \text{ lb.}$$

$$f_s = 1107 \text{ lb. per sq. in. (wires)}$$

Maximum $M_c = 200{,}000 - 372 \times 6.42 = 197{,}610$ in.-lb.

$$f_c = \frac{372}{218} + \frac{10.58 \times 197{,}610}{7477} = 281 \text{ lb. per sq. in.}$$

$$f_t = \frac{372}{218} - \frac{9.42 \times 197{,}610}{7477} = -247 \text{ lb. per sq. in.}$$

$$\text{(bars)} \qquad f_s = nf''_c = 10 \times 168.1 = 1681 \text{ lb. per sq. in.}$$

The remaining computations are made by using the procedure and strains of Problem 32, except that the prestressing steel acts on the transformed area. In the computations, after the dead load is applied, equilibrium at the section requires that the reinforcing bars be analyzed separately, for prestress and live loads ($n = 10$) as contrasted with dead-load flow ($n' = 19$ or $n' = 40$), when combining stresses. The results are tabulated below.

STRESSES IN PRESTRESSED BEAM

Due to		Line	Stresses (lb. per sq. in.)			
			Concrete		Steel (bars)	Steel (wire)
			Top	Bottom		
Prestress		1	227 T	639 C	5091 C	150,000 T
Shrinkage and flow		2	204 T	575 C	4582 C	135,000
Dead load		3	281 C	247 T	1681 T	1,107
Combined—after dead load	2 + 3 =	4	77 C	328 C	2901 C	136,107 T
Shrinkage and flow (dead)		5	269 C	213 T	2663 T	1,744
Combined—before live load	2 + 5 =	6	65 C	362 C	1919 C	136,744 T
Live load		7	562 C	494 T	3362 T	2,214
Combined—after live load	6 + 7 =	8	627 C	132 T	1443 T	138,958 T
Shrinkage and flow (dead)		9	252 C	164 T	4055 T	2,628
Combined—after several years	2 + 7 + 9 =	10	610 C	83 T	2835 T	139,842 T

CHAPTER 10

TWO-WAY AND FLAT SLABS

Floor systems of reinforced concrete have been devised for many types of construction. As previously defined, the broad expanse of floor is spoken of as a slab. The discussion in Chapter 3 was confined to slabs whose tension steel spans in one direction only. These one-way slabs are designed as rectangular beams and are supported by beams spanning perpendicularly to the slab steel. A discussion of the design of one-way joist floors is also given in Chapter 7 as an example of tee-beam design.

It is possible to design certain floor slabs, whose panels are rectangular in shape, with two-way or four-way steel. In the two-way system the steel is placed in perpendicular bands or strips. Four-way steel has diagonal bands in addition.

In this text the term *two-way slab* will refer to a floor panel supported by beams or girders on all four edges and reinforced with two-way steel. The term *flat slab* will denote a floor panel which has no supporting beams or girders. The column is enlarged at the top to give a supporting capital. Flat slabs may be reinforced with two-way or four-way steel.

TWO-WAY SLABS

203. Statically Indeterminate Slabs. The framing plan for beam and girder floor construction is often so arranged that each panel is supported on all four sides by a beam or girder. If the panel is square, or nearly so, there may be an economic advantage in designing a two-way slab with steel running longitudinally and transversely. The design of such a slab involves a statically indeterminate analysis, as any given square foot of floor is supported both by north-south and east-west steel. Such a panel will deflect much as a rectangular net does whose four sides are tightly held. The previously discussed one-way slabs have the much less complicated deflections of the same net, or a sheet of paper, supported on two opposite sides only. The division of the load between the two perpendicular sets of bars is dependent on their relative resistance to deflection and the fact that both sets must have the same deflection at their intersections. A general theoretical discussion of such slabs with a survey of tests on two-way slabs can be found in a paper, "Moments and Stresses in Slabs," by H. M. Westergaard and W. A. Slater in the 1921 *Proceedings of the American Concrete Institute.* Dr. Westergaard continues the discussion in 1926 *A.C.I. Proceedings* with "Formulas for the Design of Rectangular Slabs and the Supporting Girders." A theoretical discussion is also given in a paper, "The Calculation of Flat Plates by the Elastic Web Method," by Joseph A. Wise in the 1928 *A.C.I. Proceedings.* This is followed by "Design of Reinforced Concrete Slabs" in the 1929 *A.C.I. Proceedings.*

These derivations involve advanced mathematics and result in complicated equations; consequently even the simple case of one panel simply supported on its four edges does not give equations easily applicable to commercial use. Continuous slabs give even more intricate results.

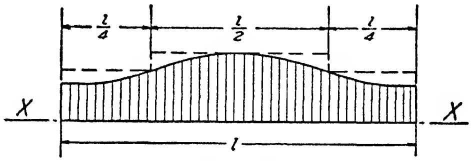


FIG. 128

Such analyses are valuable to justify some simple empirical method for commercial design. Dr. Westergaard finds from tests that there is a relative yielding of the slab at regions of high stress and a redistribution of the internal moment of resistance over a section the width of the panel due to plastic flow. This moment of resistance may vary theoretically as shown in Figure 128 but the redistribution produces a reduction of the maximum value and an increase at points where the moment per inch of width is lower in value. Therefore, the bending moments given by the complex derivations are not realized in practice and the designer uses empirical methods based on such exhaustive analysis as Dr. Westergaard's with due modification to accord with the results of tests of full-sized two-way slabs.

In the past there have been a number of such empirical methods, such as the New York or Boston Codes. At present the two in general use are the A.C.I. and Joint Committee recommendations. The choice between possible empirical methods should be made to fulfill the two basic requirements of any empirical method: (1) economical and safe results, not too far from the theoretical; (2) simple procedure easily applied by a designer who only occasionally uses this type of design.

Using the empirical methods the procedure consists of specification of the moments and shear forces to be used in the design. It is customary to design the slab as though it consisted of strips one unit wide spanning in the north-south or east-west direction. Each strip is designed as though it were a one-way slab and, for bending moment or shear, is assumed to be loaded with an average uniform load consisting of part of the actual load on the panel. The sum of these average loads on the north-south and east-west strips may not equal the total load w pounds per square foot. This is due to the fact that the assumption of the strips is empirical and does not correctly picture the transfer of the loads to the supporting columns.

The design of a one-way slab also assumes strips of unit width but the adjacent strips are loaded similarly and have the same deflections, so there are no shear forces or couples perpendicular to the strip acting on the sides of a given strip which are due to the greater or less deflection of the adjacent strips. The deflection of the two-way slab has a double curvature instead of the trough, or cylindrical, curvature of a one-way slab; therefore the adjacent strips do not deflect the same and shear forces and couples perpendicular to the sides of the strip exist. The empirical methods ignore these shear forces and couples and correct for them by using a reduced load carried by the strip to its supporting beam. It is also customary to design all strips in the central half-width alike and to make some reduction of the steel area in the end quarter-width. Two-way construction is most economical for square panels and does not compare favorably with one-way slabs for rectangular panels whose length is greater than 1.5 the short span.

204. 1940 Joint Committee Two-Way Slab. The 1940 Joint Committee two-way slab applies "to slabs (solid or ribbed), isolated or continuous, supported on all four sides by walls or beams, in either case built monolithically with the slabs. The recommended coefficients, as in the case of the design provisions for flat slabs, are based partly on analysis and partly on test data. (In general, the coefficients and methods given in these recommendations are based upon the coefficients proposed by H. M. Westergaard, M. Am. Soc. C. E. Some modifications of these coefficients have been made and the series extended to include cases not covered by Dean Westergaard. In making these modifications and extensions full consideration has been given to the results of available test data.)"

The moment coefficients are given in Table 5 of the Joint Committee report given in the Appendix. They apply to unit strips taken in the center half-width of the panel. The maximum moment coefficients c_1, whether in the long or short direction, are given in the form $M = c_1 w S^2$, where

c_1 = coefficient from Table 5
w = total load per unit area

S = short span (Spans are taken as the center-to-center distance or as the clear span plus twice the thickness of the slabs, whichever value is less.)

$$m = \text{ratio of } \frac{\text{short span}}{\text{long span}}.$$

The moments are assumed to be constant for all strips in the center half-width and a constant reduced value for the end quarter-widths.

The maximum positive moment is assumed to occur at the center line of the strip, and the maximum negative moment occurs along the edges of the panel at the face of the supporting beams. The bending moments in the strips in the end quarter-widths should average two thirds of the corresponding moments given in Table 5 for the center width. The details of shear stresses, corner reinforcement, and two unequal negative moments on opposite sides of a common supporting beam will be discussed in Problem 35.

ILLUSTRATIVE PROBLEM 35

205. Design of a Joint Committee Two-Way Slab. Figure 129 gives the framing plan of a beam and girder floor system. The slabs are to be designed as two-way

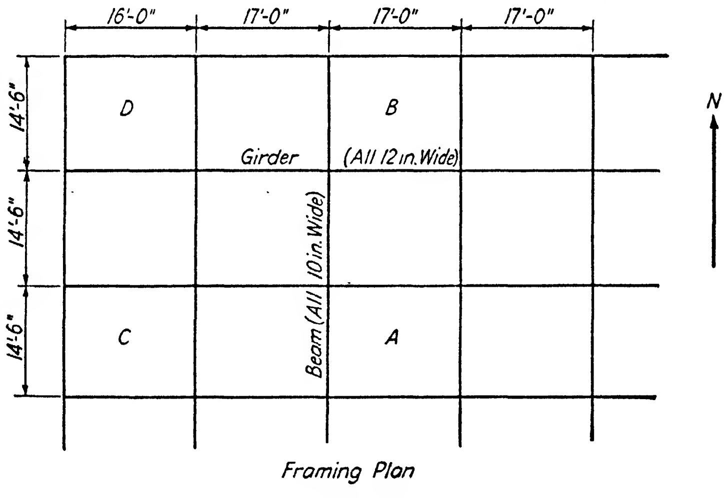


FIG. 129

slabs. The discussion will consider panels A, B, C, and D. The live load equals 130 lb. per sq. ft. and a concrete strength of $f'_c = 2000$ [1] lb. per sq. in. is assumed. All girder stems are 10 in. wide and all beam stems 8 in. wide.

[1] See footnote to Article 32.

Minimum Depth. By J.C. Article 814 (see Appendix) the minimum thickness for a 5-in. slab in panel D is

$$t = \left[14.33 + 16.0 - \frac{(14.33 + 16.0)}{10}\right] \frac{12}{72} \sqrt[3]{\frac{2500}{2000}} = 4.89 \text{ in.}$$

As this is greater than 4 in. and greater than the minimum thickness of any other panel, it is the deciding value. This equation insures that the deflection is not too great.

Assume that the slab has a constant thickness of 5 in. throughout the floor. The total load is $130 + 63 = 193$ lb. per sq. ft. It should be obvious, and Table 5 confirms it, that greater moments for each panel will be taken by the strips in the short direction. The tabulation below lists the moments in the center width for the short spans. Note that the moment coefficient c_1 is determined from Case 2 for panels B and C and from Case 3 for panel D.

NORTH-SOUTH SPANS (SHORT SPANS)

Panel	A	B	C	D
Short span, S ft.	14.33	14.33	14.33	14.33
Long span, L ft.	17	17	16	16
m	0.843	0.843	0.897	0.897
Moment Coefficient c_1				
Positive	0.033	0.039	0.036	0.043
Negative	0.045	0.052	0.048	0.057
Discontinuous negative		0.026		0.028

Since all moments are expressed as $M = c_1wS^2$, the maximum numerical value is given for the strip having the greatest value of c_1; namely, the negative moment at the interior support of panel D. This value is modified by the fact that the coefficient c_1 for the adjacent panel (another C panel) is only 0.048. The slab has constant thickness and the two short spans are the same, so the $\frac{I}{l}$ stiffnesses are the same. The difference in coefficients equals $0.057 - 0.048 = 0.009$. One third of the difference reduces the D panel coefficient to $c_1 = 0.054$ and increases the coefficient of the adjacent panel to $c_1 = 0.051$. The minimum depth for a strip 1 ft. wide equals

$$\text{Negative } d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{0.054 \times 193 \times (14.33)^2 \times 12}{157 \times 12}} = 3.69 \text{ in.}$$

Assuming $\frac{1}{2}$-in. bars, minimum $h = 3.69 + 0.75 + 0.25 = 4.69$ in. For positive bending the fireproofing for glacial gravel equals 1.50 in. The minimum depth equals

$$\text{Positive } d = \sqrt{\frac{0.043 \times 193 \times (14.33)^2 \times 12}{157 \times 12}} = 3.30 \text{ in.}$$

$$\text{Minimum } h = 3.30 + 1.5 + 0.25 = 5.05 \text{ in.}$$

If the contractor were willing to obtain an aggregate "free from disruptive action under high temperatures," the fire protection could be reduced to 1 in. and the slab depth to 5 in. Let us assume that he can economically do so to save a half inch of concrete throughout the floor. Use 5-in. slab with 1 in. of fireproofing.

Diagonal Tension. J.C. Articles 813 and 815 state that the individual panel load shall be distributed to the supporting beam as shown in Figure 130. In the center of the slab there are a few 1-ft. slab-strips wholly in the areas marked I. The maximum shear stress for one of those strips equals

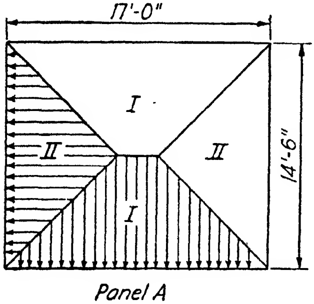


FIG. 130

$$v = \frac{193 \times 14.33}{2 \times 12 \times 0.87 \times 4} = 34 \text{ lb. per sq. in.}$$

It is, of course, not desirable to use diagonal tension reinforcement in a slab. The allowable shear stress $v_c = 40$ lb. per sq. in., so the computed value is safe, and the 5-in. slab will be used.

Design of Steel for Panel B. The determination of steel areas and placement will be illustrated by computations for panel *B*, Figure 129. The short-span steel will be placed outermost for both positive and negative bending in all panels. The steel foreman and the inspector can easily check its proper placement. In the center half-width the steel area equals

$$A_p = \frac{0.039 \times 193 \times (14.33)^2 \times 12}{20{,}000 \times 0.87 \times 3.75} = 0.29 \text{ sq. in.}$$

The modified coefficient for the negative moment at the interior support is $c_1 = 0.050$.

$$\text{Interior support } A_n = \frac{0.050 \times 193 \times (14.33)^2 \times 12}{20{,}000 \times 0.87 \times 4.0} = 0.35 \text{ sq. in.}$$

$$\text{Exterior support } A_n = 0.18 \text{ sq. in.}$$

If $\frac{1}{2}$-in. round bars at 8-in. spacing are used for positive steel, two thirds of the bars will be bent up over each support. The other third will be straight (Fig. 131). The remaining steel at the interior support will come from the adjacent span.

The long-span steel (east-west) will be figured for moment coefficients of $c_1 = 0.031$ for positive bending and $c_1 = 0.041$ for negative bending (J.C. Table 5), assuming $\frac{1}{2}$-in. round bars.

$$A_p = \frac{0.031 \times 193 \times (14.33)^2 \times 12}{20{,}000 \times 0.87 \times 3.25} = 0.26 \text{ sq. in.} \quad \text{Use 9-in. spacing}$$

$$A_n = \frac{0.041 \times 193 \times (14.33)^2 \times 12}{20{,}000 \times 0.87 \times 3.50} = 0.28 \text{ sq. in.}$$

Once again, two thirds of the bars will be bent up at both ends and one third will be straight (Fig. 131). This gives some excess area at the supports but a simple arrangement to be placed in the slab.

The moments in the quarter-widths near the beam supports can be taken as two thirds of the values given by J.C. Table 5. The steel area will be two thirds as much, or the spacing 1.5 that of the center part. This steel is so arranged that there is usually one spacing where the two bands join that is intermediate, say about 1.25 the center spacing.

206. Corner Steel. J.C. Article 811 (see Appendix) states that extra reinforcement should be placed at "exterior corners" of a panel to prevent cracks in diagonal

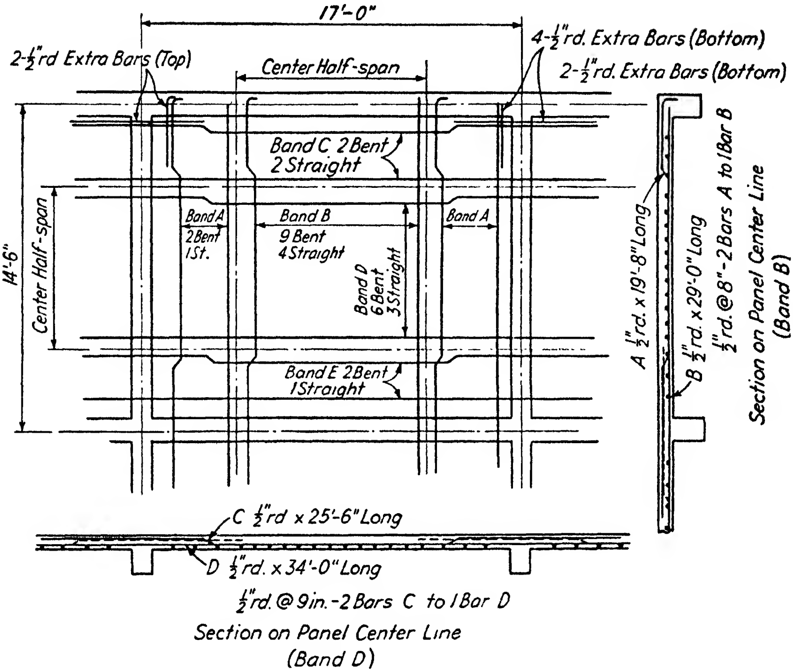


FIG. 131

directions. In Figure 129 panels B, C, and D have discontinuous edges and "exterior corners." The theoretical analysis of slabs supported on four edges demonstrates that extra steel is needed to overcome the tendency of the corner to lift. Cracks tend to appear in the top of the slab perpendicular to the diagonal and in the bottom parallel to the diagonal. If additional steel is placed perpendicular to these potential cracks, there will be four rows of steel in a thin slab, so it is customary to add extra north-south and east-west bars in the region of the corner (quarter-span). The area of steel, if placed perpendicular to the potential crack, should be equal to the short-span steel area resisting the positive moment in the center half-span. The areas of steel, if placed parallel to the edges of the panel, are equal to this "positive moment area" multiplied by the sine of the angle that the bar makes with the potential crack. This is illustrated in Figures 132 and 133.

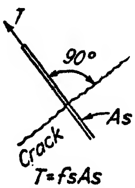
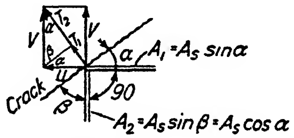


FIG. 132



$U = f_sA_1 = f_sA_s \sin\alpha$ $\quad V = f_sA_2 = f_sA_s \cos\alpha$

$T_1 = U \sin\alpha = f_sA_s \sin^2\alpha$ $\quad T_2 = V \cos\alpha = f_sA_s \cos^2\alpha$

$T_1 + T_2 = f_sA(\sin^2\alpha + \cos^2\alpha) = f_sA_s = T$

FIG. 133

The positive steel area for the center half of panel B equals 0.29 sq. in. per ft. width for the short-span steel (north-south). The diagonal of this panel makes an angle with the 17-ft. side whose sine equals 0.65 and whose cosine is 0.762. The potential crack in the top of the slab is perpendicular to the diagonal and there are two $\frac{1}{2}$-in. bars bent up in the quarter-width for both short- and long-span steel. There is available $0.196\left[\frac{0.65 \times 4 \times 4}{17} + \frac{0.76 \times 4 \times 2}{14.33}\right] = 0.203$ sq. in. per ft. This is not enough, but one extra bar in each direction will give the required 0.29 sq. in. The potential crack in the bottom of the slab is parallel to the diagonal and there are two $\frac{1}{2}$-in. bars in the long-span and one in the short-span direction. The available area equals $0.196\left[\frac{0.762 \times 4 \times 1}{17} + \frac{0.65 \times 4 \times 2}{14.33}\right] = 0.106$ sq. in. per ft. The addition of four extra bars in the short-span and two extra in the long-span direction will give a total area of 0.32 sq. in. per ft.

207. Beam Size. *Long-Span East-West Girder.* The long-span girder on the interior edge of panel B carries a load from panel B similar to area I shown in Figure 130. An equivalent uniform load to give the same positive maximum bending moment in the beam can be computed from an equation in J.C. Article 815. The uniform load w_b equals

$$w_b = \frac{wS}{3}\left[\frac{3 - m^2}{2}\right] = 1060 \text{ lb. per ft.}$$

From adjacent panel	=	1060 lb. per ft.
Estimated stem weight	=	180 lb. per ft.
Total		2300 lb. per ft.

Let us assume this beam to be supported by columns 20 in. square. The maximum negative moment equals

$$M_n = \frac{wl^2}{11} = \frac{2300 \times (15\frac{1}{3})^2 \times 12}{11} = 590{,}000 \text{ in.-lb.}$$

The minimum depth (in order not to use compression steel at the support) equals

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{590{,}000}{157 \times 10}} = 19.4 \text{ in.}$$

Minimum $h = 19.4 + 2.5 = 21.9$ in. Use 10 in. by 17 in. stem.

Each panel brings to this beam a load similar to the shaded area I in Figure 130 which shows the portion of the panel B load assigned to this girder. By the equation in J.C. Article 815, the total load equals

$$W_l = \frac{wS^2}{4}\left[\frac{2 - m}{m}\right] = 13{,}650 \text{ lb.}$$

The adjacent panel brings in the same load, half of the total being supported at each end. $V = 13{,}650 + 180 \times 15.33 \times 0.5 = 15{,}030$ lb.

$$v = \frac{V}{bjd} = \frac{15{,}030}{10 \times 0.87 \times 19.5} = 89 \text{ lb. per sq. in.} \qquad \text{Safe.}$$

A tee-beam section with stem 10 in. by 17 in. is satisfactory. The design of the steel for bending moment and diagonal tension can be completed by following the procedures described in Chapter 7. If it is desired to obtain moments or shears other than the maximum values, the beam loads should be set up with area I of Figure 130 from each adjacent panel used as panel loads.

Short-Span North-South Beam. According to the provisions of J.C. Article 815, the uniform w_b on the short-span beam is

$$w_b = \frac{wS}{3} = \frac{193 \times 14.5}{3} = 930 \text{ lb. per ft.}$$

$$\text{From adjacent panel } w_b = 930 \text{ lb. per ft.}$$

$$\text{Estimated stem weight} = \underline{140} \text{ lb. per ft.}$$

$$\text{Total} = 2000 \text{ lb. per ft.}$$

The maximum negative moment at the first interior column equals

$$M_n = \frac{wl^2}{10} = \frac{2000 \times (12.83)^2 \times 12}{10} = 395{,}000 \text{ in.-lb.}$$

The minimum depth in order to have no compression steel equals

$$d = \sqrt{\frac{395{,}000}{157 \times 8}} = 17.8 \text{ in.}$$

$$\text{Minimum } h = 17.8 + 2.5 = 20.3 \text{ in.} \qquad \text{Use 8 in. by 16 in. stem.}$$

The total load to the beam from this panel equals

$$W_s = \frac{wS^2}{4} = \frac{193 \times (14.33)^2}{4} = 9950 \text{ lb.}$$

The maximum shear force at the interior support equals

$$V = 0.575[2 \times 9950 + 140 \times 12.83] = 12{,}500 \text{ lb.}$$

$$\text{Maximum shear stress } v = \frac{12{,}500}{8 \times 0.87 \times 18.5} = 97 \text{ lb. per sq. in.}$$

A tee beam with an 8-in. by 16-in. stem is satisfactory; the steel can now be computed. The bending of bars and the shear stresses for diagonal tension should be computed by using moment and shear diagrams for a beam load from each panel equal to area II of Figure 130.

208. One-Way Slab. A comparison of one-way and two-way designs can be made with the use of panel *B*. A one-way slab would span in the short direction. If the methods of Chapter 3 are used the design results in a 6-in. slab with ½-in. round bars spaced at 6 in. for the positive steel. There are no short-span beams. The long-span girder, figured on the same assumptions as above, has a stem 12 in. by 18 in. The steel can be computed as described in Chapter 7.

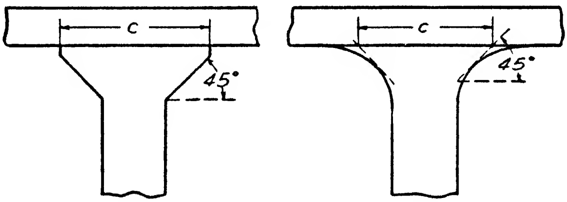
COMPARISON OF DESIGNS

	Two-Way J.C.	One-Way
Volume of slab concrete (17′ × 14.5′), cu. ft.	102.7	123.0
Volume of beam stems, cu. ft.	29.5	23.0
Total concrete, cu. ft.	132.2	146.0
Slab steel, lb.	470	500
Extra "corner steel," lb.	50	

This comparison does not include the beam steel, but does include the necessary temperature steel for the one-way slab. The use of a two-way slab results in a saving in both concrete and steel in the interior panels. For the conditions of panel *B* there is a saving in concrete with the two-way system, but the extra corner steel gives a greater total weight of steel.

FLAT SLABS

209. Flat Slabs. Flat slabs are a type of floor slab developed in the United States which are peculiar to reinforced concrete construction. The slab is not supported by beams or girders but transmits its loads directly to the columns. The columns are enlarged at their tops into capitals to give additional rigidity to the slab-column connection (Fig. 134). Formerly the "mushroom" top (Fig. 134*b*) was widely used, but



(*a*) (*b*)

FIG. 134

the restriction that the capital diameter *c* is that portion within a 45° tangent to the curve led to the adoption of the 45° cone (Fig. 134*a*) as the usual capital. In addition, the forms for mushroom capitals are more difficult to construct.

Interior columns may be round, square, octagonal, or hexagonal, but they are usually round. Exterior columns are usually rectangular. The capital for an exterior column is sometimes only a bracket which projects from the column toward the interior column. The exterior capital may project in three directions; this gives greater rigidity to the exterior column in restraining the end span of the continuous slab, but the construction is harder to form and may project into the window area.

The slab is much thicker than the beam and girder floor slabs and it has more steel, but the omission of beams and girders gives

(a) Shorter story heights for a given clear height.
(b) Better fire protection with a flat ceiling and better play for a sprinkler system.
(c) Economical form design covering flat surfaces.
(d) A uniform surface for a ceiling from which to hang piping, shafting, etc.
(e) Favorable costs compared with beam and girder floors for panels approximately square with medium to long spans and medium to heavy loads. The panels with a large ratio $\frac{l}{b}$, or panels with light load or short spans, are cheaper with ribbed floor or some form of beam and girder construction.

The flat slab is often constructed with a deepening of the slab as it nears the column capital. This deeper portion is known as the *dropped*

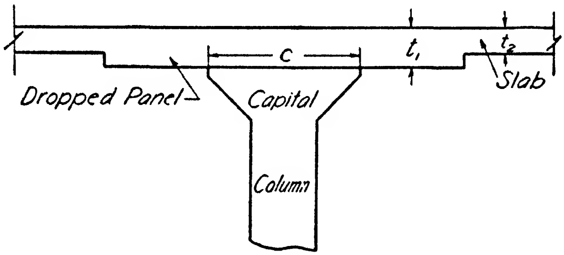


FIG. 135

panel (Fig. 135). The extra cost of the forms for a dropped panel floor is usually offset by the saving in concrete in the thinner slab. Figures 136 and 137 show sections through flat slab and dropped panel floors.

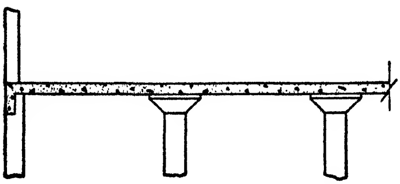


FIG. 136

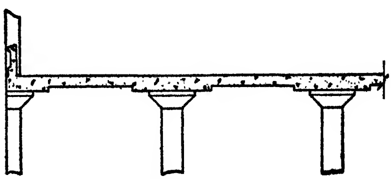


FIG. 137

210. Flat Slab Systems. Special flat slab systems have been protected by patents. Most of these patents have expired. The usual systems of steel arrangement now in use are some form of two-way or four-way steel in a flat slab or dropped panel floor. One type uses three-way steel with the columns located at the apices of equilateral triangles.

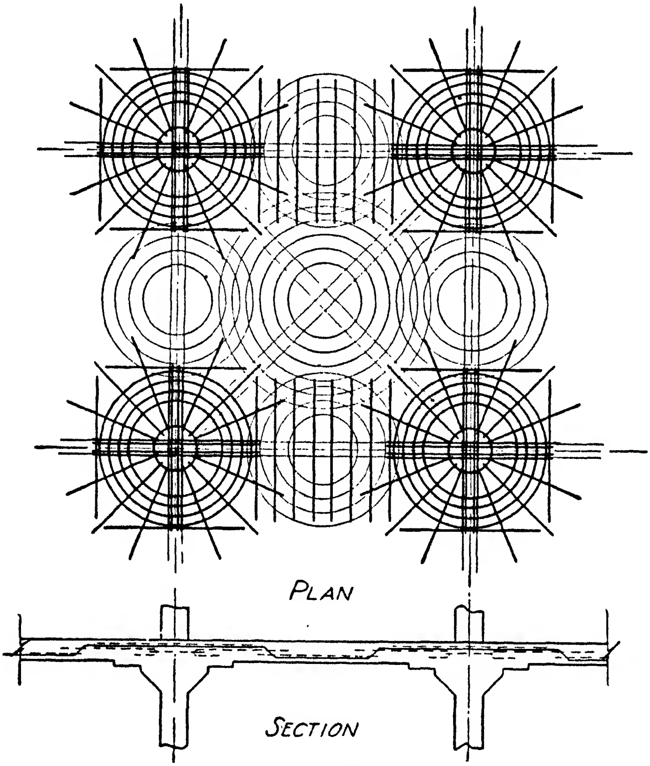


FIG. 138

This is not a very convenient column arrangement for aisles or for the diffusion of light in many buildings. A unique steel arrangement is the Smulski "S.M.I." system of Figure 138. The darker lines represent negative steel in the top of the slab. The S.M.I. design gives less weight of steel than the usual two-way or four-way system.

211. A.C.I. Flat Slab. In the past the building laws of different cities have varied greatly in their flat slab specifications. The Chicago, New York, or Philadelphia regulations gave different minimum thicknesses and led to variable steel arrangements for economy. The Joint Committee Report and the A.C.I. Code give recommendations for a flat slab design which is conservative yet of reasonable economy. Since it bids fair to be the basis for future regulations the author will use the A.C.I. slab for his illustrative designs (see A.C.I. Arts. 1000–1009 in the Appendix).

The A.C.I. regulations are to be regarded as empirical specifications based on a careful analysis modified by the results of tests. The problems of moment distribution, deflection, bond, and anchorage of the steel are handled by empirical specifications instead of a theoretical analysis.

212. Statical Analysis of a Flat Slab. Let us consider a square interior panel with a span of l between column centers. It is loaded with a uniform load of w pounds per square foot, and the adjacent panels are likewise interior panels with the same load. We shall take as a rigid body the half panel (Fig. 139) bounded by the center line BC, the panel edges EB, FG, HC, and the curved lines, FE and GH, at the perimeter of the column capitals. The column capitals have a diameter c.

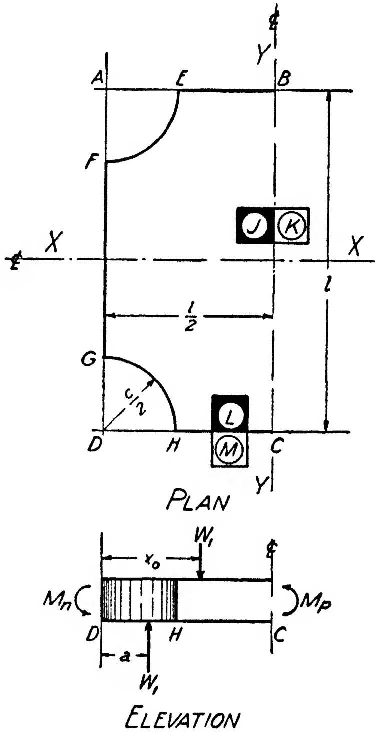


FIG. 139

Since the panel is loaded symmetrically, a square foot of floor at J will deflect the same amount as its opposite K, the other side of the center line. Therefore, there is no tendency for a sliding of the two by each other at the section BC, and there can be no shear on BC. The same line of reasoning holds for the symmetrically placed areas at L and M adjacent to the panel edge HC. Therefore, there is no shear along BE, FG, and HC. There is shear on the curved edges EF and GH, because the column capital is more rigid than the adjacent slab and deflects less. Therefore the load W_1 on the half panel is supported by the shear on the curved lines EF and GH.

The resultant load W_1 acts on the axis of symmetry XX at the center of gravity of the half panel. The half-panel area is the rectangular area $ABCD$ minus the two quadrants, AEF and DGH. Taking moments about the edge AD, the center of gravity is a distance x_o from AD equal to

$$x_o = \frac{\Sigma M}{A} = \frac{\left(\frac{l^2}{2}\right)\frac{l}{4} - \frac{\pi c^2}{8}\frac{2c}{3\pi}}{\frac{l^2}{2} - \frac{\pi c^2}{8}} = \frac{3l^3 - 2c^3}{3(4l^2 - \pi c^2)}$$

If the load equals w pounds per square foot,

$$W_1 = wA = \frac{w}{8}(4l^2 - \pi c^2) \tag{180}$$

The distribution of the shearing forces along the column capital perimeters is not known. If we deal with the average force, which is equivalent to the assumption of uniform distribution, the resultant force on EF or GH equals $\frac{W_1}{2}$ and acts at a distance $a = \frac{c}{\pi}$ from the diameter AD. These two forces give a resultant W_1 acting on the XX axis at a distance a from AD.

The only other external loadings on the rigid body are bending couples at half-panel edges. These are in vertical planes perpendicular to the edges. Those along the center line BC are positive bending moments M_p. Those along the panel edges BE, FG, and HC are negative moments. The negative moments perpendicular to the circular arcs EF and GH can be resolved into component couples parallel to those on the edge FG and the edge BE. Let M_n be the sum of the couples on edge FG plus the parallel components on the curved arcs.

The load W_1 and the resultant W_1 of the supporting forces do not act at the same point, though both are on the XX axis. They form a couple equal to

$$M = W_1(x_o - a) = W_1\left[\frac{3l^3 - 2c^3}{3(4l^2 - \pi c^2)} - \frac{c}{\pi}\right] = W_1\left[\frac{3\pi l^3 - 12cl^2 + \pi c^3}{3\pi(4l^2 - \pi c^2)}\right]$$

Substituting from equation 180 the value of W_1,

$$M = \frac{w}{24\pi}(3\pi l^3 - 12cl^2 + \pi c^3) = w\left(\frac{l^3}{8} - \frac{cl^2}{2\pi} + \frac{c^3}{24}\right) \tag{181}$$

This couple tends to turn the half panel about some axis parallel to the YY axis. If the half panel is in equilibrium the couples M_p and M_n must balance the couple M. In Figure 139 M is clockwise and both M_p and M_n are anti-clockwise. Therefore,

$$M = M_p + M_n \tag{182}$$

The couples along BE and the components on EF parallel to them balance those on HC and the component couples on GH.

Equation 181 can be written as

$$M = \frac{wl^3}{8}\left[1 - \frac{4}{\pi}\frac{c}{l} + \frac{1}{3}\left(\frac{c}{l}\right)^3\right]$$

This is approximated by the equation

$$M = M_p + M_n = \frac{wl^3}{8}\left(1 - \frac{2}{3}\frac{c}{l}\right)^2 = \frac{Wl}{8}\left(1 - \frac{2}{3}\frac{c}{l}\right)^2 \qquad (183)$$

$$W = \text{load on whole panel} = wl^2$$

This approximate analysis by the principles of statics gives values greater than the results of tests or of Professor Westergaard's more complete analysis. This is due to the fact that the statical derivation does not consider the actual deflections of the panel nor the effect of plastic flow on the moment distributions. The more exact analyses, modified by the study of tests, have led to a reduction of the sum $M_p + M_n$ but the general form of the statical moment equation has been used. The coefficient of the moment sum has been reduced as follows.

	Moment Coefficient	Percentage of Statical Analysis
Statical analysis	0.125	100
1917 A.S.C.E.	0.107	85
1921 Joint Committee	0.09	72

The coefficient 0.09 of the reduced moment was recommended by Professor Westergaard. The total moment equals

$$M = 0.09Wl\left(1 - \frac{2}{3}\frac{c}{l}\right)^2 \qquad (184)$$

213. Division of Moments of Resistance. Equation 182 states that the external bending moment M is resisted by the internal moments of resistance at the sections of maximum positive and negative bending moment. The actual variation of moment in these sections can be computed analytically and has also been checked by tests. Figure 140 shows the moment variation as computed by Professor Westergaard for the case where the column capital $c = 0.25l$. The shaded areas should add up to M_n and M_p of equation 182. For design it is customary, as with the two-way slabs, to divide the panel into two parts, or strips, and assume a constant moment throughout each strip. The middle half-

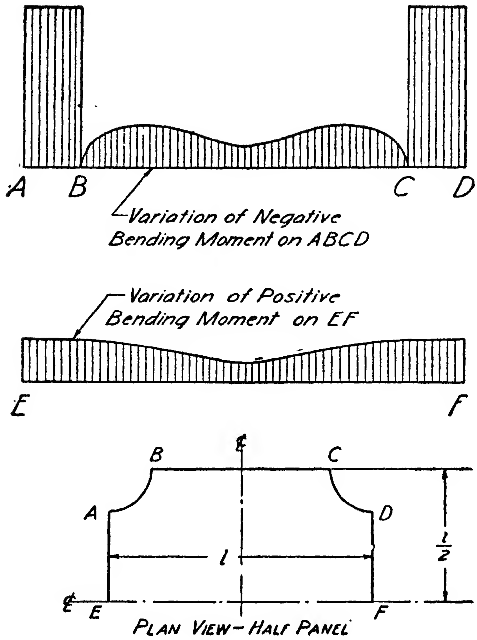


FIG. 140

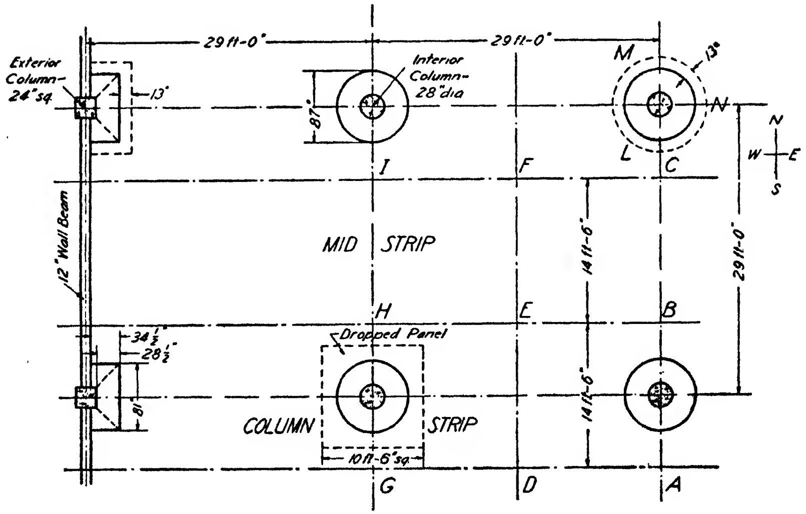


FIG. 141

span is called the *mid-strip*, and the half-span whose center is the column center line is called the *column strip* (Fig. 141). The A.C.I. Code specifies the bending moment to use in each strip. The amounts vary with two-way or four-way steel and with flat slab or dropped panel slabs. Some latitude is also given the designer, but in general

$$M_p = \tfrac{3}{8}M \qquad \text{and} \qquad M_n = \tfrac{5}{8}M$$

The steel which supplies the tension force for these couples will run in an east-west direction (Fig. 139). The north-south steel will resist moments found by taking the half-panel cut by the axis XX instead of by YY. In a square panel it will be the same as the east-west steel.

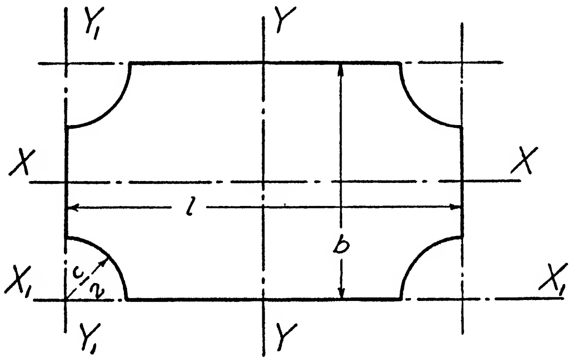


FIG. 142

214. Rectangular Panels. The analysis of Article 212 holds for a rectangular panel, if it is also an interior panel surrounded by similar interior panels. In that case the total bending moment on sections Y_1Y_1 and YY for the east-west steel equals (Fig. 142)

$$M_l = 0.09Wl\left(1 - \frac{2}{3}\frac{c}{l}\right)^2$$

where M_l = moment to be taken by long-span steel.

The total moment to be resisted at sections XX and X_1X_1 equals

$$M_b = 0.09Wb\left(1 - \frac{2}{3}\frac{c}{b}\right)^2$$

where M_b = moment to be taken by short-span steel.

215. Order of Procedure for Design. The suggested order of procedure for design requires, as usual, a check of the concrete dimensions before any steel is figured. The procedure is

A. Concrete Dimensions: 1. By empirical formulae.
2. Checked for fiber stress.
3. Checked for diagonal tension and shear.

B. Steel Area: 4. For bending moment.
5. For minimum and maximum areas.
6. Placing regulations to cover bond and anchorage.

ILLUSTRATIVE PROBLEM 36

216. Design of Dropped Panel Slab with Two-Way Steel. Design the interior and exterior panels of a two-way flat slab using 2000-lb. concrete for the floor system of Problem 17. In Figure 61 (Art. 121) assume that the alternate column rows are left out, giving column spacings of 29 ft. in both directions. The live load is 130 lb. per sq. ft.

Assume interior columns of 28 in. diameter; the exterior columns will be 24 in. square to fulfill the requirement of 1928 A.C.I. Article 1105*b* that the least dimension of exterior columns must be not less than one fifteenth the average center-to-center span. The plan view of the exterior and adjacent interior panel is given in Figure 141.

217. Minimum Thickness. A.C.I. Article 1006 (see Appendix) gives a requirement to avoid excessive deflection.

$$\text{Minimum } t = \frac{l}{40} = \frac{29 \times 12}{40} = 8.7 \text{ in.} \qquad \text{Assume 9-in. slab.}$$

218. Depth to Satisfy Fiber Stress. The column capital c is usually assumed to be $0.20l$ to $0.25l$, with $0.225l$ giving a good trial size. Assume a column capital $c = 80$ in. $= 0.23l$. In the past a minimum dropped panel width of $0.35l$ was required. Let us assume a square drop of 10 ft. 6 in. on a side. Formerly the dropped panel thickness t_1 had to be within $1.25t$ and $1.5t$, where t is the thickness of the slab. This would require t_1 to be between $11\frac{1}{4}$ and $13\frac{1}{2}$ in. The present A.C.I. Article 1006 merely permits a maximum increase $t_1 - t$ of one fourth the distance from the edge of the capital to the edge of the dropped panel. The maximum thickness $t_1 = t + \frac{63 - 40}{4} = 9 + 5.75 = 14.75$ in. Let us assume a thickness $t_1 = 11.5$ in. The load W on the panel equals

$$\begin{aligned}
\text{Live load} &= 130 \times (29)^2 = 109{,}000 \text{ lb.} \\
\text{9-in. slab} &= 113 \times (29)^2 = 95{,}000 \\
\text{2.5-in. drop} &= 31 \times (10.5)^2 = 3{,}000 \\
\hline
\text{Total load} &= 207{,}000 \text{ lb.}
\end{aligned}$$

By A.C.I. Article 1003,

$$M_o = M_n + M_p = 0.09 \times 207{,}000 \times (29 \times 12)(1 - \tfrac{2}{3} \times 0.23)^2 = 4{,}650{,}000 \text{ in.-lb.}$$

This moment is divided between the positive and negative moments. In each case it is assumed to be constant in the column head and mid-sections. The suggested division is listed in the three tables in A.C.I. Article 1004 and is tabulated below for this problem as a decimal part of M_o.

	Negative Moment				Positive Moment	
	Column		Mid		Column	Mid
Interior panel	0.50		0.15		0.20	0.15
Exterior panel	Int.	Ext.	Int.	Ext.		
East-west span	0.55	0.45	0.165	0.10	0.25	0.19
North-south span	0.125 *		0.15		0.05 *	0.15

* Half-strip.

The coefficients for the interior panel and the mid-strips in the exterior panel parallel to the discontinuous edge (north-south) are taken from Table 1004*a*. The north-south strips in the exterior panel are spanning an interior span. The coefficients for the half column-head strip adjacent to the discontinuous edge are taken from A.C I. Table 1004*c*. The coefficients for the strips in the exterior panel perpendicular to the discontinuous edge (east-west) are taken from A.C.I. Table 1004*b*.

The exterior span, perpendicular to the exterior wall, is often made less than the interior spans to offset the increase in moment coefficients.

Slab. It is apparent from Figure 141 that the sections along FE, ED and IH cut through the thin slab only. The maximum moment coefficient for these sections, positive and negative for the mid-strips and positive for the column strips, is 0.25 for the east-west column-strip positive moment. The minimum depth for this strip, 14.5 ft. wide, is

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{0.25 \times 4{,}650{,}000}{157 \times (14.5 \times 12 \times 0.75)}} = 7.5 \text{ in.}$$

Notice that A.C.I. Article 1003*c* states that compressive stresses are figured by using only three fourths the width of the strip. This is an arbitrary method of insuring a sufficient depth and supersedes empirical formulae for minimum depth formerly used. Assuming $\frac{3}{4}$-in. rounds, the minimum thickness $t = 7.5 + 1.5 + 0.38 = 9.4$ in. Change slab depth to 9.5 in. Corrected weight $W = 212{,}000$ lb. and corrected $M_o = 4{,}760{,}000$ in.-lb.

Dropped Panel. The maximum negative moment in the column strip acts on the section GH (Fig. 141). This section is shown in Figure 143. The shaded compression area is tee shaped, but the broader part is near the neutral axis. We have not developed an analytical treatment for this case of a tee-beam section. The areas GP

and QH beyond the dropped panel are not large and have low stresses. We shall be on the safe side, and not too wasteful, if we neglect them and figure the compression area as a rectangle PQ, the width of the dropped panel. A.C.I. Article 1003c confirms this.

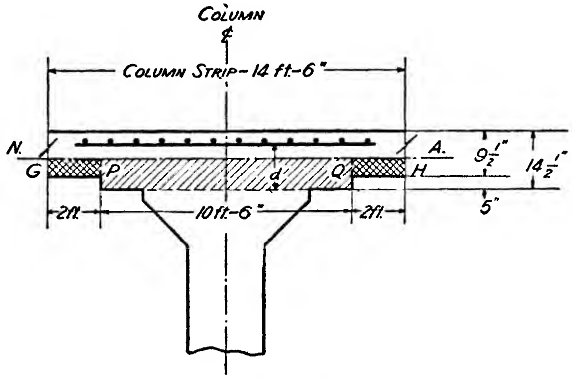


FIG. 143

The maximum coefficient for the negative moment in the column strips is 0.55 for the east-west strip in the exterior span. There are two layers of steel crossing at the top of the dropped panel but we shall assume a steel placement schedule that puts the east-west steel above the north-south. The clearance $t_1 - d = 0.75 + 0.38 = 1.13$ in.

$$\text{Minimum } d = \sqrt{\frac{0.55 \times 4{,}760{,}000}{157 \times (126 \times 0.75)}} = 13.27 \text{ in.}$$

The minimum thickness $t_1 = 13.27 + 1.13 = 14.4$ in. The A.C.I. Article 1006 maximum thickness $t_1 = 9.5 + 5.75 = 15.25$ in. Use a drop panel thickness $t_1 = 14.5$ in. The corrected weight $W = 216{,}000$ lb. and corrected $M_o = 4{,}850{,}000$ in.-lb.

219. Diagonal Tension and Shear. The designer does not wish to supply diagonal tension steel in a slab. Therefore, the slab depth must be great enough to give shear

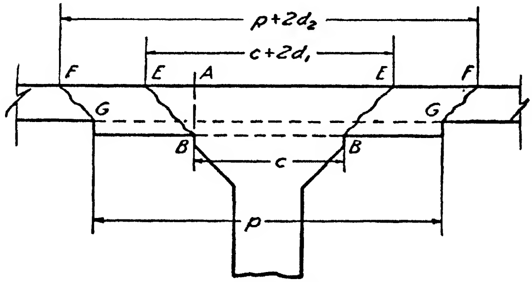


FIG. 144

stresses, or "equivalent tension," that the concrete can carry. If this is done the true shear (punching shear) will be satisfactory, since its allowable value is $v = 0.06f'_c$ for sections crossed by longitudinal steel. The true shear is a tendency of the slab to slide by the column on section AB (Fig. 144). Diagonal tension will not occur at

AB, but at the first section from which a crack can run through the slab. This is at section E. Assuming a 45° angle for the crack EB, the section E at which the crack starts is a distance d from the capital. The section E is therefore the perimeter of a circle $c + 2d$ in diameter. A.C.I. Article 807 specifies that this perimeter be checked for shear by using $d = t - 1.5$. Each interior column supports the slab halfway to the next column. This is an area 29 ft. square. The shear force V at section E will be due to the load on the portion of the area outside the perimeter LMN (Fig. 141) whose diameter equals $80 + 2 \times 13 = 106$ in. $= 8.83$ ft. The average load per square foot can be taken as $w = \dfrac{W}{A} = \dfrac{216{,}000}{841} = 257$ lb. per sq. ft.

$$v = \frac{V}{bjd} = \frac{257\left[841 - \dfrac{\pi(8.83)^2}{4}\right]}{(\pi \times 106) \times 0.87 \times 13} = 53 \text{ lb. per sq. in.}$$

The allowable shear stress can vary between $v = 0.025f'_c$ and $v = 0.03f'_c$, according to the percentage of column-strip negative steel passing directly over the column capital. This steel will be uniformly spaced over the column-strip width of 14.5 ft. The per cent crossing the column capital will be found from the ratio of the widths 80 in. to 174 in., or 46 per cent. The allowable stress (A.C.I. Article 807) equals

$$v = 0.025f'_c + \tfrac{21}{25}(0.005f'_c) = 50 + 0.84 \times 10 = 58.4 \text{ lb. per sq. in.}$$

The diagonal stresses are safe in the dropped panel.

Slab. The slab is checked for diagonal tension at section F (Fig. 144); $d = t_2 - 1.5 = 9.5 - 1.5 = 8.0$ in. is used. The perimeter of section F is that of a square with sides of $126 + 2 \times 8 = 142$ in. $= 11.83$ ft.

$$v = \frac{257[841 - (11.83)^2]}{4 \times 142 \times 0.87 \times 8.0} = 46 \text{ lb. per sq. in.}$$

Allowable stress $v = 0.03f'_c = 60$ lb. per sq. in. Safe.

A.C I. Article 807*b* states that 50 per cent of the negative steel in the column strip must be within the width of the drop panel. Since this steel is evenly spaced the relative amount of steel will be found by the ratio of widths, 10.5 ft. to 14.5 ft., which is 72.5 per cent.

The concrete has now been checked for deflection, fiber stress, and diagonal tension; so the slab thickness of 9.5 in. and dropped panel thickness of 14.5 in. will be definitely adopted.

220. Steel Areas for Fiber Stress. Table A gives the computations for steel areas. An average value of d has been used in the interior panel wherever the north-south and east-west steel are on the same side of the slab. In the exterior panel it is assumed that the north-south steel is placed first (lowest) with the east-west steel above it.

The steel placement is shown in Figure 145. The present code merely specifies that the bars shall be evenly spaced across the full width of the strip and shall provide for bending moment and bond stresses at all sections. More definite instructions for the inexperienced designer are given in the extract from the 1928 A.C.I. Code in the Appendix. These recommendations provide for a suitable steel distribution to avoid cracks in the slab and to avoid bond computations.

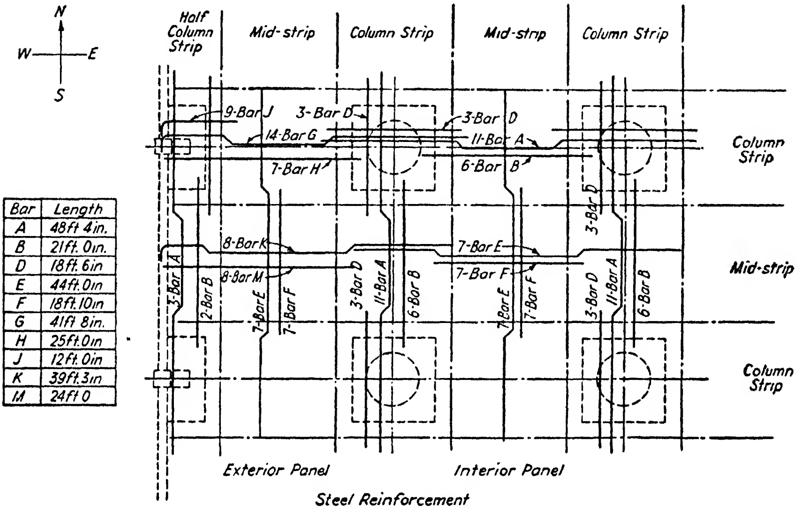
TABLE A. COMPUTATIONS FOR FLAT SLAB STEEL.

	INTERIOR PANEL				EXTERIOR PANEL STEEL PARALLEL TO WALL				EXTERIOR PANEL STEEL PERPENDICULAR TO WALL					
M_o = 4,850,000 in.-lb.	Column Strip		Mid-Strip		Column Strip		Mid-Strip		Column Strip			Mid-Strip		
	Positive	Negative	Pos.	Neg.	Pos.	Neg.	Pos.	Neg.	Pos.	Int. Neg.	Ext. Neg.	Pos.	Int. Neg.	Ext. Neg.
Moment Coefficient	0.20	0.50	0.15	0.15	0.05*	0.125*	0.15	0.15	0.25	0.55	0.45	0.19	0.165	0.10
Moment, in.-kips	970	2,425	728	728	243	606	728	728	1,213	2,670	2,180	922	800	485
9.5" 5"	d	d	d	d	d	d	d	d	d	d	d	d	d	d
Assuming $\frac{3}{4}$-in round bars. Depth d, in.	7.62	13.00	7.25	8.37	7.62	13.37	6.87	8.37	7.62	12.62	12.62	7.62	8.37	8.37
$A_s = \dfrac{M}{20{,}000 \times 0.87 \times d}$ sq.in.	7.32	10.72	5.77	5.00	1.83*	2.61*	6.08	5.00	9.15	12.16	9.93	6.95	5.50	3.33
Minimum $A_s = 0.0025 \times 174 \times d$	3.32		3.16	3.65	1.66*		2.99	3.65	3.32			3.32	3.65	3.65
Minimum $A_s = 0.0025 \times 126 \times d$		4.09				2.10*				3.97	3.97			
Use A_s, sq.in.	7.32	10.72	5.77	5.00	1.83*	2.61*	6.08	5.00	9.15	12.16	9.93	6.95	5.50	3.65
Number of bars	17	25	14	12	5*	6*	14	12	21	28	23	16	13	8

*For half-strip

Placement. The steel in the column strips should provide 0.4 of the positive area in the form of bar A of Figure 146. At least one third of the area should be straight bars in the bottom as bar B. Since the negative areas are quite large, the minimum of bars B will be used and all the rest will be bent as bars A. Counting the bars A that come from the adjacent panel, the full area required is usually not satisfied and it is necessary to add additional top bars as bar D.

In the mid-strip at least one half of the steel must be like bars E of Figure 146.



Bar	Length
A	48 ft 4 in.
B	21 ft. 0 in.
D	18 ft. 6 in
E	44 ft. 0 in
F	18 ft. 10 in
G	41 ft 8 in.
H	25 ft. 0 in
J	12 ft. 0 in
K	39 ft. 3 in
M	24 ft 0

FIG. 145

Interior Panel. In the column strip for positive steel use $\frac{17}{3} = 6$ bars B. Use $17 - 6 = 11$ bars A. Use no bars C. At the line of maximum negative moment there are now $2 \times 11 = 22$ bars A. Add 3 bars D. Total negative bars equal $2 \times 11 + 3 = 25$ bars.

In the mid-strip for positive steel use $\frac{14}{2} = 7$ bars E and 7 bars F. At the line of maximum negative moment there will be $2 \times 7 = 14$ bars E; only 12 are needed.

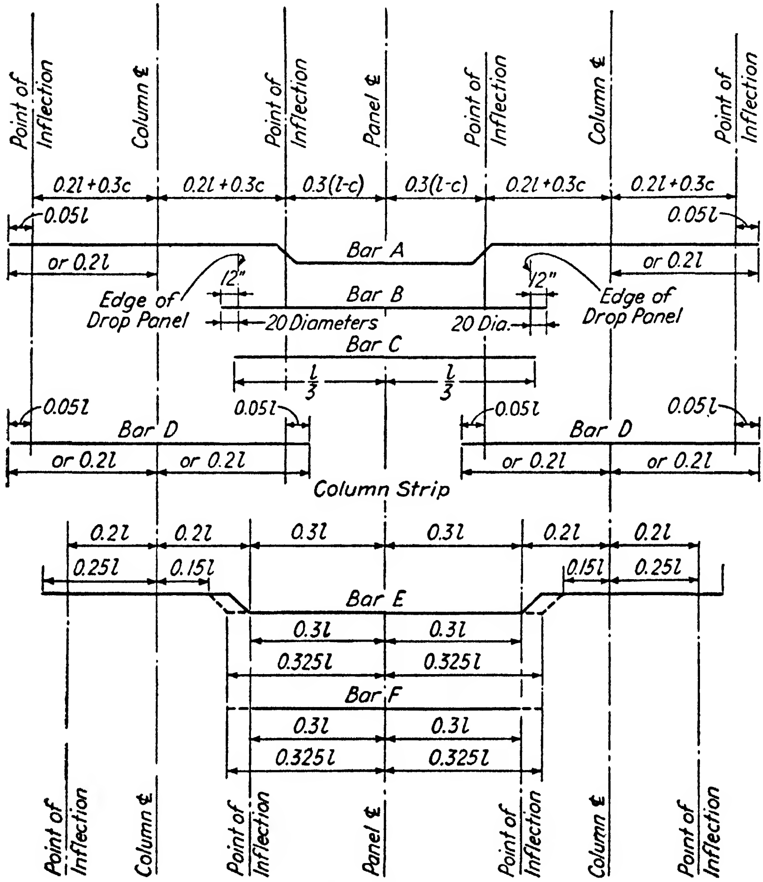
Exterior Panel. East-West Steel. In the column strip for positive steel use $\frac{21}{3} = 7$ bars H and 14 bars G (Fig. 147). At the exterior support there will be 14 bars G; add 9 bars J for a total of 23 bars. At the interior line of maximum bending moment there are 14 bars G plus 11 bars A. Add 3 bars D to give a total of 28 bars.

In the mid-strip for positive steel use $\frac{16}{2} = 8$ bars K, plus 8 bars M. At the exterior support there will be 8 bars K, which is sufficient. At the interior line of maximum moment there are 8 bars K and 7 bars E for a total of 15 bars; 13 are needed.

Exterior Panel. North-South Steel. The half-column strip on the interior column side will be identical with that of the interior panel; this is also true for the steel in the mid-strip.

The half-column strip adjacent to the wall beam will have 2 bars B and 3 bars A for positive steel. At the line of maximum negative moment there will be $3 + 3 = 6$ bars A, which is the required amount.

The greatest spacing of these bars occurs for the east-west steel in the mid-strip of the exterior panel at the exterior support. Eight bars are spaced evenly in a width of 174 in., or an average spacing of about 22 in. This does not exceed 3 × 9.5 = 28.5 in. (A.C.I. Article 1008*b*).

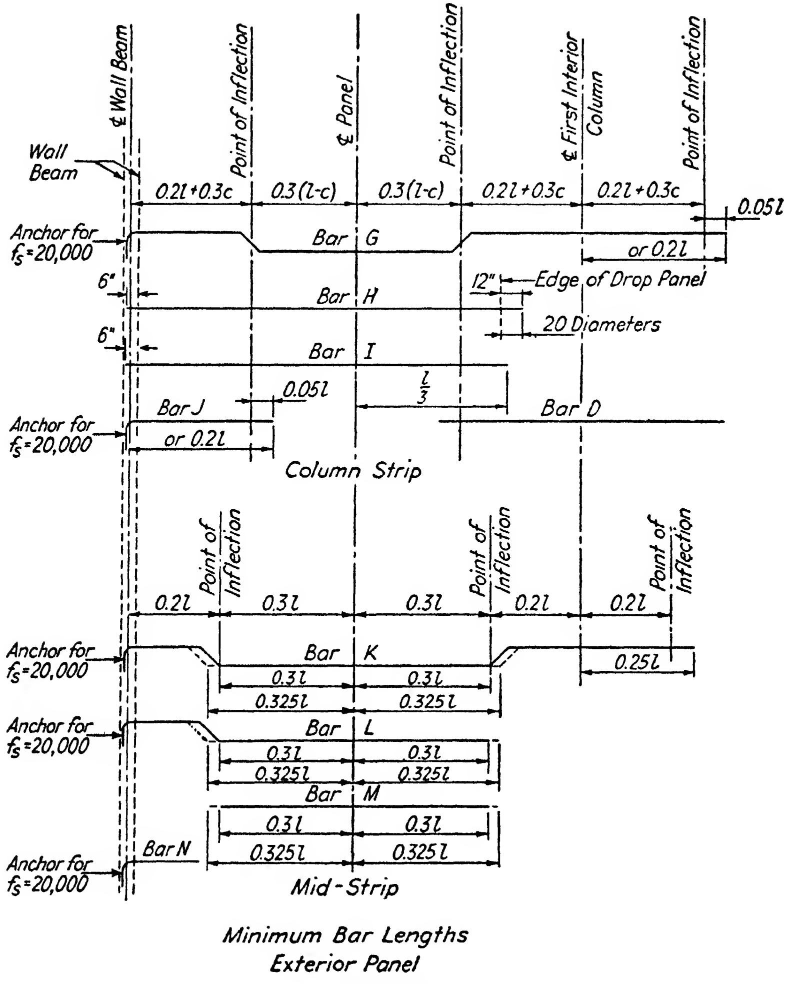


Mid-strip
Minimum Bar Lengths
Interior Panel

FIG. 146

Length of Bars. The present A.C.I. Code does not give detailed instruction for determining bar lengths to satisfy bending moment and bond. Until experience is gained the designer may well follow the recommendations of the 1928 A.C.I. Code. These are summarized in Figures 146 and 147. In certain cases two minimum anchorages are given, so the greater value is used. Bars proportioned to these requirements are not checked for bond or anchorage.

221. Wall Beam and Columns. It has been assumed that the wall beam has a depth greater than 1.5 times the slab thickness ($1.5 \times 9.5 = 14.25$ in.). By A.C.I. Table 1004*c* this beam is designed for whatever wall or window loads are brought directly upon it plus a uniform load of one quarter of the total live and dead load



Minimum Bar Lengths
Exterior Panel

FIG. 147

on the exterior panel. The beam has a flange on one side only, whose minimum lengths are given by A.C.I. Article 705*b*. The section is angle shaped and not symmetrical about the plane of loading. It is designed by the usual procedure for tee beams and possible shear stresses due to torsion are usually neglected.

The interior columns are usually circular and, in this case, the exterior columns are rectangular. The methods of design of columns are discussed in Chapter 11. If the building is considered as an elastic frame there will be bending moments in the interior columns as well as the exterior. Methods of elastic frame analysis are discussed in Chapter 14.

ILLUSTRATIVE PROBLEM 37

222. Design of Flat Slab with Four-Way Steel. Problem 36 illustrated the usual flat slab design using dropped panels and two-way steel. For comparison of results the floor system of Problem 36 will be designed for a slab of constant thickness with four-way steel. Let us tentatively adopt a column capital $c = 80$ in. diameter.

Minimum thickness $t = \dfrac{l}{36} = 9.66$ in. Assume a 10-in. slab.

Interior panel: $W = 214{,}000$ lb. and $M_o = 4{,}810{,}000$ in.-lb.

MOMENT COEFFICIENTS OF M_o

	Negative				Positive	
	Column Strip		Mid-Strip		Column	Mid
Interior panel	0.46		0.16		0.22	0.16
Exterior panel	Int.	Ext.	Int.	Ext.		
East-west span	0.50	0.41	0.176	0.10	0.28	0.20
North-south span	0.115 *		0.16		0.055 *	0.16

* Half-strip.

Minimum depth for fiber stress ($0.50M_o$): $d = 10.83$ in. $t = 11.96$ in.

Increase to $t = 13$ in. $W = 246{,}000$ lb. $M_o = 5{,}530{,}000$ in.-lb.

Then $d = 11.63$ in. and $t = 12.76$ in.

223. Diagonal Tension. Average load $w = 293$ lb. per sq. ft. Shear stress on a perimeter of $80 + 2 \times 11.5 = 103$ in. is

$$v = \frac{293[841 - 0.785(8.58)^2]}{(\pi \times 103) \times 0.87 \times 11.5} = 70.7 \text{ lb. per sq. in.}$$

As before, allowable $v = 58.4$ lb. per sq. in. There are three possible remedies.

1. Increase the capital diameter.
2. Change the mix.
3. Deepen the slab.

Increase of Capital Diameter. This change may remedy designs that are not greatly overstressed. If the capital is made the usual maximum value $c = 0.25l = 87$ in., the computed shear stress $v = 65.5$ lb. per sq. in. Now 50 per cent of the column-strip negative steel crosses the column capital and the allowable stress $v = 0.03f'_c = 60$ lb. per sq. in. In this problem the increase in column capital is not sufficient.

Increase of Capital Diameter and Depth of Slab. If the maximum capital diameter of 87 in. is used, an increase of slab thickness to 14.5 in. gives a shear stress of $v = 60$ lb. per sq. in. and an allowable stress $v = 60$ lb. per sq. in.

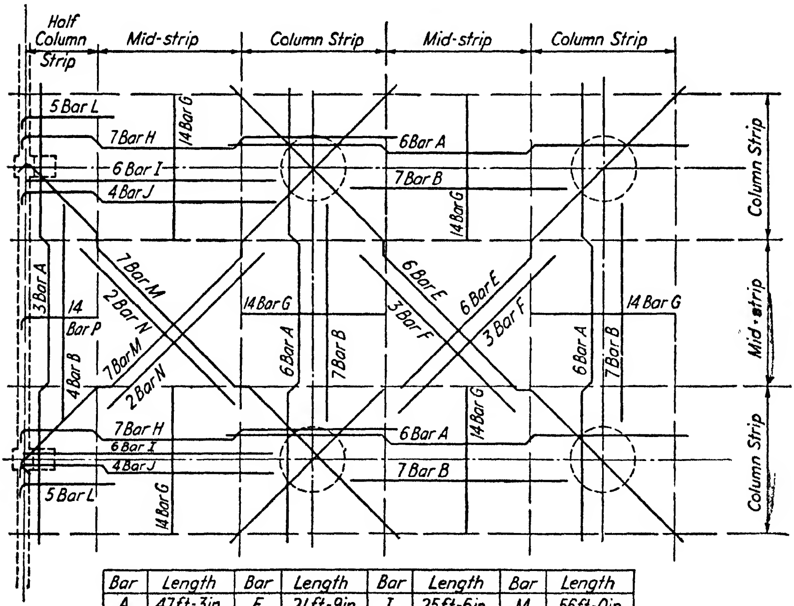
224. Change of Mix. Assume a mix $f'_c = 2500$ lb. per sq. in. and the original column capital of $c = 80$ in. The shear stress remains $v = 70.7$ lb. per sq. in. The allowable stress equals 73 lb. per sq. in.

The choice is between

1. An increase of the mix to 2500-lb. concrete.
2. A capital $c = 87$ in. and slab $t = 14.5$ in., using 2000-lb. concrete.

The decision would be given to the cheaper of the two. It is possible that the increased cost of the mix will be less than the cost of an increase of slab depth of 1.5 in. with the resulting decrease in steel areas. However, in order to compare the present results with Problem 36 we shall continue to use the 2000-lb. concrete, column capital $c = 87$ in., and slab thickness $t = 14.5$ in. It will be noticed that the slab thickness is now the same as the drop panel thickness of Problem 36.

225. Steel Areas for Fiber Stress. Table B gives the computed steel areas. The corrected value of $M_o = 5{,}700{,}000$ in.-lb. In the interior panel an average value of



Bar	Length	Bar	Length	Bar	Length	Bar	Length
A	47 ft.-3 in.	F	21 ft.-9 in.	I	25 ft.-6 in.	M	56 ft.-0 in.
B	21 ft.-9 in.	G	14 ft.-6 in.	J	27 ft.-3 in.	N	21 ft.-9 in.
E	65 ft.-0 in.	H	41 ft.-0 in.	L	11 ft.-2 in.	P	10 ft.-0 in.

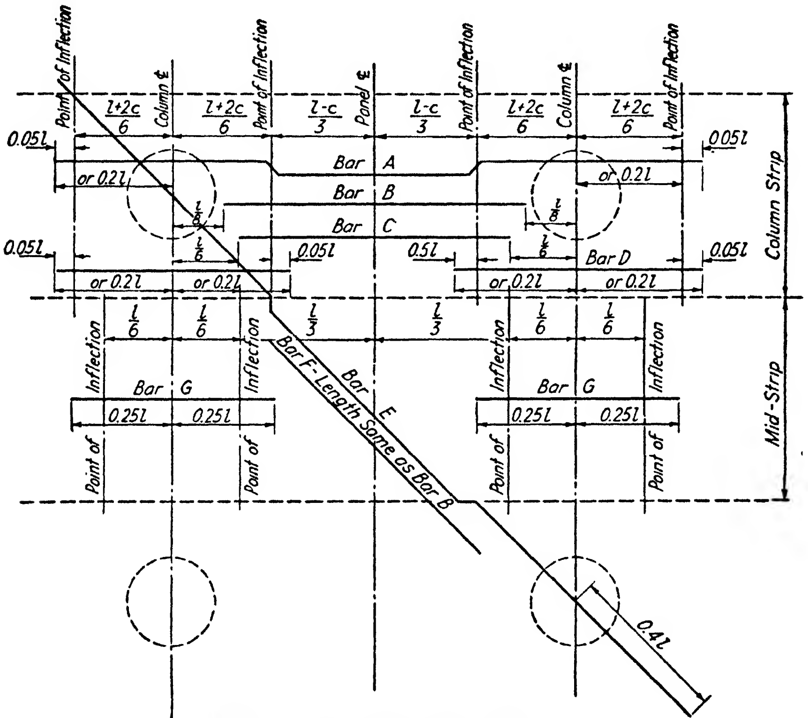
FIG. 148

TABLE B. COMPUTATIONS FOR FLAT SLAB STEEL

	INTERIOR PANEL				EXTERIOR PANEL STEEL PARALLEL TO WALL				EXTERIOR PANEL STEEL PERPENDICULAR TO WALL					
M_0 = 5,700,000 in-lb.	Column Strip		Mid-Strip		Column Strip		Mid-Strip		Column Strip			Mid-Strip		
	Positive	Negative	Pos.	Neg.	Pos	Neg.	Pos	Neg.	Pos.	Int Neg.	Ext Neg.	Pos	Int. Neg	Ext. Neg
Moment Coefficient	0.22	0.46	0.16	0.16	0.055*	0.115*	0.16	0.16	0.28	0.50	0.41	0.20	0.176	0.10
Moment, in-kips	1,255	2,625	912	912	314	656	912	912	1,600	2,850	2,340	1,140	1,005	570
14.5"	d	d	d	d	d	d	d	d	d	d	d	d	d	d
Assuming $\frac{3}{4}$-in. round bars. Depth d, in.	12.62	12.25	12.25	13.37	12.62	11.87	12.25	13.37	12.62	12.12	12.12	12.25	13.37	13.37
$A_s = \dfrac{M}{20{,}000 \times 0.87 \times d}$ sq. in	5.71	12.33	4.28	3.92	1.43*	3.18*	4.28	3.92	7.28	13.50	11.07	5.35	4.32	2.45
Minimun $A_s = 0.0025 \times 174 \times d$	5.50	5.33	5.33	5.81	2.75*	2.56*	5.33	5.81	5.50	5.29	5.29	5.33	5.81	5.81
Use A_s, sq. in.	5.71	12.33	5.33	5.81	2.75	3.18	5.33	5.81	7.28	13.50	11.07	5.35	5.81	5.81
Direct Band, number of bars	13	12**		14	7	6**		14	17	13**	15		14	14
$A_s \cos\theta$ for diagonal bands	-----	7.03	5.33	---	---	0.53	5.33	---	---	7.75	4.37	5.35	---	---
Diagonal band area, sq. in	-----	9.42	7.53	---	---	0.75	7.53	---	---	10.97	6.18	7.57	---	---
Number of diagonal bands	-----	4	2	---	---	2	2	---	---	4+	2	2	---	---
Area in one band	-----	2.48	3.77	---	---	0.16	3.77	---	---	2.83	3.09	3.79	---	---
No. of bars, diagonal band	-----	6	9	---	---	7**	9	---	---	7	7**	9	---	---

*Half-strip **Already determined +2 bands from interior panel the depth d is used for column-strip negative and mid-strip positive steel, so that all north-south and east-west bands will be alike. In the exterior panel it is assumed that the east-west steel is placed above the north-south. Notice that at several sections the requirement of a minimum per cent determines the steel area used.

The steel arrangement consists of direct bands in the north-south and east-west column strips (Fig. 148). There are also four diagonal bands crossing the sections



Minimum Bar Lengths - Interior Panel

FIG. 149

of maximum negative moment in the column strips and two diagonal bands crossing the line of positive moment in the mid-strip. The negative mid-strip steel consists of short bars placed in the top of the slab.

Interior Panel. The order of procedure is:

1. *Positive column-strip steel.* This is supplied wholly by a direct band. Since there are four diagonal bands also crossing the line of negative moment, only the minimum of bar A (Fig. 149) will be bent up in the direct band in the endeavor to balance the number of bars in direct and diagonal bands at the column. Use $13 \times 0.4 = 6$ bars A, and 7 bars B (or $\frac{13}{8} = 5$ bars B and 2 bars C).

2. *Negative column-strip steel* is supplied by two direct bands and four diagonal bands from this and the adjacent panels. There are 12 bars A crossing perpendicularly the line of maximum negative moment. The remaining area to be supplied

by four diagonal bands equals $12.33 - 5.30 = 7.03$ sq. in. The pull in a diagonal bar in the diagonal direction equals f_sA_s pounds (Fig. 150). This is resolved into component pulls in the east-west and north-south directions. They are respectively $f_sA_s \cos \theta_1$ and $f_sA_s \cos \theta_2$. The required area in one diagonal band is then computed and equals 6 bar E. This is a good balance with the 6 bar A in the direct band.

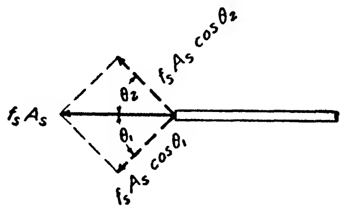


FIG. 150

3. The *positive mid-strip steel* is supplied by two diagonal bands. Each band must have an area $\frac{5.33}{2} = 2.67 = A_s \cos 45°$. The area $A_s = 3.77$ sq. in. This is given by 9 bars. The minimum number of bar E is $0.4 \times 9 = 4$ bars. From the design of paragraph 2 we use 6 bars E and 3 bars F.

4. The *negative mid-strip steel* consists of 14 bars G in the top of the slab.

Exterior Panel, East-West Strips. The same procedure is followed for the exterior panel (Fig. 151).

1. For *positive column strip steel* use $0.4 \times 17 = 7$ bars H; $\frac{17}{3} = 6$ bars I; and 4 bars J. (If fewer types of bars are preferred, use 7 bars H and 10 bars I).

2. For *negative steel* at the *interior column* we have provided 6 bars A + 7 bars H in the direct bands. In addition there are two diagonal bands from the interior panel, each bringing in 6 bars E. The total area of these bars equals 9.50 sq. in. The bars M in the two diagonal bands of the exterior panel must supply an area of $13.50 - 9.50 = 4.00 = A_s \cos 45°$. Then $A_s = 5.66$; use 2.83 sq. in. in each band, or 7 bars M.

At the *exterior support* there are 7 bars H, 4 bars J, and 14 bars M. Their area equals $0.442(11 + 14 \times 0.707) = 9.25$ sq. in. The total area required is 11.07 sq. in.; supply the remainder as 5 bars L.

3. The *positive mid-strip steel* area of 5.35 sq. in. is supplied by two diagonal bands. Each band should have 9 bars; use 7 bars M and 2 bars N.

4. The *negative mid-strip steel* will be supplied as 14 bars P.

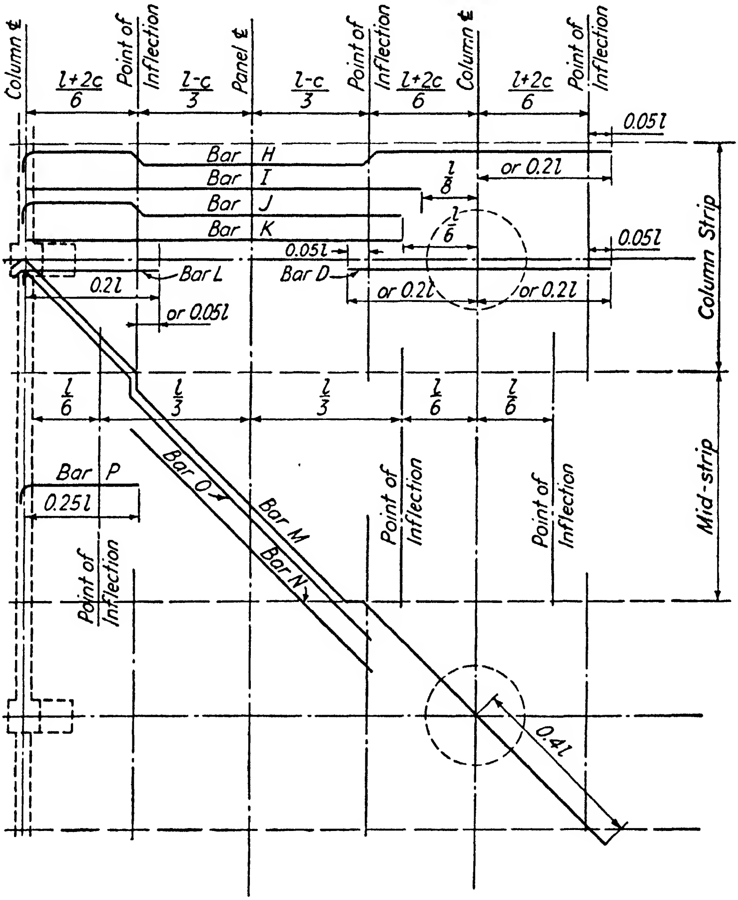
Exterior Panel, North-South Steel. The north-south steel in the exterior panel is spanning an interior span. The diagonal bands have been determined by the east-west areas and the north-south direct bands will supply whatever is needed in addition. The direct column-strip steel over the first interior columns will be made the same as the interior panels.

1. The *half column-strip* adjacent to the wall beam will have as *positive steel* $7 \times 0.4 = 3$ bars A and 4 bars B.

2. The *negative steel* area in the *half column-strip* consists of $3 \times 2 = 6$ bars A plus $7 \times 2 \times 0.707 = 9.9$ bars M, giving a total of $15.9 \times 0.442 = 7.00$ sq. in. The required area equals 3.18 sq. in.

3. The *positive mid-strip steel* should equal 5.33 sq. in. There are available 18 bars in the two diagonal bands, or 18 × 0.707 × 0.442 = 5.62 sq. in.

4. The *negative mid-strip steel* will consist of the usual 14 short bars *G*. The steel is dimensioned in Figure 148. The bands are 0.4*l* wide. If bands of this width are



Minimum Bar Lengths - Exterior Panel

FIG. 151

plotted on the panel plan, it will be seen that they cover the whole surface. The economical order of placing the steel should be determined by consultation with the steel foreman or the superintendent on the job.

226. Comparison of Design Results. The following tabulation compares the results of the two designs of Problems 36 and 37. It shows that there is 613 cu. ft. less concrete in the two panels for the drop panel floor but the steel in the drop panel type is in excess by 290 lb. It is probable that the cost of the excess steel plus the extra form cost of framing around the drop panel will not equal the saving in concrete cost. Therefore, the drop panel two-way design is probably the more economical.

Quantities for 1 Interior and 1 Exterior Panel	Drop Panel; 2-Way Steel	Flat Slab; 4-Way Steel
Slab concrete, cu. ft.	1422	2035 *
Steel—interior panel, lb.	3440	3280
Steel—exterior panel, lb.	3530	3400
Total Steel—two panels, lb.	6970	6680

* Also an 87-in. capital, instead of 80 in.

A discussion of the proper moments to use for flat slabs whose spans in either direction are markedly unequal is given in Chapter 14.

CHAPTER 11

COLUMNS

227. Columns. The column in many respects is the most important unit of a structural frame. A slab panel or a beam may often fail without serious consequences, but the failure of a column endangers the whole structure. Therefore, columns must be carefully designed and conservative allowable stresses must be used.

The floor systems discussed in previous chapters are supported by columns or walls. A wall may be regarded as a column of great width. Therefore, the design of floor supports may be covered by a general discussion of columns.

In reinforced concrete construction the type of column used varies from plain concrete to structural steel columns. The types may be listed as:

1. Plain concrete.
2. Concrete reinforced with longitudinal bars and ties (Fig. 152*a*).
3. Concrete reinforced with longitudinal bars and spiral steel (Fig. 152*b*).
4. Composite columns of structural steel or cast iron within the spiral steel of a type 3 column (Fig. 152*c*).
5. Combination columns of structural steel covered with concrete for fireproofing. The steel column is usually wrapped with wire for bond with the concrete (Fig. 152*d*).
6. Structural steel.

228. Stresses in Columns. It has been previously stated in the discussion of the use of compression steel in beams in Chapter 6 that tests of members in compression show the marked effect of shrinkage and plastic flow upon the residual steel stresses at the end of 2 years, or more, of loading. Until about 1935 it was the custom to design members for the stresses due to the loads and to specify an allowable stress low enough so that the readjustments due to shrinkage and flow would not cause failure of the member. The reinforced concrete column is a typical compression member, so it is natural that the substitution of empirical methods of design based on tests for the former practice should be first employed for columns. The present A.C.I. and Joint Committee Codes recommend such empirical equations based on a comprehensive

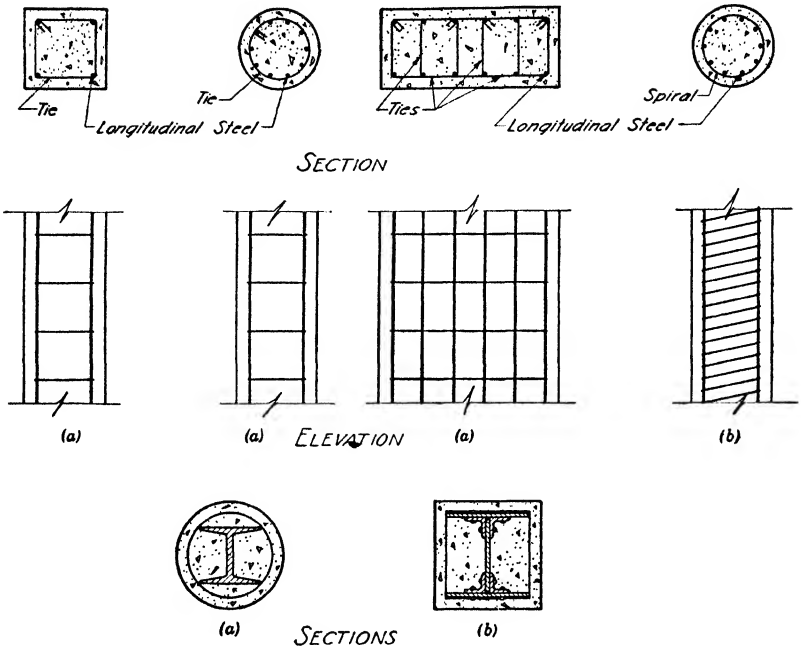


FIG. 152

series of tests made by the American Concrete Institute cooperatively at the University of Illinois and Lehigh University about 1930. The separate analysis of columns for shrinkage and plastic flow stresses is discussed in Chapter 9; the analysis for so-called "elastic stresses" due to loads is covered in this chapter. Therefore the residual stresses at any age can be computed, if the necessary strain coefficients are known, for comparison with the present empirical equations. In this chapter the derivation of the elastic stresses will be given first, then the empirical equations.

229. Distribution of Stresses Due to Loads. The analyses of the cross sections for maximum fiber stresses are divided into two cases:

A. A normal force N applied at the center of gravity of the section. It is assumed for this loading that the stresses are uniformly distributed.

B. A normal force N acting at some distance e from the center of gravity. The stresses are assumed to be uniformly varying.

   1. The neutral axis is outside the section and all stresses are compressive.

2. The neutral axis lies within the section and there is tension over part of the section.

In this chapter the stress analysis will be discussed for the different types of reinforced concrete columns when subjected to any of the three cases of stress distribution.

PLAIN CONCRETE

230. Plain Concrete Columns. Plain concrete columns are seldom used, because such compressive members are limited in height to 5 or 6 times the least thickness t to avoid the possibility of bending or buckling. If such a short strut is used, the load must be nearly axial as there can be no tension in any section. When the neutral axis is at the edge of the section, the greatest eccentricity of loading for a rectangular section equals $e = \frac{t}{6}$. The resultant N of the uniformly varying stress acts at $\frac{2t}{3}$ from the neutral axis and hence $\frac{t}{6}$ from the center line of a rectangular section.

COLUMNS WITH LONGITUDINAL STEEL AND TIES

231. Stresses Due to Axial Loads. The derivation assumes, as in the beam analyses, that there is a partnership between the concrete and steel. Both are in compression, and their strains e must be the same; otherwise the steel will slip. Equating strains gives

$$e_s = \frac{f'_s}{E_s} = e_c = \frac{f_c}{E_c}$$

$$f'_s = \frac{E_s}{E_c} f_c = nf_c \tag{185}$$

Assume any column section with a normal load N acting at the center of gravity of the section (Fig. 153). The force N is supported by the sum of the compressive forces in the steel and the concrete. Let

A = total area of the column

p = steel ratio $= \frac{A_s}{A}$

$$N = f_cA_c + f'_sA_s = f_c(A - A_s) + nf_cA_s$$

$$N = f_cA[1 - p + np] = f_cA[1 + (n - 1)p] \tag{186}$$

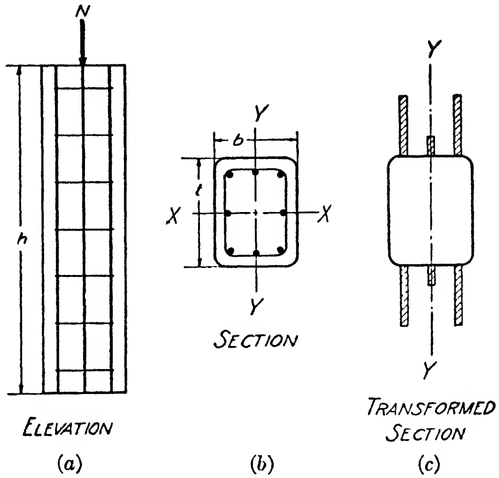


FIG. 153

232. Transformed Section. The method of solution by use of the transformed area (Art. 22) gives identical results. When a section of "equivalent concrete" (Fig. 153c) is used, the steel is removed and $nA_s = npA$ areas of concrete substituted in the proper position to give the same area and moment of inertia. One area fills the hole left by the steel, and $(n - 1)A_s$ areas are added as fins symmetrically placed about the YY axis, if bending or buckling is expected about this axis. The total area A_g now equals the area of the original column plus the fins, or

$$A_g = A + (n - 1)A_s = A + (n - 1)pA = A[1 + (n - 1)p]$$

As the stress is uniform,

$$N = f_c A_g = f_c A[1 + (n - 1)p]$$

233. Empirical Formulae for Axial Load. The A.C.I. tests determined that the load N' carried at failure by a tied column was closely approximated by the equation

$$N' = 0.85f'_c A_c + f_y A_s \tag{187}$$

Since f'_c is the compressive strength of a 6-in. cylinder 12 in. high, the reduction to $0.85f'_c$ allows for the greater length-thickness ratio of the usual column. Failure occurs when the concrete fails in compression and the steel has reached the yield point stress f_y. Failure may be due only to application of load, or it may be due to the long-time application of a less load, or to thorough drying out (shrinkage) plus a long-time load application.

For purposes of design the failure equation is modified by factors of safety:

$$N = A(0.18f'_c + 0.32f_y p) \tag{188}$$

The stresses in concrete and steel no longer obey equation 185, as shrinkage and flow have changed this elastic relation.

234. Long Columns. Columns which are exceptionally long will fail at loads less than that given by equation 187. The design load that such a long column can safely carry is obtained from equation 188 by use of a load-reduction formula, involving the slenderness ratio, similar to the familiar stress reduction formulae used in steel and timber design. In A.C.I. Article 1107, a long column is defined as one whose length exceeds ten times its least cross-sectional dimension and

$$N_l = N\left(1.3 - 0.03\frac{h}{t}\right) \tag{189}$$

where N_l = safe load for long column
N = safe load for short column by equation 188
h = unsupported height
t = least cross-section dimension.

235. Economy. It is possible, by varying the mix, steel, and size, to design a great number of column sections, all of which will carry a given load safely, but only one will be the cheapest. For the true reinforced concrete column (types 2 and 3) the illustrative problems will discuss economical design. The contractor is interested in the cheapest construction costs for his columns; but the owner, who is purchasing floor space, is also interested in having small columns. This interest is, of course, quite apart from the occasional necessity of having definite clearances for machinery, aisles, etc. The cost comparison from the owner's point of view should include allowance for all floor surface in excess of that given by the column of least cross-sectional area.

Many city building laws allow a reduction of the live load for certain buildings if the column in question carries more than one floor. The reductions specified by the New England Building Officials Conference are given in the Appendix.

ILLUSTRATIVE PROBLEM 38

236. Design of Column with Longitudinal Steel and Ties. *Axial Load.* Design an interior column whose unsupported length is 16 ft. for an axial load at the top section of 300,000 lb.

The interior columns of beam and girder floors are usually square; flat slab columns are round. Assume a square column. Equation 188 has three unknowns:

size, mix, and steel ratio. Assume for the first analysis the rich mix of $f'_c = 5000$ lb. per sq. in. and a yield point of the steel $f_y = 50{,}000$ lb. per sq. in.

Long or Short Column. The ratio $\frac{h}{t}$ must not exceed 10 for short columns. Columns thicker than $\frac{h}{10} = 19.2$ in. are short columns.

Maximum and Minimum Areas. The bottom section has the greatest load N. Assuming the column to be "short" and to weigh 3000 lb., equation 188 becomes

$$303{,}000 = A[900 + 16{,}000p]$$

The steel ratio can vary between 1 and 4 per cent. Substituting the limiting ratios,

$$\text{If } p = 0.01,\ A = 286 \text{ sq. in.},\ t = 16.9 \text{ in.}$$

$$\text{If } p = 0.04,\ A = 197 \text{ sq. in.},\ t = 14.1 \text{ in.}$$

A 15-in. or 16-in. column can be used. Either one is a "long" column. Let us adopt a 16-in. column, weighing 4300 lb. By equation 189 this column will be designed as a short column carrying the increased load N of

$$304{,}300 = N[1.3 - 0.03 \times \tfrac{192}{16}]$$

$$N = 324{,}000 \text{ lb.}$$

Then

$$324{,}000 = 256(900 + 16{,}000p)$$

$$p = 0.0228 \qquad \text{and} \qquad A_s = 0.0228 \times 256 = 5.84 \text{ sq. in.}$$

In order to keep a symmetrical arrangement of steel, the bars in a square column should be in multiples of four. In a round column any number over four with a minimum diameter of $\frac{5}{8}$ in. may be evenly spaced around a perimeter. The possibilities for this column are

Four $1\frac{1}{4}$-in. squares with area = 6.25 sq. in.

Eight 1-in. rounds with area = 6.28 sq. in.

Twelve $\frac{7}{8}$-in. rounds with area = 7.21 sq. in.

Use four $1\frac{1}{4}$-in. squares. Assuming in this case that the total height of the column is 18 ft., the steel is 24 diameters longer (A.C.I. Art. 1103*c*), or 20 ft. 6 in. long. A.C.I. Article 1104*b* states that the ties shall not be smaller than $\frac{1}{4}$ in. This is a light bar to restrain a $1\frac{1}{4}$-in. rod from buckling. For this design we shall use as a single tie for four rods:

$\frac{1}{4}$-in. ties with $\frac{5}{8}$-in. bars

$\frac{3}{8}$-in. ties with $\frac{3}{4}$-, $\frac{7}{8}$-, and 1-in. round bars

$\frac{1}{2}$-in. ties with 1-, $1\frac{1}{8}$-, and $1\frac{1}{4}$-in. square bars

Use $\frac{1}{2}$-in. ties, whose spacing must not exceed

16 bar diameters	= 20 in.
48 tie diameters	= 24 in.
Least column dimension	= 16 in.

Use 16-in. spacing. Figure 154*a* shows the steel arrangement in the section. If 8 bars are used additional ties must be added as in Figure 154*b*; the ties for 12 bars

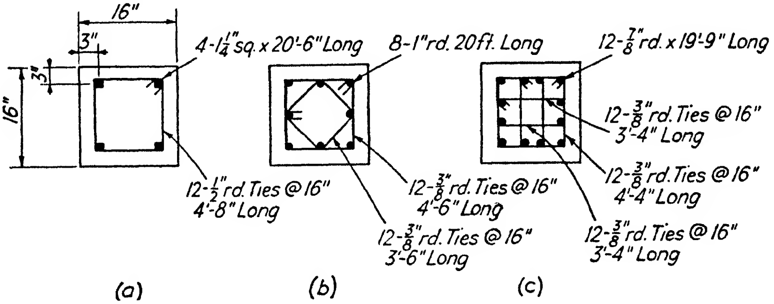


FIG. 154

are shown in Figure 154*c*. These additional ties add to the labor and cost of wiring the column steel, in addition to interfering with pouring concrete down the 16-ft. height. There is every incentive to use a few large-sized bars. When additional ties are used the tie sizes can be made smaller than for a single tie. It is suggested that the tie size be then made large enough so that it does not reduce the tie spacing: in other words,

$\frac{1}{4}$-in. ties with $\frac{5}{8}$- and $\frac{3}{4}$-in. bars

$\frac{3}{8}$-in. ties with $\frac{3}{4}$- to $1\frac{1}{8}$-in. bars

$\frac{1}{2}$-in. ties with $1\frac{1}{4}$-in. bars

We have been designing a section such as *AA* in Figure 155. The steel in this column ends at this section and cannot carry load. It does reinforce the sections such as *BB* which are nearly as heavily loaded. At section *AA* the steel from below must supply the reinforcement and must have an area at least equal to $A_s = 5.84$ sq. in.

237. Cost. Let us assume:

1. Concrete costs 45 cents per cubic foot. This includes materials, labor, and plant costs for mixing and placing.

2. Forms cost 18 cents per square foot. This includes materials, and labor cost for making, erecting, stripping, and repairing.

3. Steel costs 4 cents per pound. This includes material, labor, and plant charges for bending and placing.

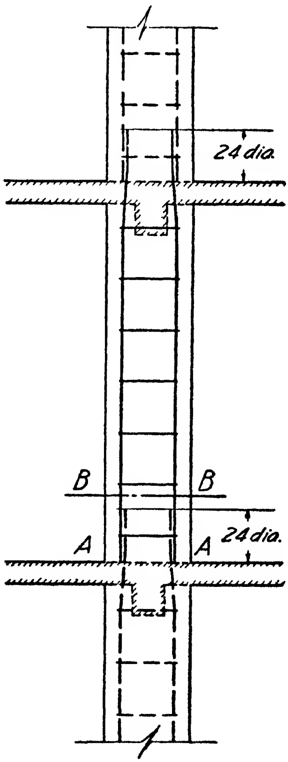


FIG. 155. Column Elevation.

TABLE C. SQUARE COLUMNS TO SUPPORT 300,000 POUNDS

Mix, strength f'_c lb. per sq. in.	2000	2000	2500	2500	3000	3000	4000	4000	5000	5000
Size, in.	24	18	22	18	21	17	17	16	16	15
Weight, lb.	9600	5400	8100	5400	7400	4800	4800	4300	4300	3800
Long column	....	Yes	....	Yes	....	Yes	Yes	Yes	Yes	Yes
Minimum steel ratio p	0.011	0.0376	0.0117	0.032	0.0109	0.0351	0.0236	0.0341	0.0228	0.0357
Area A_s, sq. in.	6.35	12.18	5.65	10.37	4.41	10.10	6.83	8.72	5.84	8.00
Use	8—1 sq.	8—1¼ sq.	4—1¼ sq.	8—1¼ sq.	8—⅞ rd.	8—1⅛ sq.	8—1 sq.	8—1⅛ sq.	4—1¼ sq.	8—1 sq.
Length, ft.	20.67	21.33	21.33	21.33	19.75	20.25	20.0	20.25	20.5	20.0
Ties, number and size	12—⅜	11—½	10—½	11—½	14—⅜	12—⅜	12—⅜	12—⅜	12—½	13—⅜
Spacing, in.	16	18	20	18	14	17	16	16	16	15
Length, ft., outer	7.17	5.25	6.67	5.25	6.17	4.83	4.83	4.50	4.67	4.17
second set	5.50	4.17		4.17	5.00	4.00	3.83	3.67		3.33
Cost, dollars concrete	25.20	14.17	22.40	15.00	21.50	14.10	15.20	13.43	14.40	12.66
forms	25.92	19.44	23.76	19.44	22.68	18.36	18.36	17.28	17.28	16.20
steel	24.76	39.00	19.90	39.00	15.28	29.48	23.32	29.36	18.96	23.44
Contractor's comparison	75.88	72.61	66.06	73.44	59.46	61.94	56.88	60.07	50.64	52.30
Excess floor area	4.88	1.38	3.60	1.38	3.00	0.89	0.89	0.43	0.43	
Owner's comparison	80.76	73.99	69.66	74.82	62.46	62.83	57.77	60.50	51.07	52.30
Total load N by final design, kips	335.4	310.0	317.6	338.0	315.1	306.0	323.5	329.0	310.5	303.0

Cost per column for 18 ft. height equals

1. Concrete $\frac{16 \times 16}{144} \times 18 \times 0.45$		=	\$14.40
2. Forms $4 \times \frac{16}{12} \times 18 \times 0.18$		=	17.28
3. Steel,			
longitudinal, $4 \times 20.5 \times 5.31$	= 436 lb.		
ties, $12 \times 4.67 \times 0.67$	= 38 lb.		
	474 lb. $\times$ 0.04	=	18.96
Total		=	\$50.64

For comparison, Table C gives the designs for 2000-, 2500-, 3000-, 4000-, and 5000-lb. concretes using high and low steel ratios. It will be noticed that when a size larger than the maximum limit is used, steel equal to $p = 0.01$ must be used. In certain cases the commercial steel is much in excess of the computed amount. The steel must have the minimum spacings of A.C.I. Article 1103*b* (see Appendix). Assuming the ratio of total building cost to total floor area to be \$2.00 per sq. ft., the cost comparison includes an allowance for the "excess floor area." Assume also that

2000-lb. concrete costs 35 cents per cubic foot
2500 " " " 37 " " " "
3000 " " " 39 " " " "
4000 " " " 42 " " " "
5000 " " " 45 " " " "

It will be noticed that the stronger mixes are cheaper than the weaker. It is also usually true for a given mix that the low steel ratios are cheaper, because compression steel is not economical. Professor Lyse[1] proves that the statement "the richer mixes are cheaper" may be made general regardless of the load.

238. Elastic Stresses. Equation 186 enables one to compute the stresses due to loads only. It is as easily manipulated as the present empirical equation 188. For example, if one checks the design in Table C for the 3000-lb. concrete by equation 186, the results for the 21-in. column with eight $\frac{7}{8}$-in. round bars are

$$307{,}400 = f_c \times 441 \left[1 + 9 \times \frac{4.81}{441}\right]$$

$$f_c = 634 \text{ lb. per sq. in.} \qquad \text{and} \qquad f'_s = nf_c = 6340 \text{ lb. per sq. in.}$$

The allowable stress due to loads was formerly taken as $f_c = 0.225f'_c = 675$ lb. per sq. in. and the designs would be identical by the two methods.

[1] "Relation between Quality and Economy of Concrete," *Jour. A.C.I.*, March–April, 1933, p. 325.

The 17-in. column with eight $1\frac{1}{8}$-in. square bars would have computed elastic stresses of

$$304{,}800 = f_c \times 289\left[1 + 9 \times \frac{10.13}{289}\right]$$

$$f_c = 803 \text{ lb. per sq. in.} \qquad \text{and} \qquad f'_s = 8030 \text{ lb. per sq. in.}$$

In this case the elastic stress considerably exceeds the former allowable of 675 lb. per sq. in. and the present empirical method permits the use of less steel.

239. Stresses Due to Load, Shrinkage, and Flow. These same columns are analyzed for the stresses due to loads, shrinkage, and flow in Problem 29 (Chapter 9). There it is shown that there is ultimately a very considerable decrease of concrete stress and a greater increase of steel stress. The effects of shrinkage and flow are so marked that computations of stresses due to loads only do not give a comprehensive picture of stress conditions at any time during the life of the column. For this reason, even though the former method gave safe columns, it was abandoned in favor of the present empirical method.

ECCENTRIC LOADS

STRESSES DUE TO LOADS

240. Eccentric Loads—Rectangular Section. *Case I. Compression on Entire Section. Neutral Axis Outside Section.* Assume a rectangular section with equal areas of steel on the two broad faces. This is the usual condition for exterior columns. The load N acts on the axis of symmetry XX with an eccentricity of e (Fig. 156). The transformed section of Figure 156*c* will be substituted for the reinforced section by adding nA_s areas of concrete for the steel. One area fills the steel holes, and $(n - 1)A_s$ areas are added as fins in the same relative position about the center line. Let $A_s = pbt$.

Take the portion of the column above the section AA as the rigid body, disregarding the weight of this portion. The force at this section equals N and it is the resultant of the stresses which are assumed to be uniformly varying with the neutral axis at a distance kt from the side of the maximum stress f_c. Then, by Figure 156*b*,

$$\frac{f_c}{kt} = \frac{f^{\text{II}}_c}{kt - d'} = \frac{f^{\text{III}}_c}{kt - d} = \frac{f^{\text{IV}}_c}{kt - t} \tag{190}$$

$$N = \frac{f_c + f^{IV}{}_c}{2} bt + f^{II}{}_c(n-1)\frac{p}{2}bt + f^{III}{}_c(n-1)\frac{p}{2}bt$$

$$N = \frac{bt}{2}\left[f_c + f_c\left(\frac{kt-t}{kt}\right) + (n-1)pf_c\left(\frac{kt-d'}{kt} + \frac{kt-d}{kt}\right)\right]$$

$$N = \frac{f_c bt}{2k}(2k-1)[1 + (n-1)p] \tag{191}$$

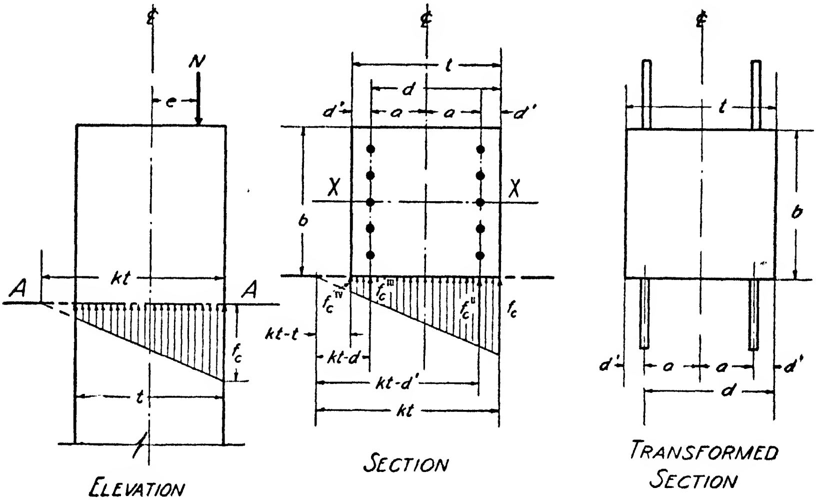


FIG. 156

It is also true that the sum of the moments of the stresses about the center line equals Ne. We shall assume that the stress on the rectangle bt consists of a uniform stress $f^{IV}{}_c$ plus stresses uniformly varying from zero to $f_c - f^{IV}{}_c$.

$$Ne = f^{IV}{}_c bt \times 0 + \left(\frac{f_c - f^{IV}{}_c}{2}\right) bt\left(\frac{t}{6}\right) + f^{II}{}_c(n-1)\frac{p}{2}bta - f^{III}{}_c(n-1)\frac{p}{2}bta$$

$$Ne = \frac{bt}{2}\left[\left(f_c - f_c\frac{(kt-t)}{kt}\right)\frac{t}{6} + (n-1)paf_c\left(\frac{kt-d'}{kt} - \frac{kt-d}{kt}\right)\right]$$

$$M = Ne = \frac{f_c bt^2}{12k}\left[1 + 12(n-1)p\left(\frac{a}{t}\right)^2\right] \tag{192}$$

Multiplying equation 191 by e and equating to equation 192,

$$\frac{f_c bte}{2k}(2k-1)[1+(n-1)p] = \frac{f_c bt^2}{12k}\left[1+12(n-1)p\left(\frac{a}{t}\right)^2\right]$$

$$k = \frac{1}{2} + \frac{1+12(n-1)p\left(\frac{a}{t}\right)^2}{12\left(\frac{e}{t}\right)[1+(n-1)p]} \tag{193}$$

Equations 191, 192, and 193 contain as variables the terms f_c and n governed by the mix, the steel ratio p, and the column dimensions b and t. A direct design resulting in a definite solution is not possible. The designer assumes a size, mix, and steel ratio, solves for k by equation 193, and for the maximum stress f_c in equation 191 or 192. Diagrams 11 to 15 (in the Appendix) can also be used to solve these equations.

Approximate Equations. Many texts add nA_s areas of concrete fins for the transformed section. This results in equations:

$$N = \frac{f_c bt}{2k}(2k-1)(1+np) \tag{194}$$

$$M = Ne = \frac{f_c bt^2}{12k}\left[1+12np\left(\frac{a}{t}\right)^2\right] \tag{195}$$

$$k = \frac{1}{2} + \frac{1+12np\left(\frac{a}{t}\right)^2}{12\left(\frac{e}{t}\right)(1+np)} \tag{196}$$

The error is on the unsafe side but it is not great. For instance, assume $n = 6$ for 5000-lb. concrete and the maximum steel ratio $p = 0.04$. A column of minimum thickness $t = 12$ in. will have a ratio

$$\left(\frac{a}{t}\right)^2 = \left(\frac{3}{12}\right)^2 = 0.0625$$

If, for Case I, a median value of the eccentricity is taken as $\frac{e}{t} = 0.10$, k will be about 1 per cent too great, N about 3 per cent too great, and M about 4 per cent too great. These maximum errors are very small, and equations 194, 195, and 196 may be used. The author considers, however, that the numerical value of $n - 1$ can be substituted in computations as easily as n, so equations 191, 192, and 193 will be used for problem solutions.

241. Eccentric Loads—Rectangular Section. *Case II. Neutral Axis within Section.* Assume a rectangular section with equal areas of steel on the b faces. The load N acts on the XX axis of symmetry with an eccentricity e (Fig. 157). The transformed section of Figure 157c will

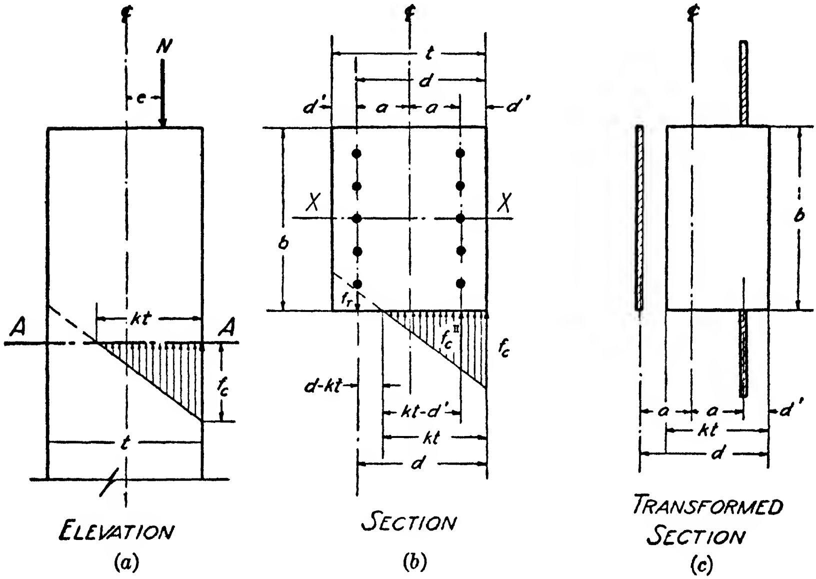


FIG. 157

be substituted for the reinforced section by adding nA_s areas of concrete for the steel. On the compression side one area fills the steel hole, and $(n - 1)A_s$ areas are added as fins. The remaining compression area is now a rectangle b wide and kt deep. On the tension side there is only a fin with an area of nA_s.

Take as a rigid body the portion of the column above section AA, neglecting the weight of this portion. The force at the section must equal N and it is the resultant of the compressive stresses on the rectangle $b(kt)$ which vary uniformly from zero to f_c, plus the force on the compressive fins, and minus the force on the tensile fin. From Figure 157b,

$$\frac{f_c}{kt} = \frac{f''_c}{kt - d'} = \frac{f_t}{d - kt} \tag{197}$$

$$N = \frac{f_c}{2} b(kt) + f''_c(n-1)\frac{p}{2}bt - f_t n \frac{p}{2} bt$$

$$N = \frac{bt}{2}\left[f_c k + f_c\left(\frac{kt - d'}{kt}\right)(n-1)p - f_c\left(\frac{d - kt}{kt}\right)np\right]$$

$$N = \frac{f_c bt}{2k}\left[k^2 + np(2k-1) - p\left(k - \frac{d'}{t}\right)\right] \tag{198}$$

Taking the sum of the moments of the forces about the center line,

$$Ne = \frac{f_c}{2} b(kt)\left(\frac{t}{2} - \frac{kt}{3}\right) + f''_c(n-1)\frac{p}{2}bta + f_t n \frac{p}{2} bta$$

$$Ne = \frac{bt}{2}\left[\frac{f_c kt}{2} - \frac{f_c k^2 t}{3} + \frac{f_c(kt - d')}{kt}(n-1)pa + \frac{f_c(d - kt)}{kt} npa\right]$$

$$M = Ne = \frac{f_c bt^2}{12k}\left[3k^2 - 2k^3 + 12np\left(\frac{a}{t}\right)^2 - 6p\frac{a}{t}\left(k - \frac{d'}{t}\right)\right] \tag{199}$$

Multiplying equation 198 by e and equating to equation 199

$$\frac{f_c bte}{2k}\left[k^2 + np(2k-1) - p\left(k - \frac{d'}{t}\right)\right] =$$

$$\frac{f_c bt^2}{12k}\left[3k^2 - 2k^3 + 12np\left(\frac{a}{t}\right)^2 - 6p\frac{a}{t}\left(k - \frac{d'}{t}\right)\right]$$

Then

$$k^3 + 3\left(\frac{e}{t} - \frac{1}{2}\right)k^2 + 3p\left[\frac{e}{t}(2n-1) + \frac{a}{t}\right]k =$$

$$3p\left\{n\left[2\left(\frac{a}{t}\right)^2 + \frac{e}{t}\right] + \frac{d'}{t}\left[\frac{a}{t} - \frac{e}{t}\right]\right\} \tag{200}$$

Equation 200 must be solved for k and its value substituted in either equation 198 or 199 for the fiber stress f_c.

242. Approximate Equations. Equations 198, 199, and 200 are very long. The error from using fins of nA_s on the compression side affects only the compression fins, and the error is less than in Case I. The author recommends for Case II the use of this approximation which gives the equations usually employed. The results are

$$N = \frac{f_c bt}{2k}[k^2 + np(2k-1)] \tag{201}$$

$$M = \frac{f_c bt^2}{12k}\left[k^2(3 - 2k) + 12np\left(\frac{a}{t}\right)^2\right] \tag{202}$$

$$k^3 + 3\left(\frac{e}{t} - \frac{1}{2}\right)k^2 + 6np\frac{e}{t}k = 3np\left[2\left(\frac{a}{t}\right)^2 + \frac{e}{t}\right] \tag{203}$$

243. Plots. Diagrams 11 to 15 (in the Appendix) are given to solve equations 191 and 193 for Case I. Four diagrams have been supplied for interpolation between $\frac{d'}{t} = 0.05$ and $\frac{d'}{t} = 0.25$. If $d' = 3$ in. this covers the range from $t = 12$ in. to $t = 60$ in. From equation 191 it is seen that the constant C_1, in $f_c = \frac{N}{bt} C_1$, equals

$$C_1 = \frac{f_c bt}{N} = \frac{2k}{(2k - 1)[1 + (n - 1)p]} \tag{204}$$

Diagrams 16 to 20 solve equations 202 and 203 for Case II of rectangular sections. The constant C_2, in $f_c = \frac{M}{bt^2} C_2$, equals

$$C_2 = \frac{f_c bt^2}{M} = \frac{12k}{k^2(3 - 2k) + 12np\left(\frac{a}{t}\right)^2} \tag{205}$$

In order to condense the scale the eccentricity ratio is plotted as $\frac{t}{e}$.

244. Stresses Due to Shrinkage and Flow. Empirical Equations. The normal stresses on a cross section of a column loaded eccentrically are uniformly varying, as are the normal stresses of a beam. The shrinkage and flow stresses can be computed in the same manner as discussed in Chapter 9 for beams. Ignorance of the presence and length of cracks and the readjustment of stresses does not enable the designer to compute final stresses with confidence in the accuracy of the results, even though it is recognized that the final stresses are widely different from those due to loads only. Logically, empirical equations based on test data should be employed for eccentric column design; but test data are still scarce, though tests of eccentric loading are accumulating. At present, in default of a better method, both the A.C.I. and Joint Committee Codes recommend that eccentrically loaded columns be designed by computation of the stresses due to loads only; which is the case just abandoned for axially loaded columns.

245. Allowable Concrete Stresses. Some years ago beams were designed with an allowable concrete stress $f_c = 0.40f'_c$, whereas axially loaded columns used an allowable stress due to loads $f_c = 0.20f'_c$. Columns eccentrically loaded, whose load stresses were equal to $f_c = \frac{N}{A} \pm \frac{(Ne)y}{I}$, were given an allowable stress $f_c = 0.30f'_c$, even if the bending stress were very small or very great compared to the axial. This was not logical, for columns with slight eccentricity of load should have an allowable stress approaching that of an axially loaded column and those with a great eccentricity should approach the allowable stress of beams. The recent A.C.I. and Joint Committee Codes have corrected this by a more complicated but more logical method of obtaining the allowable concrete stress due to loads. The allowable stress f_a for axially loaded columns is computed as

$$f_a = \frac{N}{A} = \frac{0.18f'_c + 0.32f_y p}{1 + (n-1)p} \tag{206}$$

$$\text{Allowable stress for beams equals } f_b = 0.45f'_c \tag{207}$$

$$\text{Ratio } C = \frac{f_a}{f_b} \qquad \text{and the term } D = \frac{t^2}{2R^2} \qquad \text{(208) and (209)}$$

where R is the least radius of gyration of a column section. The section has the steel symmetrically arranged. For a rectangular section,

$$R^2 = \frac{I}{A} = \frac{\frac{bt^3}{12} + (n-1)pbta^2}{bt[1 + (n-1)p]} = \frac{t^2 + 12(n-1)pa^2}{12[1 + (n-1)p]} \tag{210}$$

where a is distance from center line of section to center of the steel.

$$\text{Maximum allowable stress } f_p = f_a\left[\frac{t + De}{t + CDe}\right] \tag{211}$$

This stress varies from $f_p = f_a$, when $e = 0$, to $f_c = f_b$, when e approaches infinity (a couple acting on section).

246. Design Procedure. There are so many variables in the stress equations that designs are made by selecting a mix, size, and steel ratio and computing the maximum stresses f_c and f_s. The size is usually determined by the desire to use the same size as the column above, or to make a 4-in. to 6-in. increase and hold for several stories in order to use the same forms. It is well to use first the maximum steel ratio p, as a "sighting shot," and to indicate whether the size and mix are reasonable. The steel ratios may vary from $p = 0.01$ to $p = 0.04$.

In general, ratios greater than $\frac{e}{t} = 0.27$ give Case II solutions, and ratios less than $\frac{e}{t} = 0.17$ are Case I. The values intermediate may be either, depending on the mix and steel ratio used. For large values of $\frac{e}{t}$ in the Case II solutions the steel stress f_s should be checked as well as the concrete stress f_c.

ILLUSTRATIVE PROBLEM 39

247. Design of Column with Rectangular Section and Longitudinal Steel Loaded Eccentrically. *Case II.* In Problem 10 (Chapter 6) the end span of a beam, 16 in. by 32 in., was supported by an exterior column, whose section was assumed to be $b = 16$ in. and $t = 20$ in. Check this assumption, and complete the design of the column. The unsupported height of the column is 18 ft. 2 in.

The beam design gave at the exterior support a supporting force V and bending moment M_n equal to

$$V = 2705 \times 0.5 \times 26.5 = 35{,}900 \text{ lb.} \qquad \text{(Art. 92)}$$

$$M_n = \frac{wl^2}{16} = 1{,}425{,}000 \text{ in.-lb.}$$

In addition assume that the wall beams bring to the center line of the column a load of 10,000 lb.

If the joint is taken as a rigid body, the forces acting at section AA are shown in Figure 158. Therefore, $N = 45{,}900$ lb. and

$$M = 35{,}900 \times 10 + 1{,}425{,}000 = 1{,}784{,}000 \text{ in.-lb}$$

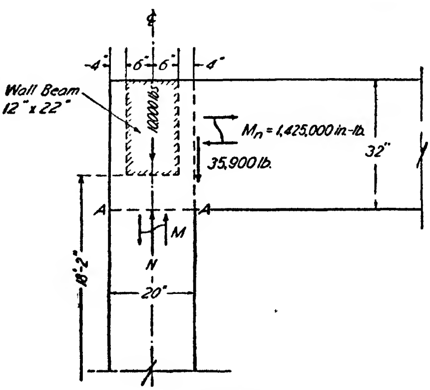


FIG. 158

The force and couple can be combined to give a resultant $N = 45{,}900$ lb. acting with an eccentricity e equal to

$$e = \frac{M}{N} = \frac{1{,}784{,}000}{45{,}900} = 38.8 \text{ in.}$$

$$\text{Eccentricity ratio } \frac{e}{t} = \frac{38.8}{20} = 1.94. \qquad \text{This is Case II.}$$

If this is a long column, the force N at section AA will be arbitrarily increased for design purposes, but the eccentricity e will remain unchanged. The ratio $\frac{h}{t} = \frac{218}{20} =$ 10.9, which exceeds the limiting ratio of 10. Then, by A.C.I. Article 1107 (see Appendix),

$$45{,}900 = N[1.3 - 0.03 \times 10.9] = 0.973N$$

$$N = 47{,}200 \text{ lb.}$$

Maximum Concrete Stress. In this problem a trial section has been assumed. It remains to adopt a mix and steel ratio in order to compare computed stresses with the allowable. It is advantageous to try first the maximum steel ratio $p = 0.04$ for each mix because, if the trial proves unsafe, it is not necessary to investigate that mix further. Let us assume a concrete strength $f'_c = 4000$ lb. per sq. in. and $p = 0.04$. By equation 203,

$$k^3 + 3(1.94 - 0.5)k^2 + 6 \times 7.5 \times 0.04 \times 1.94k = 3 \times 7.5 \times 0.04[2(\tfrac{7}{20})^2 + 1.94]$$

$$k^3 + 4.32k^2 + 3.49k = 1.97 \qquad \text{and} \qquad k = 0.375$$

By equation 202 the maximum concrete stress due to loads equals

$$f_c = \frac{47{,}200 \times 38.8 \times 12 \times 0.375}{16 \times 400[(0.375)^2(3 - 0.75) + 12 \times 7.5 \times 0.04(0.35)^2]} = 1700 \text{ lb. per sq. in.}$$

Allowable Concrete Stress. The allowable concrete stress for axial loads, by the equations in Article 245, equals

$$f_a = \frac{0.18 \times 4000 + 16{,}000 \times 0.04}{1 + 6.5 \times 0.04} = \frac{1360}{1.26} = 1080 \text{ lb. per sq. in.}$$

$$C = \frac{f_a}{f_b} = \frac{1080}{0.45 \times 4000} = 0.6$$

$$R^2 = \frac{400 + 12 \times 6.5 \times 0.04 \times 49}{12 \times 1.26} = 36.5 \qquad \text{and} \qquad D = \frac{400}{2 \times 36.5} = 5.48$$

$$f_p = 1080 \left[\frac{20 + 5.48 \times 38.8}{20 + 0.6 \times 5.48 \times 38.8}\right] = 1700 \text{ lb. per sq. in.}$$

In this case a 4000-lb. concrete with 4 per cent of steel is a possible solution.

Maximum Steel Stress. By equation 197 this equals $f_s = nf_t = 7.5 \times 1700 \times \frac{17 - 0.375 \times 20}{0.375 \times 20} = 16{,}100$ lb. per sq. in. This is safe, as the allowable stress equals

20,000 lb. per sq. in. The steel area amounts to $A_s = 0.04 \times 16 \times 20 = 12.8$ sq. in., or five $1\frac{1}{8}$ in. square bars in each 16-in. face. It is impossible to place these bars with the required spacing of 3 diameters ($3\frac{3}{8}$ in.) on centers.

The maximum steel that can be placed is four 1-in. square bars on each face, or $p = 0.025$. Using $f'_c = 6000$ lb. per sq. in. and $p = 0.025$, the computed stress $f_c = 2570$ lb. per sq. in. and allowable stress $f_p = 2500$ lb. per sq. in. This is not satisfactory. Moreover the maximum steel stress $f_s = 24{,}400$ lb. per sq. in., so the steel is overstressed.

Let us increase the column width to 21 in., using the same depth of 20 in. and a concrete strength $f'_c = 4000$ lb. per sq. in. Eight $1\frac{1}{8}$-in. square bars can be placed in this width, so $p = 0.0242$. The maximum stresses are $f_c = 1660$ lb. per sq. in. and $f_s = 19{,}900$ lb. per sq. in.; the allowable concrete stress $f_p = 1672$ lb. per sq. in. This is satisfactory. The steel arrangement is shown in Figure 159.

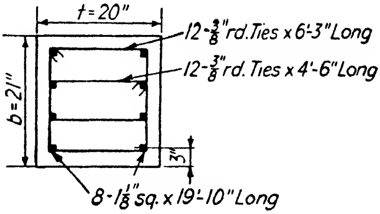


FIG. 159

ILLUSTRATIVE PROBLEM 40

248. Solution by Plots. *Computed Stresses.* Solve Problem 39 by means of Diagrams 16 to 20 (see Appendix). For the first solution the data are

$$N = 47{,}200 \text{ lb.}; \; e = 38.8 \text{ in.}; \; b = 16 \text{ in.}; \text{ and } t = 20 \text{ in.}$$

$$f'_c = 4000 \text{ lb. per sq. in.}; \; p = 0.04.$$

$$\frac{t}{e} = 0.515; \; \frac{d'}{t} = \frac{3}{20} = 0.15; \text{ and } np = 0.30.$$

On Diagram 18 the intersection of $\frac{t}{e} = 0.515$ and $n_p = 0.30$ gives $C_2 = 5.95$. Also $k = 0.38$ approximately.

$$f_c = \frac{M}{bt^2} C_2 = \frac{47{,}200 \times 38.8 \times 5.95}{16 \times 400} = 1700 \text{ lb. per sq. in.}$$

The final design used $b = 21$ in.; $t = 20$ in.; and $p = 0.0242$. For $\frac{d'}{t} = 0.15$, Diagram 18 is entered at $\frac{t}{e} = 0.515$ and $np = 0.181$. $C_2 = 7.6$ and $k = 0.33$ approximately.

$$f_c = \frac{47{,}200 \times 38.8 \times 7.6}{21 \times 400} = 1660 \text{ lb. per sq. in.}$$

The values of C_2 and k cannot be read from the plots as accurately as they can be computed but there is little divergence from the results of the computations in Problem 39 and much time is saved.

Allowable Stresses. Diagram 21 (in the Appendix) gives the value of the term D of equation 209, if $\frac{d'}{t}$ and $(n - 1)p$ are known or assumed. The values of the allowable axial stress f_a of equation 206 are difficult to plot as the numerator consists of the essential part of the empirical equation 188, whose steel term is independent of the ratio n of the moduli of elasticity, whereas its denominator contains the essential part of the equation for the transformed area, whose steel term contains the product $(n - 1)p$. The ratio $\frac{f_a}{f'_c}$ should be computed. Diagram 22 gives the allowable stress f_p as the ratio $\frac{f_p}{f'_c}$, if $\frac{f_a}{f'_c}$ and the product $D\frac{e}{t}$ are known.

For the first solution in Problem 39, $f'_c = 4000$ lb. per sq. in. and $f_a = 1080$ lb. per sq. in The ratio $\frac{f_a}{f'_c} = 0.27$. Also $\frac{e}{t} = 1.94$; $\frac{d'}{t} = 0.15$; and $(n - 1)p = 0.26$. By Diagram 21, $D = 5.48$. Using Diagram 22, $D\frac{e}{t} = 10.62$ and $\frac{f_p}{f'_c} = 0.425$. Therefore $f_p = 0.425 \times 4000 = 1700$ lb. per sq. in. This checks the solution in Problem 39.

For the final design, $f'_c = 4000$ lb. per sq. in.; $f_a = 955$ lb. per sq. in. and $\frac{f_a}{f'_c} = 0.239$. Also $\frac{e}{t} = 1.94$; $\frac{d'}{t} = 0.15$; and $(n - 1)p = 0.157$. By Diagram 21, $D = 5.63$. Using Diagram 22, $D\frac{e}{t} = 10.92$ and $\frac{f_p}{f'_c} = 0.418$. Then $f_p = 1672$ lb. per sq. in. The solution of allowable stress by these plots saves much time.

ILLUSTRATIVE PROBLEM 41

249. Design of Column with Rectangular Section and Longitudinal Steel Loaded Eccentrically. *Case I.* Let us assume that the design of the column stack of Problem 39 is continued. Several floors below, the load from above equals 250,000 lb. and the floor loads of Problem 39 come in again. Figure 160 shows the forces at the junction of columns and beam. For the first trial both columns meeting at this floor will be given the previous section of $b = 21$ in. and $t = 20$ in. with the steel placed at $d' = 3$ in. from each face. The story height above this floor is 20 ft.; that below is 18 ft. At section CD (Fig. 160), $N = 295{,}900$ lb. Taking moments about the center of this section, $M_U + M_L = 1{,}425{,}000 + 35{,}900 \times 10 = 1{,}784{,}000$ in.-lb.

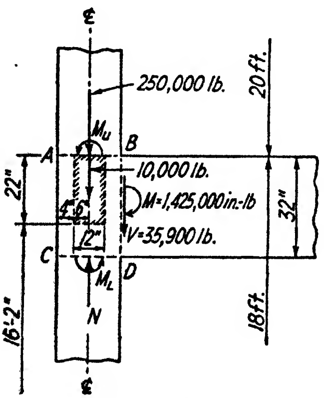


FIG. 160

The division of moment between M_U and M_L will be in proportion to their stiffnesses $\frac{I}{h}$. If the unknown steel areas are neglected and outside dimensions are dealt with, both columns will have the same moment of inertia I. Then

$$\frac{M_U}{M_L} = \frac{I_U}{h_U} \times \frac{h_L}{I_L} = \frac{16.17}{18.17} = \frac{1}{1.125}$$

$$M_U = 840{,}000 \text{ in.-lb.} \qquad \text{and} \qquad M_L = 944{,}000 \text{ in.-lb.}$$

Design of Section AB of Upper Column. At this section $N = 250{,}000$ lb., $M = 840{,}000$ in.-lb., and $e = 3.36$ in.

$$\frac{e}{t} = \frac{840{,}000}{250{,}000 \times 20} = 0.168. \qquad \text{This is Case I.}$$

However, this is a "long" column, as the ratio $\frac{h}{t} = 10.9$. The design will use a force $N = \frac{250{,}000}{0.973} = 257{,}000$ lb. and the eccentricity $e = 3.36$ in. Assume a 4000-lb. concrete and $p = 0.011$. By equations 193 and 191,

$$k = \frac{1}{2} + \frac{1 + 12 \times 6.5 \times 0.011 \times (0.35)^2}{12 \times 0.168 \times 1.0715} = 1.012$$

$$f_c = \frac{257{,}000}{21 \times 20} \times \frac{2.024}{1.024 \times 1.0715} = 1128 \text{ lb. per sq. in.}$$

The allowable stress will be found as in Problem 40, $\frac{d'}{t} = 0.15$; $(n - 1)p = 0.0715$; $f_a = 836$ lb. per sq. in. By Diagram 21, $D = 5.82$. By Diagram 22, for $\frac{f_a}{f'_c} = 0.209$ and $\frac{De}{t} = 0.977$, $\frac{f_p}{f'_c} = 0.284$; so $f_p = 1136$ lb. per sq. in. It is not necessary to check Case I solutions for the maximum steel stress. In this case it amounts to

$$f'_s = nf_c\left(\frac{kt - d'}{kt}\right) = 7.5 \times 1128 \times \left(\frac{20.24 - 3}{20.24}\right) = 7200 \text{ lb. per sq. in.}$$

$$A_s = 0.011 \times 420 = 4.62 \text{ sq. in.}$$

Use six 1-in. round bars. The steel arrangement is shown in Figure 161*a*. Before the design is finally adopted the section at the top of the column should be checked for its normal load N and bending moment M from the floor above. The steel of Figure 161 ends at the section AB (Fig. 160) we have just designed. It cannot reinforce that section, though it will those sections a short distance above. Therefore, we must make sure that at least 4.62 sq. in. of steel comes through from the column below and is extended far enough into this column to reinforce section AB.

Design of Lower Column. Section CD (Fig. 160). $N = 295{,}900$ lb. and $M = 944{,}000$ in.-lb.; $\frac{e}{t} = \frac{3.19}{20} = 0.16$. This is Case I and a "short column." Assuming the same mix as in the column above of $f'_c = 4000$ lb. per sq. in., the steel ratio is finally found to be $p = 0.02$. For this ratio $k = 1.050$ and $f_c = 1192$ lb. per sq. in.

The allowable stress is obtained for values of $(n - 1)p = 0.13$; $\frac{d'}{t} = 0.15$, giving $D = 5.69$. $D\frac{e}{t} = 0.91$; $f_a = 922$ lb. per sq. in.; and $\frac{f_a}{f'_c} = 0.2305$ gives $\frac{f_p}{f'_c} = 0.299$, so $f_p = 1196$ lb. per sq. in.

$A_s = 0.02 \times 420 = 8.40$ sq. in. Use six $1\frac{1}{4}$-in. square bars. The steel arrangement is shown in Figure 161*b*. It will be noticed that it has been possible to use the

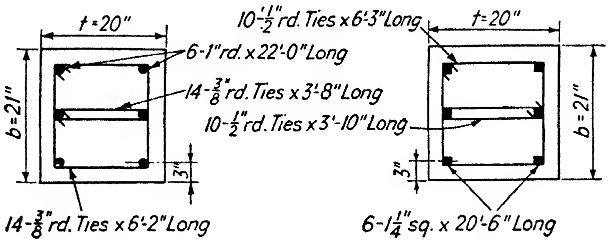


FIG. 161

same section as the roof column with the same strength of mix and a reduction of the steel ratio. The increase of load has been balanced by the decrease in eccentricity.

ILLUSTRATIVE PROBLEM 42

250. Case I. Solution by Plots. Solve Problem 41 for maximum concrete stress f_c by means of Diagrams 11 to 15.

Upper Column. $b = 21$ in.; $t = 20$ in.; $N = 257{,}000$ lb.; $e = 3.36$ in.; $\frac{e}{t} = 0.168$; and $\frac{d'}{t} = 0.15$. If the same mix $f'_c = 4000$ lb. per sq. in. and the final steel ratio $p = 0.011$ are adopted (Diagram 13), for $\frac{e}{t} = 0.168$ and $(n - 1)p = 0.121$, the value of $C_1 = 1.84$.

$$f_c = \frac{N}{bt}C_1 = \frac{257{,}000}{420} \times 1.84 = 1126 \text{ lb. per sq. in.}$$

This checks the computation in Problem 41.

Lower Column. $b = 21$ in.; $t = 20$ in.; $N = 295{,}900$ lb.; $e = 3.19$ in.; $\frac{e}{t} = 0.1595$; and $\frac{d'}{t} = 0.15$. The mix $f' = 4000$ lb. per sq. in. and the final steel ratio $p = 0.02$, so $(n - 1)p = 0.13$. From Diagram 13, $C_1 = 1.695$,

$$f_c = \frac{295{,}900}{420} \times 1.695 = 1192 \text{ lb. per sq. in.}$$

This also checks the computed f_c in Problem 41. The allowable stress f_p was obtained in that problem by means of Diagrams 21 and 22.

251. Critical Section. These designs have been made for certain selected sections. Each story should be designed for the section of greatest stresses. Figure 162 shows the original exterior column stack in full lines. The dashed lines show its probable deformation when the floors are loaded. Each column has a point of inflection near its center.

Since the columns are loaded at their ends only, the bending moment in the column will vary uniformly from the value of M_L at its top to M_U of the floor below at its bottom. The designer can usually tell by inspection which is the section of greatest stress. At the bottom section the steel in this story ends and cannot, therefore, take any stress. The steel from below which runs into this column must act at this section and should not be less than the computed area. For example, the computed area for section AB (Fig. 160) is 4.62 sq. in. Six 1-in. round bars are used in the upper column and they end at this section. The steel from below consists of six $1\frac{1}{4}$-in. square bars with an area of 9.38 sq. in., which is satisfactory. Care should be used to check this steel when the column below is greatly increased in size or its mix is increased in strength.

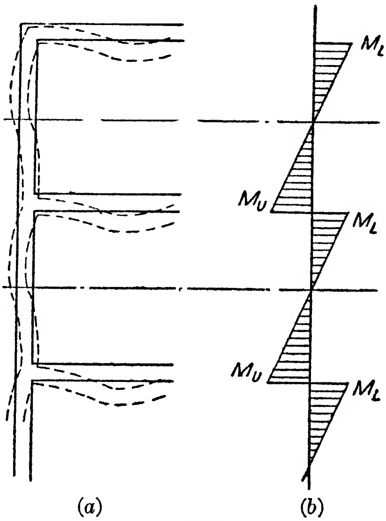


FIG. 162

252. Economical Design of Columns in Bending. Comparative designs for the same loading using different values of f'_c and p can be made with a tabulation similar to that in Table C (Art. 237) for axial loads. The results usually show that a minimum steel ratio p is cheapest for columns in bending. This leads to the general statement that longitudinal steel in a column is not an efficient reinforcement.

The effect of the mix is less marked than with axial loads. Medium to high-strength mixes give the cheapest designs.

253. Eccentric Loads. Circular Section. *Case I. Compression on Entire Section. Neutral Axis Outside Section* (Fig. 163). Assume a circular section of radius R whose longitudinal steel is uniformly spaced on a perimeter of radius r. The neutral axis lies outside the section at a distance of $k(2R)$ from the particle with the maximum compressive stress f_c. The values of k must be greater than unity for Case I. The steel area equals A_s, and the steel ratio $p = \dfrac{A_s}{\pi R^2}$.

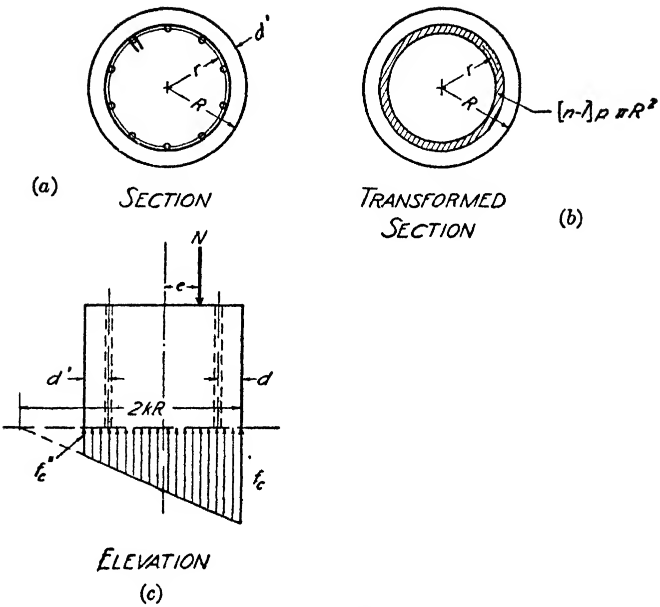


FIG. 163

The transformed area can be constructed by removing the steel and filling the holes with concrete. The remaining area $(n - 1)p\pi R^2$ is supplied as an annular ring superposed on the section. The transformed area equals

$$A = \pi R^2 + (n - 1)p\pi R^2$$

$$A = \pi R^2[1 + (n - 1)p] \tag{212}$$

This is, of course, the same result as that for axial loads. The moment of inertia of the transformed area about a diameter equals

$$I = \frac{\pi R^4}{4} + [(n - 1)p\pi R^2]\left(\frac{r^2}{2}\right) = \frac{\pi R^2}{4}[R^2 + 2(n - 1)pr^2]$$

The maximum stress equals

$$f_c = \frac{N}{A} + \frac{My}{I} = \frac{N}{\pi R^2[1 + (n - 1)p]} + \frac{4NeR}{\pi R^2[R^2 + 2(n - 1)pr^2]}$$

$$f_c = \frac{N}{\pi R^2}\left\{\frac{1}{[1 + (n - 1)p]} + \frac{4Re}{[R^2 + 2(n - 1)pr^2]}\right\} \tag{213}$$

or

$$f_c = \frac{N}{\pi R^2}(T + S) \tag{214}$$

The minimum stress f''_c equals

$$f''_c = \frac{N}{A} - \frac{My}{I} = \frac{N}{\pi R^2}(T - S) \tag{215}$$

If f_c and f''_c are known the neutral axis distance $k(2R)$ can be computed. Diagram 23 (in the Appendix) gives the values of T for given values of n and p.

Equation 213 can best be used by an assumption of mix, size, and steel ratio and by a computation of the fiber stress f_c. If Case I holds, the term T must be greater than term S in equation 214.

254. Eccentric Loads. Circular Section. *Case II. Tension on Part of Section. Neutral Axis within Section* (Fig. 164). If the neutral axis

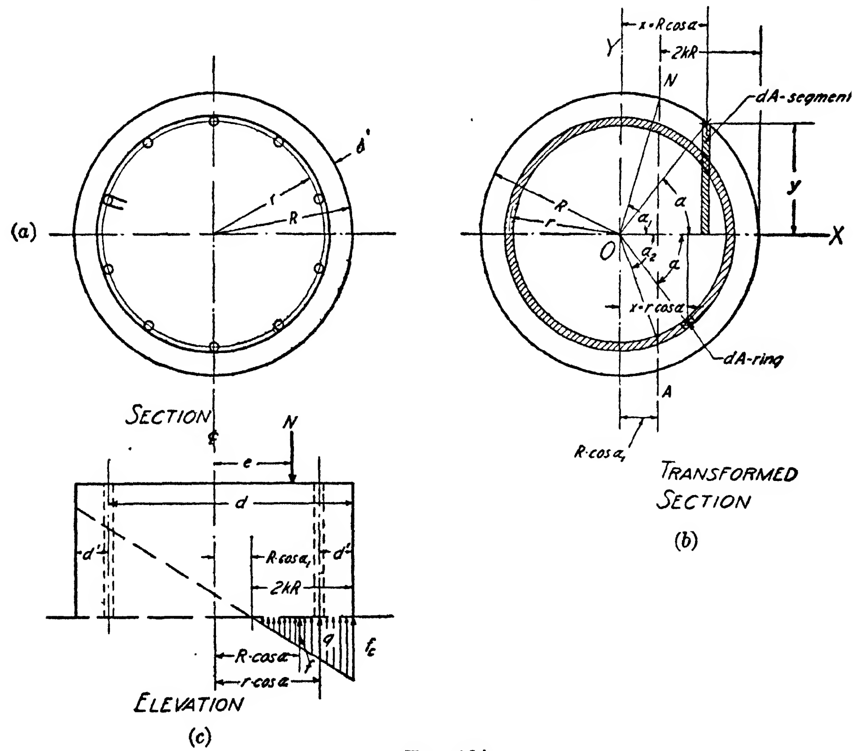


FIG. 164

lies within the section, only the concrete on the compression side is considered in stress computations. In constructing the transformed section the steel is removed and the holes on the compression side are filled with concrete. An arc of an annular ring of concrete, whose area is

$(n - 1)$ times the area of the steel on the compression side, is superposed. On the tension side there is only the arc of an annular ring of concrete whose area is n times the area of steel on the tension side. It is not necessary for accuracy to keep this difference in ring width. Therefore a complete ring of nA_s can be used without much error, as has been done with Case II for rectangular sections.

The transformed area now consists of a circular segment subtending an angle of $2\alpha_1$, and an annular ring whose area is

$$nA_s = np\pi R^2$$

The resultant compression force N_c of the uniformly varying stresses on the segment equals

$$N_c = \int f\,dA$$

where f = intensity of stress on the area dA, which is a distance x from the OY axis, where $x = R\cos\alpha$. From Figures 164b and 164c,

$$\frac{f}{f_c} = \frac{x - R\cos\alpha_1}{2kR}$$

$$f = \frac{f_cR(\cos\alpha - \cos\alpha_1)}{2kR} = \frac{f_c(\cos\alpha - \cos\alpha_1)}{2k}$$

If $x = R\cos\alpha$, $dx = R\sin\alpha\,d\alpha$ (numerically), and

$$dA = y\,dx = R^2\sin^2\alpha\,d\alpha$$

then,

$$N_c = \int f\,dA = \frac{f_cR^2}{2k}\int_{-\alpha_1}^{\alpha_1}(\cos\alpha - \cos\alpha_1)\sin^2\alpha\,d\alpha$$

$$N_c = \frac{f_cR^2}{2k}\left[\frac{\sin^3\alpha}{3} - \cos\alpha_1\left(\frac{\alpha}{2} - \frac{\sin\alpha\cos\alpha}{2}\right)\right]_{-\alpha_1}^{\alpha_1}$$

$$N_c = \frac{f_cR^2}{2k}\left[\frac{2}{3}\sin^3\alpha_1 + \sin\alpha_1\cos^2\alpha_1 - \alpha_1\cos\alpha_1\right]$$

The annular ring substituted for the steel has an area equal to

$$A_s = np\pi R^2 = 2\pi rt \qquad (216)$$

where t = thickness of ring.

The intensity q of the uniformly varying stresses acting on the ring can be found by the relation

$$\frac{q}{f_c} = \frac{r \cos \alpha - R \cos \alpha_1}{2kR}$$

or

$$q = \frac{f_c}{2kR}(r \cos \alpha - R \cos \alpha_1)$$

For values of α from 0 to α_2 the ring is in compression, from α_2 to π it is in tension. The intensity q acts on an area $dA = tr\, d\alpha$. The resultant force equals:

$$N_s = \int q\, dA = \frac{f_c rt}{2kR} \int_{-\pi}^{\pi} (r \cos \alpha - R \cos \alpha_1)\, d\alpha$$

$$N_s = \frac{f_c rt}{2kR} [r \sin \alpha - R\alpha \cos \alpha_1]_{-\pi}^{\pi}$$

$$N_s = -\frac{f_c rt}{2kR} (2\pi R \cos \alpha_1)$$

$$N_s = -\frac{\pi np R^2 f_c \cos \alpha_1}{2k}$$

The minus sign signifies that the resultant is a tensile force if $k < 0.5$, or $\alpha_1 < 90°$.

The total normal force N equals

$$N = N_c + N_s = \frac{f_c R^2}{6k}[2 \sin^3 \alpha_1 + 3 \cos \alpha_1 (\sin \alpha_1 \cos \alpha_1 - \alpha_1 - \pi np)] \quad (217)$$

Combining the terms that deal with α_1 only,

$$N = \frac{f_c R^2}{6k}[V - 3\pi np \cos \alpha_1] \quad (218)$$

where $V = 2 \sin^3 \alpha_1 + 3 \cos \alpha_1 (\sin \alpha_1 \cos \alpha_1 - \alpha_1)$.

The moment M of the external force N about the diameter OY of the column section equals $M = Ne$. The sum of the moment of the internal stresses must balance this moment. The moment M_c of the stresses in the circular segment equals

$$M_c = \int fx\,dA = \frac{f_cR^3}{2k}\int_{-\alpha_1}^{\alpha_1} (\cos\alpha - \cos\alpha_1)\cos\alpha\sin^2\alpha\,d\alpha$$

$$M_c = \frac{f_cR^3}{2k}\left[\left(\frac{\alpha}{8} - \frac{\sin 4\alpha}{32}\right) - \cos\alpha_1\frac{\sin^3\alpha}{3}\right]_{-\alpha_1}^{\alpha_1}$$

$$M_c = \frac{f_cR^3}{k}\left[\frac{\alpha_1}{8} - \frac{\sin 4\alpha_1}{32} - \frac{\cos\alpha_1\sin^3\alpha_1}{3}\right]$$

The moment M_s of the ring about the diameter OY equals

$$M_s = \int qx\,dA = \frac{f_cr^2t}{2kR}\int_{-\pi}^{\pi}(r\cos\alpha - R\cos\alpha_1)\cos\alpha\,d\alpha$$

$$M_s = \frac{f_cr^2t}{2kR}\left[\left(\frac{r\alpha}{2} + \frac{r\sin\alpha\cos\alpha}{2}\right) - R\cos\alpha_1\sin\alpha\right]_{-\pi}^{\pi}$$

$$M_s = \frac{f_cr^2t}{2kR}\left[\frac{2\pi r}{2}\right] = \frac{2\pi r^3tf_c}{4kR}$$

Substituting, $2\pi rt = \pi npR^2$,

$$M_s = \frac{\pi npRr^2f_c}{4k}$$

The total moment of resistance M about OY equals

$$M = M_c + M_s$$

$$= \frac{f_cR^3}{96k}\left[12\alpha_1 - 3\sin 4\alpha_1 - 32\sin^3\alpha_1\cos\alpha_1 + 24\pi np\left(\frac{r}{R}\right)^2\right] \tag{219}$$

Combining the terms dealing in α_1 only,

$$M = \frac{f_cR^3}{96k}\left[W + 24\pi np\left(\frac{r}{R}\right)^2\right] \tag{220}$$

From equations 217 and 219, the eccentricity $e = \dfrac{M}{N}$, or

$$\frac{e}{R} = \frac{M}{NR} = \frac{1}{16}\,\frac{\left[12\alpha_1 - 3\sin 4\alpha_1 - 32\cos\alpha_1\sin^3\alpha_1 + 24\pi np\left(\frac{r}{R}\right)^2\right]}{2\sin^3\alpha_1 + 3\cos\alpha_1(\sin\alpha_1\cos\alpha_1 - \alpha_1 - \pi np)} \tag{221}$$

This can also be written

$$\frac{e}{R} = \frac{\left[W + 24\pi np\left(\frac{r}{R}\right)^2\right]}{16(V - 3\pi np\cos\alpha_1)} \tag{222}$$

Diagram 24 (in the Appendix) gives the values of the terms V and W for values of α_1 from 0° to 180°. A problem is attacked by assuming Case I to hold. If k comes out less than unity, the designer knows that the equations for Case II are necessary. The equations do not use the term k but it can always be found since $2kR = R - R\cos\alpha_1$ or

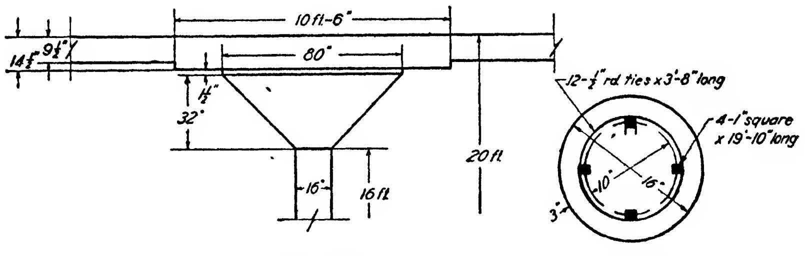
$$k = \frac{1 - \cos\alpha_1}{2} \tag{223}$$

In general, values of $\frac{e}{R}$ greater than 0.37 require a Case II solution.

Solutions for Case II are made by the assumption of size, mix, and steel ratio and the computation of the stress f_c.

ILLUSTRATIVE PROBLEM 43

255. Design of Axially Loaded Column of Circular Section (Fig. 165). Design a column to support the interior panel of the flat slab of Problem 36 (Chapter 10). Assume that the height equals 20 ft. from floor to floor.



(a) (b)

FIG. 165

Size. Each interior column supports a floor area of 29 ft. by 29 ft. If the column in question supports one floor only, the load at its base equals

Floor load (Art. 218)	216,000 lb.
Column capital (Fig. 165*a*)	7,000 lb.
Column weight for 16 ft.	4,000 lb.
	227,000 lb.

Assuming a short column with concrete strength $f'_c = 5000$ lb. per sq. in., by equation 188 (Art. 233),

$$227{,}000 = A(900 + 16{,}000p)$$

If $p = 0.01$: $A = 214$ sq. in., and diameter $D = 16.5$ in.

If $p = 0.04$: $A = 148$ sq. in., and diameter $D = 13.7$ in.

Let us adopt a column 16 in. in diameter. The unsupported length h is the distance from the top of the floor to the bottom of the capital, which is 16 ft. (A.C.I. Art. 1102a, in the Appendix). The ratio $\frac{h}{D} = 12$, so this is a long column. The load increase factor of A.C.I. Article 1107 becomes $1.3 - 0.03 \times 12 = 0.94$. Solving for the necessary steel ratio,

$$227{,}000 = 0.94 \times 201(900 + 16{,}000p)$$

$$p = 0.019 \qquad \text{and} \qquad A_s = 0.019 \times 201 = 3.82 \text{ sq. in.}$$

These bars are spaced evenly around a perimeter and there is no need for multiples of two or four for symmetry. Either four 1-in. squares or five 1-in. rounds are satisfactory.

The 1-in. square bars will be used as it is easier to provide ties for 4 bars than 5, and there will be fewer bars to handle. The steel arrangement is shown in Figure 165b.

The column has been designed for an axial load due to the live and dead loads. Analysis of the slab and columns as an elastic frame may give bending moments in interior columns for certain live load arrangements. In such a case this column design should be checked for these moments and the corresponding axial loads.

ILLUSTRATIVE PROBLEM 44

256. Design of Column of Circular Section in Bending. *Case II.* Assume that several interior panels of the flat slab floor of Problem 36 have been left open for a

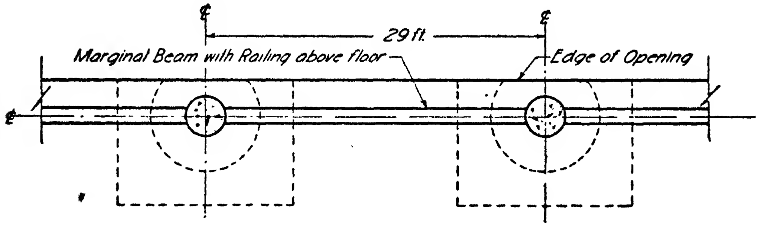


FIG. 166

crane well, and the 16-in.-diameter column designed in the previous problem is to be checked for use at the panel edges (Fig. 166). Complete its design for mix and steel.

Loads. Each column supports a half panel load.

Half panel load	108,000 lb.
Marginal beam and cantilever slab	9,000 lb.
Capital	6,000 lb.
Load at base of capital	N = 123,000 lb.

The 1928 A.C.I. Article 1105b recommends a bending moment of $\frac{Wl}{35}$ as the moment brought by the slab to the marginal column. The load W is the load on the whole panel. Our problem is a similar case, and this moment will be adopted. We shall neglect the balancing weight of the slab which overhangs into the crane well.

$$M = \frac{Wl}{35} = \frac{216{,}000 \times 29 \times 12}{35} = 2{,}150{,}000 \text{ in.-lb.}$$

$$e = \frac{M}{N} = \frac{2{,}150{,}000}{123{,}000} = 17.5 \text{ in.} \qquad \text{and} \qquad \frac{e}{R} = \frac{17.5}{8} = 2.19$$

In general, values of $\frac{e}{R}$ greater than 0.37 give Case II solutions; values below $\frac{e}{R} = 0.25$ give Case I. This is Case II. Equation 221 enables the designer to locate the neutral axis by successive assumptions of α_1. It makes for speed if he assumes $\alpha_1 = 180°$ and decreases α_1 by using the multiples of 30° and 45°. A plot of $\frac{e}{R}$ soon suggests the true α_1.

For this problem assume a 5000-lb. concrete and the maximum steel ratio $p = 0.04$.

α_1	$M = \frac{f_cR^2}{96k}C_m$	$N = \frac{f_cR^2}{6k}C_n$	$\frac{e}{R} = \frac{M}{NR} = \frac{C_m}{16C_n}$
	C_m	C_n	
180°	44.8	11.68	0.24
90°	25.9	2.00	0.81
85°	23.0	1.42	1.01
80°	21.14	0.88	1.50
78°	19.45	0.69	1.76
77°	18.92	0.578	2.05
76°	18.44	0.483	2.38

Use $\alpha_1 = 77°$ as the approximate angle for $\frac{e}{R} = 2.19$. Then $k = \frac{1 - \cos\alpha_1}{2} = 0.39$.

By equation 220,

$$f_c = \frac{96kM}{C_mR^3} = \frac{96 \times 0.39 \times 2{,}150{,}000}{18.92 \times (8)^3} = 8300 \text{ lb. per sq. in.}$$

The solution of Problem 43 showed that a column 16 in. in diameter is a long column. In the present design the load $N = 123{,}000$ lb. should be increased but assumed to act with the same eccentricity e. We have dealt with equations involving $\frac{e}{R}$ and the external bending moment M, so that the reduction of stress due to "long column" dimensions does not appear. This reduction will be applied to the allowable stress.

Allowable Stress. Using the method of A.C.I. Article 1110 (see Appendix), the radius of gyration ρ is computed by using the complete concrete area. By Article 253,

$$\rho^2 = \frac{I}{A} = \frac{R^2 + 2(n-1)pr^2}{4[1 + (n-1)p]} = \frac{64 + 2 \times 5 \times 0.04 \times 25}{4(1 + 5 \times 0.04)} = 15.4$$

The term D in A.C.I. Article 1110 can be used as $D = \frac{4R^2}{2\rho^2}$ where the eccentricity ratio equals $\frac{e}{2R}$. Note that Diagram 21 (in the Appendix) applies only to rectangular sections.

$$D = \frac{256}{2 \times 15.4} = 8.3 \quad \text{and} \quad \frac{De}{2R} = 8.3 \times \frac{2.19}{2} = 9.1$$

$$f_a = \frac{0.18 \times 5000 + 16{,}000 \times 0.04}{1.20} = 1280 \text{ lb. per sq. in.}$$

For $\frac{f_a}{f'_c} = 0.256$ and $D\frac{e}{2R} = 9.1$, Diagram 22 gives a ratio $\frac{f_p}{f'_c} = 0.419$. The allowable stress $f_p = 2095$ lb. per sq. in. for a "short column." In Problem 43 it was found that the load-increase factor, or stress-reduction factor, was 0.94. For a "long column" the allowable stress $f_p = 0.94 \times 2095 = 1970$ lb. per sq. in. Of course, in this case, it is not necessary to compute the allowable stress, as the stress $f_c = 8300$ lb. per sq. in. exceeds the compressive strength of the concrete. The section is much too small.

Revised Section. After several trials a section 24 in. in diameter with $f'_c = 6000$ lb. per sq. in. and $p = 0.035$ is selected. This is a short column.

$$\text{The necessary ratio } \frac{e}{R} = \frac{17.5}{12} = 1.457$$

An angle $\alpha_1 = 79°$ and $k = 0.404$ gives closely this value of $\frac{e}{R}$. The computed stress $f_c = 2380$ lb. per sq. in. and the allowable stress $f_p = 2390$ lb. per sq. in. The neutral axis is located at $2kR = 9.7$ in. from the greatest concrete stress. The maximum steel stress equals

$$f_s = nf_c\left(\frac{d - 2kR}{2kR}\right) = 5 \times 2380 \times \frac{11.3}{9.7} = 13{,}900 \text{ lb. per sq. in.}$$

This design is satisfactory.

$$A_s = 0.035 \times \pi \times (12)^2 = 15.8 \text{ sq. in.}$$

Use sixteen 1-in. square bars with a center-to-center spacing of approximately $3\frac{1}{2}$ in., as compared to the minimum spacing of 3 in. (Fig. 167). The requirement

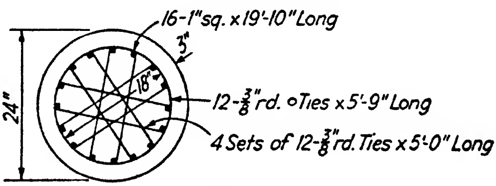


FIG. 167

of A.C.I. Article 1104b in regard to ties necessitates many additional ties which impede the efficient deposition of the concrete. The labor of installing steel and pouring concrete is greatly increased, so that the A.C.I. requirement puts a premium on the use of low steel ratios.

For comparison a second design is shown in Figure 168 using a section 28 in. in diameter with $p = 0.012$ with $f'_c = 6000$ lb. per sq. in. The computed concrete stress $f_c = 2180$ lb. per sq. in. and the allowable $f_p = 2230$ lb. per sq. in., while the maximum $f_s = 18{,}000$ lb. per sq. in.

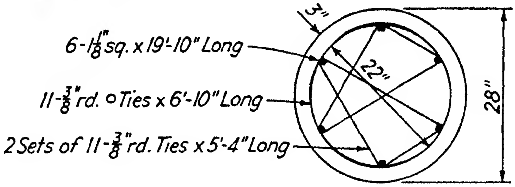


FIG. 168

The tie arrangement is much simpler and it will be easier to install the steel and pour the concrete. The volume of concrete is somewhat greater but the cost is probably less.

ILLUSTRATIVE PROBLEM 45

257. Design of Column of Circular Section in Bending. *Case I.* Assume that the design of the column stack of Problem 44 is continued. A few floors down, the load from above equals 500,000 lb. and the floor loads of Problem 44 are again brought in. Assume both columns to be 28 in. in diameter (Fig. 169). The upper story is 20 ft. from floor to floor, so the column has an unsupported height from floor to base of capital of 16 ft. 5½ in. The lower column has an unsupported height of 14 ft. 5½ in.

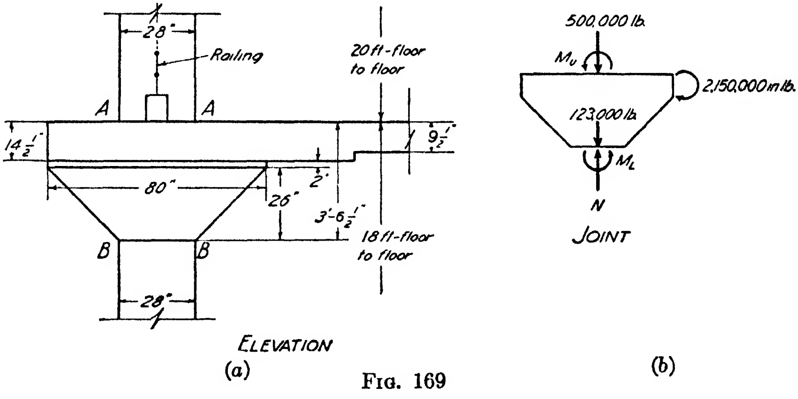


FIG. 169

Figure 169*b* shows the forces acting at the column capital. The floor moment of 2,150,000 in.-lb. will be divided between the upper and lower columns in proportion to their stiffness ratios $\frac{I}{h}$. The moment of inertia I of the columns will be taken as that of the plain concrete section. In this problem $I_U = I_L$.

$$\frac{M_U}{M_L} = \frac{I_U}{h_U} \times \frac{h_L}{h_U} = \frac{14.46}{16.46} = \frac{1}{1.14}$$

also

$$M_U + M_L = M = 2{,}150{,}000$$

Then

$$M_U = 1{,}005{,}000 \text{ in.-lb.} \qquad \text{and} \qquad M_L = 1{,}145{,}000 \text{ in.-lb.}$$

Design of Section AA of Upper Column. This is a short column.

$$N = 500{,}000 \text{ lb.} \qquad \text{and} \qquad M = 1{,}005{,}000 \text{ in.-lb.}$$

$$e = \frac{M}{N} = \frac{1{,}005{,}000}{500{,}000} = 2.01 \text{ in.} \qquad \text{and} \qquad \frac{e}{R} = 0.144$$

This is Case I. Try $f'_c = 4000$ lb. per sq. in. with $p = 0.04$. By equation 213,

$$f_c = \frac{500{,}000}{616}\left(\frac{1}{1 + 6.5 \times 0.04} + \frac{4 \times 14 \times 2.01}{196 + 2 \times 6.5 \times 0.04 \times 121}\right)$$

$$f_c = 812(0.793 + 0.435) = 995 \text{ lb. per sq. in.}$$

The allowable stress f_p can be determined by the procedure in Article 256.

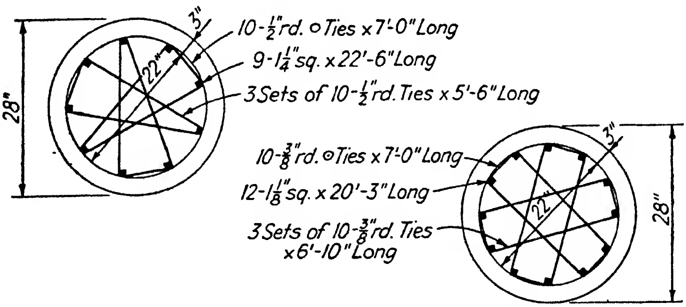
$$\rho^2 = \frac{259}{4 \times 1.26} = 51.4 \qquad \text{and} \qquad D = \frac{4R^2}{2\rho^2} = \frac{784}{2 \times 51.4} = 7.62$$

$$D\frac{e}{2R} = \frac{7.62 \times 2.01}{28} = 0.55 \qquad \text{and} \qquad \frac{f_a}{f'_c} = \frac{720 + 640}{1.26 \times 4000} = 0.27$$

From Diagram 22, $f_p = 0.316f'_c = 1264$ lb. per sq. in. There is an excess of steel and a few trials determine the correct steel ratio $p = 0.021$. For this amount $f_c = 1112$ lb. per sq. in. and $f_p = 1125$ lb. per sq. in.

$$A_s = 0.021 \times 616 = 12.94 \text{ sq. in.}$$

Use nine $1\frac{1}{4}$-in. square bars lapping 24 diameters into the columns above; or 22 ft. 6 in. long. These bars are spaced over $7\frac{1}{2}$ in. on centers, which is more than the minimum of $3 \times 1.25 = 3.75$ in. (Fig. 170*a*). At this section *AA* an equivalent area



Section A-A
Upper Column
(a)

Section B-B
Lower Column
(b)

FIG. 170

of steel must come up from the column below. It is possible, of course, that the top section of this column may be higher stressed, and the designer also examines that section.

258. Design of Section *BB* of Lower Column. (Figure 169.)

$$N = 623{,}000 \text{ lb.} \qquad \text{and} \qquad M = 1{,}145{,}000 \text{ in.-lb.}$$

$$e = \frac{M}{N} = 1.84 \text{ in.} \qquad \text{and} \qquad \frac{e}{R} = \frac{1.84}{14} = 0.131$$

Assume this to be Case I. Try $f'_c = 5000$ lb. per sq. in. and $p = 0.01$. The maximum concrete stress $f_c = 1460$ lb. per sq. in. and the allowable $f_p = 1185$ lb. per sq. in.

Several trials of additional steel result in $p = 0.024$ with $f_c = 1367$ lb. per sq. in. and $f_p = 1375$ lb. per sq. in.

$$A_s = 0.024 \times 616 = 14.79 \text{ sq. in.}$$

Use twelve $1\frac{1}{8}$-in. square bars, 20 ft. 3 in. long. This also gives more than the required area at section AA of the upper column. The section at the bottom of this lower column should be investigated before the design is adopted. It may require a larger steel ratio. The steel arrangement is shown in Figure 170*b*.

PLASTIC THEORY OF COLUMNS

259. The empirical column equations recommended by the A.C.I. and Joint Committee Codes are based on studies of test data for loads at failure. There has been previously given in this text under the heading of *plastic theory* analytical equations for the breaking loads of beams. The same analysis can be applied to columns and the derived equations should check with the A.C.I. results, as both are based on test data. The case immediately considered is that of a rectangular section subjected to a normal force on an axis of symmetry which is eccentric in regard to the perpendicular center line of the section. Assuming, as before, that a uniform concrete stress intensity of $0.85f'_c$ can be substituted for the stress conditions at failure and that the steel areas on the two opposite faces are equal ($A_s = A'_s$), the forces acting on a short portion of the column are shown in Figure 171. The steel is assumed to be stressed to its yield point f_y.

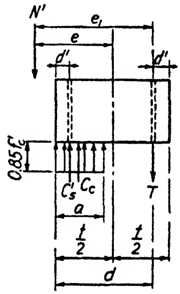


FIG. 171

Failure on Compression Side. If the steel areas on the two faces are equal, this is not a probable case. Take moments about the center of the tension steel and note that the ultimate moment for the concrete is given by equation 20 (Art. 25):

$$N'\left(e - \frac{t}{2} + d\right) = \frac{f'_c}{3} bd^2 + f_y A'_s(d - d')$$

(*Case II*) $$N' = \frac{2[f'_c bd^2 + 3f_y A'_s(d - d')]}{3(2e + 2d - t)} \qquad (224)$$

This equation applies if the eccentricity is great enough so that the compression stress f'_c occurs on only part of the area of the section. For an axial load $e = 0$, then, substituting $A_t = A_s + A'_s = 2A'_s$ and $d - d' = 2d - t$

$$N' = \frac{2f'_c bd^2}{3(2d - t)} + f_y A_t \tag{225}$$

It is known that a plain concrete column, loaded axially, has an average stress at failure of $0.85f'_c$ and fails at a load $N = 0.85f'_c bt$. The concrete term of equation 225 can be written

$$\frac{2f'_c bd^2}{3(2d - t)} = \frac{f'_c bt}{\frac{3}{2}\left(\frac{2d - t}{d^2}\right)t} = 0.85f'_c bt$$

or

$$\frac{3}{2}\left(\frac{2d - t}{d^2}\right)t = \frac{1}{0.85} = 1.178$$

(Axial Load)
$$N' = 0.85f'_c bt + f_y A_t \tag{226}$$

This is the equation determined by the Illinois-Lehigh tests for failure of axially loaded columns. Mr. Whitney recommends a factor of safety of 2.5 for spiral-steel columns and a factor of $\frac{2.5}{0.8} = 3.13$ for tied columns. When the A.C.I. equations 188 and 237 are multiplied by these factors, the ultimate load N' based on the A.C.I. equations becomes

$$N' = 0.563f'_c bt + f_y A_t \tag{226a}$$

In other words, the A.C.I. Code uses a greater factor of safety for the concrete than for the steel.

For moderate eccentricities (Case I), with compression over the whole area, equation 224 can then be rearranged to read

(Case I)
$$N' = \frac{f'_c bt}{\frac{3et}{d^2} + 1.178} + \frac{2f_y A'_s(d - d')}{2e + 2d - t} \tag{227}$$

Failure on Tension Side. This is the more probable case, as the compression side has an equal steel area and the concrete is in compression; but the tension steel has a greater moment arm about the applied force N.

Assuming the compressive force in the concrete to act d' in from the edge, take moments about the center of the compression force.

$$N'\left(e - \frac{t}{2} + d'\right) = f_y A_s (d - d')$$

$$N' = f_y A_s \frac{2(d - d')}{2e + 2d' - t} \tag{228}$$

If the resultant of the concrete stresses of $0.85f'_c$ acts other than d' from the edge the equilibrium of forces gives

$$N' + f_y A_s = 0.85f'_c ab + f_y A'_s$$

If

$$A_s = A'_s, \qquad a = \frac{N'}{0.85f'_c b}$$

Taking moments about the center of the tension steel,

$$N'\left(e - \frac{t}{2} + d\right) = 0.85f'_c ab\left(d - \frac{a}{2}\right) + f_y A'_s (d - d')$$

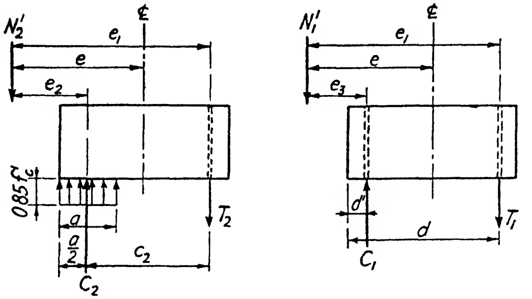
Substituting $\frac{p}{2} = \frac{A_s}{bt} = \frac{A'_s}{bt}$ and the values of a given above and $m = \frac{f_y}{0.85f'_c}$, this reduces to

(*Case II*)

$$N' = 0.85f'_c bt\left\{\sqrt{mp\left(\frac{d - d'}{t}\right) + \left(\frac{e}{t} - \frac{1}{2}\right)^2} - \left(\frac{e}{t} - \frac{1}{2}\right)\right\} \tag{229}$$

For small eccentricities, there is no tension in the steel, and no failure can occur on the tension side.

When the eccentricity ratio $\frac{e}{t}$ is large, the two terms of equation 229 within the bracket are nearly equal and it is difficult to get an accurate result with slide-rule computations. An alternate derivation covers this case. Refer to Figure 172; the load N' may be conceived as divided into two parts, N'_1 resisted by all the compression steel and some of the tension steel A_1, and N'_2 resisted by the concrete in compression and the remainder of the tension steel A_2. Then $A_s = A_1 + A_2$.



No Compression Steel
(a)

No Concrete Stresses
(b)

FIG. 172

In Figure 172*b* take moments about N'_1, noting that $C_1 = f_yA'_s$, then

$$T_1 = f_yA_1 = f_yA'_s\frac{e_3}{e_1} \quad \text{and} \quad A_1 = A'_s\frac{e_3}{e_1} = A'_s\left[1 - \left(\frac{d - d'}{e_1}\right)\right]$$

Taking moments about C_1,

$$N'_1 = \frac{T_1(d - d')}{e_3} = f_yA'_s\frac{d - d'}{e_1} \tag{230}$$

In Figure 172*a* take moments about C_2

$$N'_2 = T_2\frac{c_2}{e_1 - c_2}$$

so

$$N'_2e_1 = f_yA_2\frac{c_2e_1}{e_1 - c_2}$$

Taking moments about T_2, noting that $\frac{a}{2} = d - c_2$:

$$0.85f'_cabc_2 = N'_2e_1 = f_yA_2\frac{c_2e_1}{e_1 - c_2}$$

$$a = 2(d - c_2) = \frac{f_y}{0.85f'_c} \times \frac{A_2}{b} \times \frac{e_1}{e_1 - c_2}$$

This can be expanded to

$$4(c_2)^2 - 4c_2(e_1 + d) + [(e_1 + d)^2] = \frac{2me_1A_2}{b} - 4e_1d + [(e_1 + d)^2]$$

Taking square root,

$$c_2 = \frac{1}{2}\left\{\pm\sqrt{\frac{2me_1A_2}{b} - 4e_1d + (e_1+d)^2} + (e_1+d)\right\}$$

Now,

$$A_s = A'_s, \text{ and } A_2 = A_s - A_1 = A_s - A'_s\left[1 - \frac{(d-d')}{e_1}\right] = A_s\frac{(d-d')}{e_1}$$

Then

$$c_2 = \frac{1}{2}\left\{(e_1+d) - \sqrt{\frac{2mA_s(d-d')}{b} + (e_1-d)^2}\right\} \tag{231}$$

Also,

$$T_2 = f_yA_s - T_1 = f_yA_s - f_yA'_s\frac{e_3}{e_1} = f_yA_s\left(\frac{e_1-e_3}{e_1}\right) = \frac{f_yA_s(d-d')}{e_1}$$

Finally

$$N' = N'_1 + N'_2 = \frac{f_yA'_s(d-d')}{e_1} + f_yA_s\frac{(d-d')}{e_1}\frac{c_2}{e_1-c_2}$$

(*Case II*)

$$N' = \frac{f_yA_s(d-d')}{e_1}\left[1 + \frac{c_2}{e_1-c_2}\right] = \frac{f_yA_s(d-d')}{(e_1-c_2)} \tag{232}$$

260. Columns with Round Cores. Mr. Whitney[2] has also proposed equations for the load at failure of square columns reinforced with longitudinal steel placed on a circular perimeter and for round columns with circular placement of the steel. In view of the added assumptions based on test data these equations seem to be strictly empirical.

Square Columns with Circular Core. Diameter of steel ring equals d. Compression Failure.

$$N' = \frac{0.85f'_cA_c}{\dfrac{10.2et}{(t+0.67d)^2} + 1.51} + \frac{f_yA_t}{\dfrac{3e}{d} + 1} \tag{233}$$

Tension Failure.

$$N' = 0.85f'_ct^2\left\{\sqrt{0.67\frac{mpd}{t} + \left(\frac{e}{t} - 0.4\right)^2} - \left(\frac{e}{t} - 0.4\right)\right\} \tag{234}$$

[2] *Trans. A.S.C.E.*, Vol. 68, p. 251, 1942, "Plastic Theory of Reinforced Concrete Design," by Charles S. Whitney.

Round Column. With diameter D and steel ring diameter of d. Compression Failure.

$$N' = \frac{0.85f'_cA_c}{\dfrac{8.16De}{(0.8D + 0.67d)^2} + 1.51} + \frac{f_yA_t}{\dfrac{3e}{d} + 1} \tag{235}$$

Tension Failure.

$$N' = 0.85f'_cD^2 \left\{\sqrt{\frac{mpd}{2.5D} + \left(\frac{0.85e}{D} - 0.3\right)^2} - \left(\frac{0.85e}{D} - 0.3\right)\right\} \tag{236}$$

ILLUSTRATIVE PROBLEM 46

261. Checking Column Designs by Plastic Theory. *Axial Load. Rectangular Section.* In Problem 38 $N = 310{,}500$ lb. (Table C, Art. 237), $b = t = 16$ in., $A_t = 6.24$ sq. in., $f'_c = 5000$ lb. per sq. in., $f_y = 50{,}000$ lb. per sq. in. For tied columns the reduction factor of 0.8 recommended by Mr. Whitney and A.C.I. Article 1104*a* will be used. By equation 226 the ultimate load equals

$$N' = 0.8[0.85 \times 5000 \times 256 + 50{,}000 \times 6.24] = 1{,}120{,}000 \text{ lb.}$$

By equation 226*a* the ultimate load

$$N' = [0.563 \times 5000 \times 256 + 50{,}000 \times 6.24] = 1{,}032{,}600 \text{ lb.}$$

Since this is a "long column," both loads are multiplied by the reduction factor, 0.94. Then, the respective factors of safety are 3.40 and 3.13. Equation 226*a* automatically gives the factor of safety of 3.13, which is based on the A.C.I. formula.

Axial Load. Circular Section. In Problem 43 $D = 16$ in., $d = 10$ in., and $A_t = 4.00$ sq. in. By the methods of Problem 43 the total safe load equals

$$N = 0.94 \times 201\left(900 + 16{,}000 \times \frac{4.00}{201}\right) = 230{,}000 \text{ lb.}$$

By equation 226, the ultimate load for a tied column equals

$$N' = 0.8 \times 0.94(0.85 \times 5000 \times 201 + 50{,}000 \times 4.00) = 792{,}800 \text{ lb.}$$

The factor of safety is 3.45.

From these two examples it appears that the A.C.I. formula has a factor of safety of about 3.4 on the ultimate load determined by the plastic theory.

Eccentric Load. Case I. Rectangular Section. In Problem 41, for the *upper column*, $f'_c = 4000$ lb. per sq. in., $f_y = 50{,}000$ lb. per sq. in., $b = 21$ in., $t = 20$ in., $A_s = A'_s = 2.36$ sq. in., $d = 17$ in., $d' = 3$ in., $e = 3.36$ in. By the methods of Problem 41 the maximum safe load with an eccentricity of 3.36 in. is found to be

$N = 0.973 \times 261{,}500 = 254{,}000$ lb. for a "long column." By equation 227 the ultimate load for failure equals

$$N' = 0.973 \times 0.8 \left[\frac{3500 \times 21 \times 20}{\dfrac{3 \times 3.36 \times 20}{(17)^2} + 1.178} + \frac{2 \times 50{,}000 \times 2.36 \times 14}{2 \times 3.36 + 2 \times 17 - 20} \right]$$

$$N' = 822{,}000 \text{ lb.}$$

The factor of safety of the A.C.I. solution equals 3.24.

The *lower column* data gives $f'_c = 4000$ lb. per sq. in., $f_y = 50{,}000$ lb. per sq. in., $b = 21$ in., $t = 20$ in., $A_s = A'_s = 4.69$ sq. in., $d = 17$ in., $d' = 3$ in., $e = 3.19$ in. The maximum safe load $N = 312{,}000$ lb. By equation 227 the ultimate load for failure equals

$$N' = 0.8 \left[\frac{3500 \times 420}{\dfrac{3 \times 3.19 \times 20}{(17)^2} + 1.178} + \frac{2 \times 50{,}000 \times 4.69 \times 14}{2 \times 3.19 + 2 \times 17 - 20} \right]$$

$$N' = 988{,}000 \text{ lb.}$$

The factor of safety of the A.C.I. solution equals 3.17. Mr. Whitney recommends a factor of safety of 3.13 for tied columns; thus, the factors of safety by present methods of design are slightly greater.

Eccentric Load. Case I. Circular Section. Upper Column. In Problem 45 the essential data are $f'_c = 4000$ lb. per sq. in., $f_y = 50{,}000$ lb. per sq. in., $D = 28$ in., $d = 22$ in., $A_t = 14.04$ sq. in., $e = 2.01$ in. By the methods of Problem 45 the maximum safe load $N = 507{,}000$ lb. By equation 235 the ultimate load for failure equals

$$N' = \frac{0.85 \times 4000 \times 602}{\dfrac{8.16 \times 28 \times 2.01}{(0.8 \times 28 + 0.67 \times 22)^2} + 1.51} + \frac{50{,}000 \times 14.04}{\dfrac{3 \times 2.01}{22} + 1} = 1{,}663{,}000 \text{ lb.}$$

The factor of safety of the A.C.I. solution equals 3.28, which slightly exceeds the factor of 3.13 recommended by Mr. Whitney for tied columns.

Lower Column. The section considered in this column had dimensions of $D = 28$ in., $d = 22$ in., $A_t = 15.24$ sq. in., $e = 1.84$ in., $f'_c = 5000$ lb. per sq. in., and $f_y = 50{,}000$ lb. per sq. in. The maximum safe load $N = 632{,}000$ lb. By equation 235 the ultimate load equals

$$N' = \frac{0.85 \times 5000 \times 601}{\dfrac{8.16 \times 28 \times 1.84}{(0.8 \times 28 + 0.67 \times 22)^2} + 1.51} + \frac{50{,}000 \times 15.24}{\dfrac{3 \times 1.84}{22} + 1} = 2{,}014{,}000 \text{ lb.}$$

The factor of safety of the A.C.I. solution equals 3.19.

Eccentric Loads. Case II. Rectangular Section. In Problem 39, $f'_c = 4000$ lb. per sq. in., $f_y = 50{,}000$ lb. per sq. in., $b = 21$ in., $t = 20$ in., $A_s = A'_s = 5.06$ sq. in., $d = 17$ in., $d' = 3$ in., $e = 38.8$ in. By the design methods of Problem 39 the maximum safe load with an eccentricity of 38.8 in. is found to be $N = 46{,}400$ lb.

If failure on the compression side is assumed the ultimate load N' by equation 224 equals

$$N' = \frac{2(4000 \times 21 \times 289 + 3 \times 50{,}000 \times 5.06 \times 14)}{3(2 \times 38.8 + 2 \times 17 - 20)} = 254{,}000 \text{ lb.}$$

If failure is assumed on the tension side with compression steel taking all the compression stresses, the ultimate load by equation 228 equals

$$N' = \frac{50{,}000 \times 5.06 \times 2 \times 14}{2 \times 38.8 + 2 \times 3 - 20} = 111{,}300 \text{ lb.}$$

It is, therefore, probable that the concrete takes some stress. For tied columns with large eccentricities, Mr. Whitney recommends that equations 224, 229, 231, and 232 be used without the multiplier of 0.8. Using equation 229,

$$N' = 0.85 \times 4000 \times 21 \times 20 \left\{ \sqrt{\frac{14.7 \times 0.0241 \times 14}{20} + (1.94 - 0.50)^2} - (1.94 - 0.50) \right\}$$

$$N' = 0.85 \times 4000 \times 21 \times 20(1.527 - 1.44) = 124{,}000 \text{ lb.}$$

Comparing the value with the result of equations 231 and 232, where

$$e_1 = 45.8 \text{ in.}$$

$$c_2 = \frac{1}{2} \left\{ (45.8 + 17) - \sqrt{\frac{2 \times 14.7 \times 5.06 \times 14}{21} + (45.8 - 17)^2} \right\} = 16.15 \text{ in.}$$

$$N' = \frac{50{,}000 \times 5.06 \times 14}{(45.8 - 16.15)} = 119{,}300 \text{ lb.}$$

This result is more accurate by slide rule than the result of equation 229, as the latter contains the term (1.527 − 1.44) whose difference is small.

Surveying these results one concludes that the critical case is failure on the compression side with the concrete taking stress, the ultimate load being 0.973 × 119,300 = 116,000 lb. The factor of safety of the A.C.I. design based on this load is 2.50.

Eccentric Loads. Case II. Circular Section. In Problem 44 f'_c = 6000 lb. per sq. in., f_y = 50,000 lb. per sq. in., D = 24 in., d = 18 in., and A_t = 16.00 sq. in. By the methods of Problem 44 the maximum safe loads with an eccentricity e = 17.5 in. is N = 124,000 lb.

Assuming failure on the compression side by equation 235,

$$N' = \frac{0.85 \times 6000 \times 436}{\dfrac{8.16 \times 24 \times 17.5}{(0.8 \times 24 + 0.67 \times 18)^2} + 1.51} + \frac{50{,}000 \times 16}{\dfrac{3 \times 17.5}{18} + 1} = 649{,}000 \text{ lb.}$$

If a tension failure is assumed equation 236 gives

$$N' = 0.85 \times 6000 \times (24)^2 \left\{ \sqrt{\frac{9.79 \times 0.0354 \times 18}{2.5 \times 24} + \left(\frac{0.85 \times 17.5}{24} - 0.3\right)^2} - \left(\frac{0.85 \times 17.5}{24} - 0.3\right) \right\}.$$

$$N' = 396{,}000 \text{ lb.}$$

The ultimate load equals 396,000 lb. and the factor of safety of the A.C.I. design equals 3.19. A general comparison of the present A.C.I. designs with the ultimate loads by the plastic theory is given below.

	Type of Loading	Eccentricity Ratio $\frac{e}{t}$ or $\frac{e}{R}$	A.C.I. Safe Load lb.	Plastic Theory Ultimate Load lb.	Factor of Safety
Rectangular section	Axial	0	310,500	1,052,000	3.40
	Eccentric, Case I	0.168	254,000	822,000	3.24
	" " "	0.160	312,000	988,000	3.17
	" Case II	1.94	46,400	116,000	2.50
Circular section	Axial	0	230,000	792,800	3.45
	Eccentric, Case I	0.144	507,000	1,663,000	3.28
	" " "	0.131	632,000	2,014,000	3.19
	" Case II	1.46	124,000	396,000	3.19

Mr. Whitney recommends a factor of safety of 3.13 for tied columns with axial loads or small eccentricities of loading, and a factor of 2.5 for beams; for tied columns with large eccentricities the factor of safety should range from 3.13 to 2.5. Columns with spiral steel may be designed for a factor of safety of 2.5.

COLUMNS WITH SPIRAL STEEL

262. Spiral Steel Columns. The column with closely spaced spiral steel is almost invariably reinforced with longitudinal bars, which add to the column's resistance to bending and to shrinkage. The spiral steel restrains the concrete core from bulging as the column is loaded but does not take much stress until the lateral deformation is considerable. The lateral deformation, dependent on Poisson's ratio, is not great at working loads and therefore there is little stress in the spiral steel at working loads. This steel does reduce excessive concrete deformations and increases the ultimate load carried by the column. When the spiral steel becomes stressed, the longitudinal steel has passed its elastic limit, and the concrete is highly stressed. At failure the longitudinal steel has reached its yield point stress. Any analysis that assumes the spiral steel to be stressed must be made for conditions beyond the yield point of the column and does not lend itself to a simple analytic treatment. For this reason spiral column design formulae are empirical.

263. Axial Loads. (A.C.I. Art. 1103.) The spiral steel column loaded with an axial load is designed by the empirical equation

$$N = A(0.225f'_c + 0.4f_y p) \tag{237}$$

Equation 188 (Art. 233) for the design of tied columns is 0.8 of this equation. The spiral column of the same dimensions as a tied column will, therefore, carry an axial load 25 per cent greater. In addition, however, it is permissible to use longitudinal steel up to 8 per cent, though it is frequently difficult to place as much steel as this and comply with the minimum spacings.

Formerly the amount of spiral steel was in direct proportion to the amount of longitudinal steel, its volume being one fourth the volume of the longitudinal steel. Although it is always permissible to use as much spiral steel, the Illinois-Lehigh tests justify a different procedure. The strength added by the spiral steel at loads near failure should equal the loss of strength due to spalling off of the concrete outside the spirals. The minimum amount of spiral steel equals

$$p' = 0.45\left(\frac{A_g}{A_c} - 1\right)\frac{f'_c}{f_y} \tag{238}$$

where A_g = gross area of column

A_c = area of the core, measured to the outside diameter of the spiral

f_y = yield point of the spiral steel, but limited to 60,000 lb. per sq. in.

ILLUSTRATIVE PROBLEM 47

264. Design of Axially Loaded Spiral Column. Design the circular interior column for the flat slab floor of Problem 36 (Chapter 10), using a spiral steel column. Assume that the design is made several floors down the stack for a load at the base of the capital of 1,100,000 lb. The story height is 20 ft., and the unsupported length is 16 ft.

Size. Assume f'_c = 6000 lb. per sq. in. and that the column weighs 10,000 lb. The load at the bottom of the column equals

$$1{,}110{,}000 = A(1350 + 20{,}000p)$$

If $p = 0.01$, A = 717 sq. in., outside diameter D = 30.2 in.

If $p = 0.08$, A = 377 sq. in., outside diameter D = 21.9 in.

Diameter of column, D in.	30	22
Weight of 16-ft. length, lb.	12,000	7000
Steel ratio, p	0.011	0.078
Steel area A_s, sq. in.	7.85	29.7
Use	8—1″ sq.	19—1¼″ sq.
Spiral ratio p'	0.0178	0.0265
(Outside of spiral 2 in. from outside of column, f_y = 50,000 lb. per sq. in.)		
Volume of core, per ft. height, cu. in.	6390	3060
Volume of spirals, cu. in.	114	81
Volume $a_s \times \pi d \times$ no. of turns per ft.		
$a_s \times$ no., sq. in.	1.43	1.48
Use	⅝-in. rd. @ 2½ in.	⅝-in. rd. @ 2⅝ in.

The spirals chosen have a clear spacing exceeding $1\frac{3}{8}$ in. and less than 3 in. The center-to-center spacing does not exceed one sixth the core diameter. The column sections are shown in Figure 173. It will be noticed that it is difficult to place the steel and pour the concrete in a column containing a steel ratio approaching $p = 0.08$.

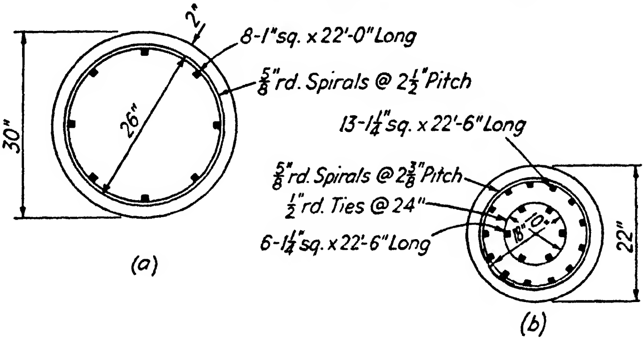


FIG. 173

265. Economy. It is evident that for a given axial load many designs may be made which use different sizes, steel ratios, or mixes. At a given floor only one of these designs is the cheapest. This cheapest column is dependent somewhat on the sizes above and below, upon the possibility of using the column forms for more than one floor, and the necessity of placing extra steel dowels at certain column junctions. In general, however, a comparative analysis, like that made in Article 237 for tied columns, shows that the cheapest column with spiral steel is that with a rich mix and high percentage of steel. The fact that the spiral steel is in tension so increases the ultimate load that it becomes economical to use high steel ratios for longitudinal steel.

266. Columns with Spiral Steel in Bending. Since the spiral steel does not take stress until failure approaches, columns in bending can be designed for working conditions by the equations used for tied columns of circular section. The allowable stress is determined by the same method, except that in place of equation 206 the axial stress

$$f_a = \frac{0.225f'_c + 0.4f_y p}{1 + (n - 1)p} \tag{239}$$

267. Composite Columns. Composite columns are defined as steel or cast-iron columns encased in concrete reinforced with longitudinal bars and spiral steel. The area of the metal column shall not exceed 20 per cent of the gross column area. If a hollow cast iron or steel column is used, its core shall be filled with concrete. The longitudinal steel ratio and the spiral steel ratio shall conform with the requirements for reinforced spiral columns.

ILLUSTRATIVE PROBLEM 48

268. Design of a Composite Column. Design a composite column to carry an axial load of 2,000,000 lb. at its base. After several trials a 14 WF 158 steel section is adopted with cross-sectional area of 46.5 sq. in. This section is shown in Figure 174.

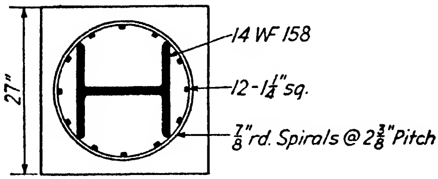


FIG. 174

If a 2-in. fireproofing clearance beyond the spiral steel is allowed, the smallest square section has an outside dimension of 27 in. Thus, the metal section is only 6.4 per cent of the gross area of the column. Assume $f'_c = 6000$ lb. per sq. in. and $f_y = 50{,}000$ lb. per sq. in. for the reinforcing steel. Equation 22 in A.C.I. Article 1105 can be set up as

$$2{,}000{,}000 = 1350(729 - 46.5 - A_s) + 20{,}000A_s + 16{,}000 \times 46.5$$

$$A_s = 17.92 \text{ sq. in.} \qquad \text{Use twelve } 1\tfrac{1}{4}\text{-in. square bars.}$$

In the computation of the spiral steel $A_g = 729 - 46.5 - 18.7 = 663.8$ sq. in. and $A_c = \pi(11.5)^2 - 65.2 = 349.8$ sq. in.

$$p' = 0.45\left(\frac{663.8}{349.8} - 1\right)\frac{6000}{50{,}000} = 0.0485$$

Spiral volume $= 0.0485 \times 349.8 \times 12 = 204$ cu. in. per ft. height

$$204 = a_s(\pi \times 22.13) \times \frac{12}{p} \qquad \text{and} \qquad \frac{a_s}{p} = 0.245$$

Use $\frac{7}{8}$-in. round bars spaced at $2\frac{3}{8}$-in. pitch.

269. Combination Columns. The last step in the progression from a plain concrete column to a plain structural steel column is the steel column encased in concrete. There are no restrictions on the area of the steel column, the only requirement being that the concrete shall cover all metal parts, except rivet heads, by at least 2.5 in. The only concrete reinforcement is welded wire mesh encircling the steel column.

The column stack in a high building usually starts with tied columns supporting the upper stories, followed by spiral steel columns and composite or combination columns as the load increases at the lower stories.

ILLUSTRATIVE PROBLEM 49

270. Design of a Combination Column. Design a combination column to carry the same axial load of 2,000,000 lb. at its base as in Problem 48. The unsupported length is 16 ft. After several trials a 14-in. plate and angle column weighing 414 lb.

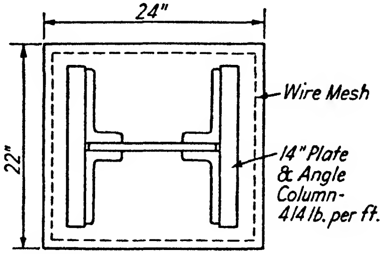


FIG. 175

per ft. (1937 *A.I.S.C. Handbook*) is adopted. The column is encased in a concrete covering 24 in. by 22 in. outside dimensions (Fig. 175). The load that can be carried by equation 23 of A.C.I. Article 1106 is

$$N = 121.7 \times 16{,}100\left[1 + \frac{(528 - 121.7)}{100 \times 121.7}\right] = 2{,}025{,}000 \text{ lb.}$$

The ratio $\dfrac{l}{r} = \dfrac{192}{4.44} = 43.3$ and the allowable steel stress $f'_r = 16{,}100$ lb. per sq. in.

CHAPTER 12

FOOTINGS AND RETAINING WALLS

The loads that are applied on a roof or floor system are brought to the columns and carried down the column stacks for transference to the soil or rock. Columns, piers, or caissons can occasionally transfer their load to bedrock without change of size, but usually there must be an expansion of the column base in order to prevent excessive settlement. This expanded portion is called the column footing. Foundations vary in complexity from the simple footing supporting one column to a raft foundation which acts as an inverted floor and supports the entire building. In some recent buildings the basement and the floor above, with the connecting columns and walls, form trusses stiffening the foundation in perpendicular directions to produce uniform settlement.

271. Allowable Soil Pressure. The designer of the foundations should not consider exclusively any individual footing. He has the problem of floating his structure on a yielding medium without the advantage enjoyed by the naval architect, whose yielding medium is water with constant elastic properties. The foundation engineer deals with sands, gravels, and clays whose elastic properties vary with their previous geological history. Even though the soil may consist wholly of sand or of clay, its homogeneity and its compressibility may vary throughout the building site. It is usually uneconomical to make enough bearing tests to be sure of the proper allowable unit pressure at the base of each footing to give a uniform settlement for the whole building. Often average values for the locality are all that are available for the designer. Unequal settlement with unsightly cracks in the finished structure and unforeseen overloading of some part is not at all unusual. It is not the function of this text to develop the science of soil mechanics, and in this discussion it will be assumed for the footings examined that the proper allowable bearing pressure has been adopted so that all the footings in the structure will settle uniformly.

Maximum allowable soil pressures can be determined approximately from the 1907 Building Code of Boston:

"In the absence of satisfactory tests of their sustaining power, the maximum allowable bearing values of the above materials shall be limited by the following unit pressures:

	Tons per Square Foot
Solid ledge rock	100
Shale and hardpan	10
Gravel, compact sand and hard yellow clay	6
Dry or wet sand of coarse or medium-sized grains, hard blue clay mixed or unmixed with sand, disintegrated ledge rock	5
Medium stiff or plastic clay mixed or unmixed with sand, or fine grained dry sand	4
Fine grained wet sand (confined)	3
Soft clay protected against lateral displacement	2

272. Distribution of Soil Pressure. The experimental data on variation of soil pressures on the bases of footings are not yet extensive enough to give the actual distributions. For wall footings which project as cantilevers, it is probable that the actual pressure varies as shown by the dash lines of Figure 176*b*, because the portion of the footing remote from the wall will deflect readily when acted on by a smaller pressure than the center values. The total pressure on the soil must support the wall load, the weight of the footing, and the weight of the soil and the basement floor loads immediately above the footing. While the soil is still unconsolidated, there can be little of the arching action that occurs through a compact soil to transfer the basement loads outside of the footing area. It is customary, however, to design footings neglecting the basement load and the soil weight, which is an assumption that arching action does occur by the time the full live load is on the floors.

In the case of the stepped footing of Figure 176*a*, the weights of the footing will vary as shown, and the assumption that the net pressure due to the wall load is uniform is reasonable. Such an assumption is probably not correct for the reinforced footing of Figure 176*b*. Nevertheless, the net pressure is commonly taken as an average or uniform pressure. The weight of the footing, soil, and the basement load immediately above a unit area of soil cause pressure on the soil but do not cause shear forces or bending moments in the footing, since the supporting soil is in line with the loads. It is the building loads, or net pressures, which cause bending and shear in the footing sections.

Simple footings for columns shown in Figure 177 tend to take a bowl shape as they deflect. The pressure distribution is complicated, and the analysis involves slope and deflection solutions for a slab supported by the column and subjected to variable pressures. As with flat slabs, these complicated derivations, not yet justified by tests, are not used for commercial designs. The net pressure is again assumed to be uniform. This assumption gives bending moments and shear forces which are somewhat too large, and the design is on the safe side.

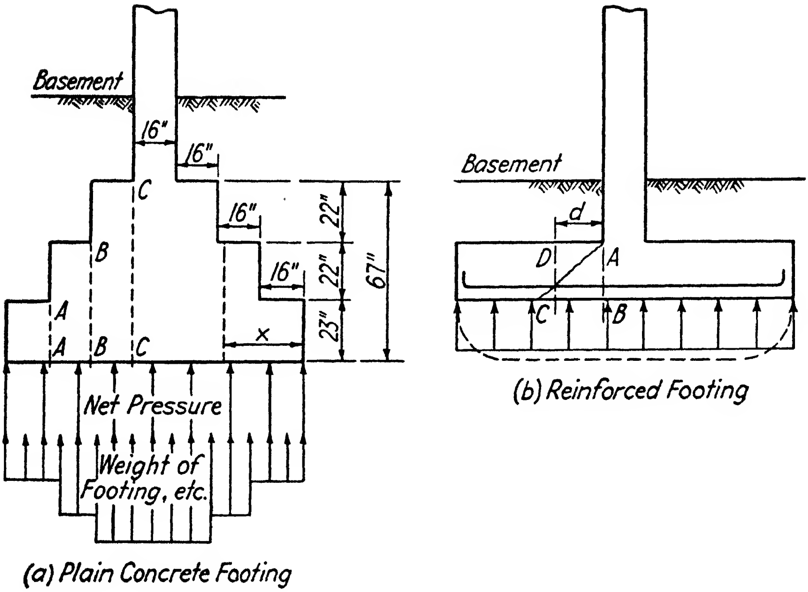


FIG. 176

273. Piles. A footing which is supported by piles, rather than by the soil pressure on its base, is considered to be loaded with a concentrated load at each pile equal to the net load carried by that pile. The footing design follows the methods described below except for this substitution of concentrated for distributed loads.

WALL FOOTINGS

274. Wall Footings. Wall footings are continuous and can be designed as cantilever beams 1 ft. wide, if the wall loads are constant. The footing may be a stepped footing of plain concrete of considerable depth, or the footing may be expanded at once to its full area and reinforced. The type adopted depends on the relative cost and particularly on the difficulty of excavating for the greater depth of the stepped footing.

ILLUSTRATIVE PROBLEM 50

275. Design of a Stepped Footing. Design a plain concrete footing to support a wall 16 in. thick loaded with 50,000 pounds per linear foot. The top of the footing is 2 ft. below ground level. Neglect basement floor load and weight of soil. Allowable soil pressure = 3 tons per square foot.

Use 2000-lb. concrete and assume the net pressure to be uniform. The total load on the soil per foot length of wall equals

$$\begin{aligned} &\text{Wall load} &&= 50{,}000 \text{ lb. per ft.} \\ &\text{Estimated footing weight} &&= \underline{\ \ 6{,}000} \text{ lb. per ft.} \\ &\quad\text{Total} &&= 56{,}000 \text{ lb. per ft.} \end{aligned}$$

$$\text{Area of footing} = \frac{56{,}000}{6000} = 9.33 \text{ sq. ft. per ft. length.}$$

Bending Moment. The footing is 9 ft. 4 in. long and projects 48 in. on each side of the wall, forming cantilever beams.

$$\text{Net pressure } p = \frac{50{,}000}{9.33} = 5360 \text{ lb. per sq. ft.}$$

The bending moment at any section x inches from the end equals (Fig. 176a):

$$M = \frac{px^2}{2 \times 12} = \frac{5360x^2}{24} = 223x^2 \text{ in.-lb.}$$

Each section is rectangular, b inches wide and t inches thick. If the allowable tension f is taken as $f = 0.03f'_c = 60$ lb. per sq. in.,

$$\text{Section modulus} = \frac{bt^2}{6} = \frac{M}{f}$$

$$t^2 = \frac{6M}{bf} = \frac{6 \times 223x^2}{12 \times 60} = 1.86x^2$$

$$t = 1.36x$$

At the section CC of maximum bending moment, $x = 48$ in. and $t = 66$ in.

Shear. The shear force V at any section x feet from the end of the footing equals

$$V = \frac{px}{12} = 447x$$

$$\text{Maximum shear stress } v_c = \frac{VQ}{bI} = \frac{3V}{2bt} = 0.02f'_c$$

$$t = \frac{1.5V}{v_c b} = \frac{1.5 \times 447x}{40 \times 12} = 1.39x$$

At section CC,

$$t = 67 \text{ in.}$$

The shear stress governs, and we adopt a total depth of 67 in.

Steps. The side of each step must have forms, and their heights should be dimensioned to avoid the necessity of ripping planks to give the form height. If we assume three steps for this design with successive "risers" of 23 in., 22 in., and 22 in., the corresponding "treads" determined by fiber stress equal

t in.	x in.	Tread in.
23	16.5	16
45	32.4	16
67	48	16

Costs. In Table D this same footing is designed for richer mixes and the comparative costs computed:

2000-lb. concrete costs 35 cents per cubic foot
2500 " " " 37 " " " "
3000 " " " 39 " " " "
4000 " " " 42 " " " "
Forms cost 10 cents per sq. ft. of riser
Excavation costs 4 cents per cu. ft.

The excavation extends 1 ft. beyond the footing end.

It will be noticed, if the costs are plotted for these data, that the richest mix considered is the cheapest, there being as yet no minimum.

TABLE D. STEPPED FOOTING

Mix, f'_c	2000	2500	3000	4000
Weight, lb.	6000	5000	5000	4000
Length, ft.	9.33	9.17	9.17	9.00
Projection, in.	48	47	47	46
Net pressure, p	5360	5450	5450	5555
By fiber stress, t, in.	$1.36x$	$1.23x$	$1.12x$	$0.982x$
By shear stress, t, in.	$1.39x$	$1.14x$	$0.946x$	$0.723x$
Thickness at wall, t, in.	67	58	53	46
First riser, in.	23	20	18	16
First tread, in.	16	16	16	16
Second riser, in.	22	19	18	15
Second tread, in.	16	15	16	15
Third riser, in.	22	19	17	15
Third tread, in.	16	16	15	15
Cost per Foot				
Concrete	$13.11	$11.80	$11.28	$10.38
Forms	1.12	0.97	0.89	0.77
Excavation	3.44	3.06	2.87	2.57
Total	$17.67	$15.83	$15.04	$13.72

ILLUSTRATIVE PROBLEM 51

276. Design of Reinforced Wall Footing. Given the wall load of Problem 50, design a reinforced concrete footing. Use a 2000-lb. concrete and assume the net pressure uniform. The total load on soil per foot length of wall equals

Wall load	50,000 lb. per ft.
Estimated footing weight	3,000 lb. per ft.
	53,000 lb. per ft.

$$\text{Area of footing} = \frac{53,000}{6000} = 8.83 \text{ sq. ft.}$$

Bending Moment. The footing is 8 ft. 10 in. long and projects 45 in. on each side of the wall as a cantilever beam.

$$\text{Net pressure } p = \frac{50{,}000}{8.83} = 5670 \text{ lb. per sq. ft.}$$

Take a width of 1 ft. of this footing as a unit beam.

$$\text{Maximum moment} = \frac{wl^2}{2} = \frac{5670(45)^2}{12 \times 2} = 479{,}000 \text{ in.-lb.}$$

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{479{,}000}{157 \times 12}} = 16.0 \text{ in.}$$

The steel is at the bottom of the footing. Assume that projecting stones, clay, muck puddles, etc., may spoil 2 in. of concrete. Using in addition 3 in. of dampproofing (A.C.I. Art. 507*a*), the minimum depth equals $h = d + 5 = 21.0$ in.

Diagonal Tension. The portion of the footing immediately under the wall acts as an extension of the wall and is in compression. Diagonal tension cracks will not continue to spread in this portion of the footing. The first crack to extend completely through the footing and, hence, the failure crack is AC of Figure 176*b*. If it is assumed to be at an angle of 45°, diagonal tension will be checked at section CD at a distance d from the face of the wall. The shear is due to the net soil pressure on the extension of the footing beyond this section. The allowable stress $v_c = 0.03f'_c$ (A.C.I. Art. 808), as it is not desirable to use diagonal tension steel.

$$d = \frac{V}{vbj} = \frac{5670(45 - d)}{60 \times 12 \times 0.87 \times 12}$$

Solving, $d = 19.3$ in. The diagonal tension stresses determine the footing depth. Use $h = 25$ in. and $d = 20$ in.

The A.C.I. Regulations do not require a determination of the maximum shear stress on section AB of Figure 176*b*. The allowable stresses and fabrication of the steel permit only special anchorage designs. If the shear stress is figured,

$$v = \frac{V}{bjd} = \frac{5670 \times 45}{12 \times 0.87 \times 20 \times 12} = 102 \text{ lb. per sq. in.}$$

The allowable shear stress (no web reinforcement) equals $v = 0.06f'_c = 120$ lb. per sq. in

Weight. The footing weighs $\frac{25}{12} \times 8.83 \times 150 = 2800$ lb. Estimate of 3000 lb. is safe.

Steel.

$$A_s = \frac{M}{f_sjd} = \frac{479{,}000}{20{,}000 \times 0.87 \times 20} = 1.37 \text{ sq. in.}$$

The bond stress is often high, so the necessary perimeters to keep within the allowable $u = 0.075f'_c$ will be computed.

$$\Sigma o = \frac{V}{ujd} = \frac{21{,}250}{150 \times 0.87 \times 20} = 8.14 \text{ in.}$$

The area and perimeter requirement can be satisfied by using $\frac{7}{8}$-in. round bars at 4-in. spacings. It will be noticed that the depth and steel areas of these cantilever footings with short-span and heavy loads are often determined by the diagonal tension and bond requirements rather than by the fiber stress requirement that usually governs for the beams supported at each end. The application of A.C.I. Article 905 gives a bar length of 10 ft. 6 in., if a 90° hook is used at each end.

If the mix has a strength $f'_c = 2500$ lb. per sq. in., the footing length is 8.75 ft., its depth $h = 22$ in., $d = 17$ in. For steel, 1-in. round bars at 4.5 in. spacing are satisfactory. With the costs of Problem 50 and a steel cost of 4 cents per pound, the 2000-lb. concrete design costs \$11.12 per ft. length and the 2500-lb. concrete \$11.00. There is no advantage in using richer mixes as the diagonal tension stress is limited to 75 lb. per sq. in., which is the maximum for $f'_c = 2500$ lb. per sq. in.

COLUMN FOOTINGS

277. Assumptions for Design Practice. American design practice for the footings for single columns and for wall footings was formerly based on the classical series of tests by Professor Talbot [1] at the University of Illinois. Car springs were used to give distributed loads on the footing base. In addition to the test report Professor Talbot made recommendations for footing design which employed the rectangular beam equations and the usual allowable stresses. These recommendations, based upon tests, were empirical, and the complicated stress analyses for this indeterminate structure were not used for commercial design.

After many years of use engineers concluded that the Talbot method gave footing depths which were greater than needed. The present A.C.I. recommendations are also empirical but somewhat simpler to apply. The depth is usually determined by the diagonal tension requirement, whereas the depth by the Talbot method was usually fixed by the maximum fiber stress. Chapter 12 of the A.C.I. Regulations states that the soil pressure shall be assumed to be uniform or that the force exerted by a pile on the footing may be assumed a concentrated force. Moments and shear forces are determined by passing a plane across the footing and dealing with the forces acting on the footing to one side of this plane.

278. Pedestals. Column footings are sometimes sloped to save concrete. The design should allow a 6-in. shelf adjacent to the column face upon which to place the column forms. The minimum thickness at the edge of the footing is 12 in. The maximum slope, between these two points, should give the necessary thickness for diagonal tension at the section d inches out. A sloped footing requires forms for the slope, and a carpenter must be in attendance during the pouring to fill in the pouring holes left in the sloped forms. The added form costs cancel some of the concrete saving.

[1] Bulletin 67, University of Illinois.

Concrete can also be saved by using a pedestal between the column and footing. This pedestal may be reinforced, forming an additional short column; or it may be merely an enlargement of the footing as are the stepped wall footings. A.C.I. Article 1207 gives recommendations for pedestal design.

ILLUSTRATIVE PROBLEM 52

279. Design of Exterior Column Footing. Design an exterior footing for the exterior column stack of Problem 41 (Chapter 11) with an allowable soil pressure of 3 tons per square foot. The basement column is $b = 21$ in. and $t = 24$ in. and it has six $1\frac{1}{4}$-in. square bars and a concrete strength $f'_c = 6000$ lb. per sq. in. This column has a load $N = 600{,}000$ lb. at its base and a moment $M_1 = 1{,}000{,}000$ in.-lb. at its upper section under the first floor. Assume that the footing fixes the lower end; therefore the moment at the bottom section is $M = \dfrac{M_1}{2} = 500{,}000$ in.-lb. (Art. 372).

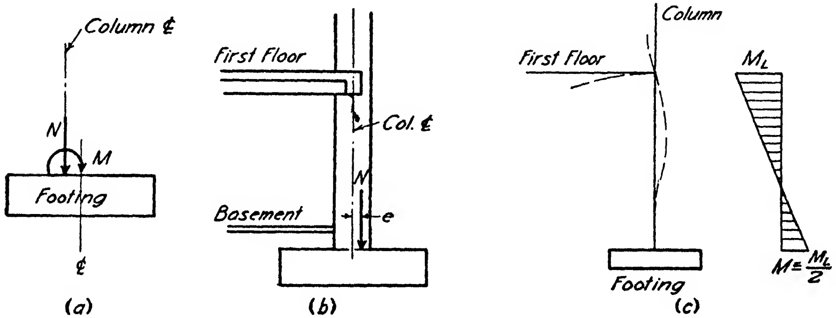


FIG. 177

The resultant of the force N and couple M (Fig. 177a) acts at $e = \dfrac{M}{N} = \dfrac{500{,}000}{600{,}000} =$ 0.83 in. from the column center line (Fig. 177b). If the force N passes through the center of gravity of the footing base, or center line, the assumption of uniform net pressure on the base of the footing is justified. If the footing is centered on the column center line, the net soil pressure varies uniformly with the maximum intensity on the outer edge, and the footing will tend to tip outward.

In this problem the force N is less than 1 in. from the column center line. If the live load on the first floor is reduced and the moment in the column decreases, the force N approaches the column center line. In order to avoid any chance of the footing tipping outward owing to a resultant force N to the right of the footing center, the exterior footing will be centered 1 in. outside the column center (approximation for 0.83 in.), and will be designed for a uniform soil pressure. The footings for interior columns are centered on the column axis, as the most severe loading for interior columns is an axial load.

280. Depth for Diagonal Tension. The shear stress v, used as a measure of diagonal tension, shall not exceed $0.03f'_c$, nor 75 lb. per sq. in. Since the depth is usually determined by this requirement, there is no incentive to use concrete stronger than $f'_c = 2500$ lb. per sq. in. Adopt this strength.

A rectangular footing will be used with the rectangular column. Taylor, Thompson, and Smulski [2] state that the cheapest proportions are given if the rectangular base projects the same distance beyond each column face, because these proportions require the minimum amount of materials. This criterion will be adopted for this design.

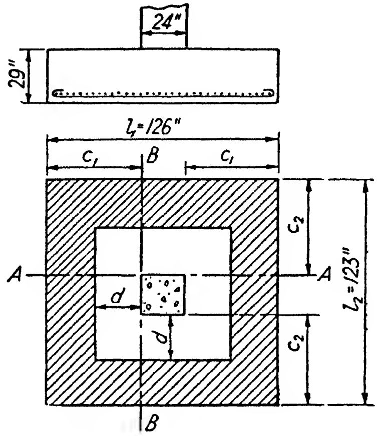


FIG. 178

In Figure 178 let the dimensions $c_1 = c_2 = c$

Column load	600,000 lb.
Weight of footing	39,000 lb.
Total load	639,000 lb.

$$\text{Area of base} = \frac{639{,}000}{6000} = 106.5 \text{ sq. ft.}$$

$$\text{Area} = 106.5 = \frac{(24 + 2c)(21 + 2c)}{144}$$

$$c = 50.7 \text{ in.} \qquad \text{Use } c = 51 \text{ in.}$$

Use $l_1 = 126$ in. and $l_2 = 123$ in. Area = 107.8 sq. ft.

A.C.I. Article 1205*a* states that the critical sections for diagonal tension shall be assumed at a distance of d from the faces of the column. The shear force on these sections is due to the net pressure on the cross-hatched area outside (Fig. 178). Since d is not known, the computation is somewhat intricate.

$$\text{Net pressure } p = \frac{600{,}000}{107.8} = 5570 \text{ lb. per sq. ft.}$$

[2] *Concrete, Plain and Reinforced,* Vol. I, p. 510.

$$v = 75 = \frac{V}{bjd} = \frac{5570\left[107.8 - \frac{(24 + 2d)(21 + 2d)}{144}\right]}{2[(24 + 2d) + (21 + 2d)] \times 0.87d}$$

$$151.6d + 13.48d^2 = 15{,}019 - 90d - 4d^2$$

$$d = 23.2 \text{ in.} \qquad \text{Use } d = 24 \text{ in.}$$

Use $h = d + 5 = 29$ in. Weight of footing = 39,000 lb.

281. Depth by Bending Moment. In Figure 178 the bending moments at sections AA and BB are

$$M_A = \frac{wl^2}{2} = 5570 \times \frac{126 \times (51)^2}{2 \times 144} = 6{,}350{,}000 \text{ in.-lb.}$$

$$M_B = 5570 \times \frac{121 \times (51)^2}{2 \times 144} = 6{,}100{,}000 \text{ in.-lb.}$$

The sections are rectangular; therefore

$$d = \frac{M}{Kb} = \sqrt{\frac{6{,}350{,}000}{196 \times 126}} = \sqrt{\frac{6{,}100{,}000}{196 \times 121}} = 16.1 \text{ in.}$$

This depth is much less than the diagonal tension requirement. Use $d = 24$ in.; $h = 29$ in.

282. Steel. Let us assume $d = 24$ in. as an average for the two levels of steel. Place the steel crossing section AA lowermost with an assumed $d = 24.5$ in.; its area equals

$$A_s = \frac{M}{f_sjd} = \frac{6{,}350{,}000}{20{,}000 \times 0.87 \times 24.5} = 14.9 \text{ sq. in.}$$

For $d = 23.5$ in., the steel crossing section BB equals

$$A_s = \frac{6{,}100{,}000}{20{,}000 \times 0.87 \times 23.5} = 14.9 \text{ sq. in.}$$

Bond is checked at the face of the column. For the steel crossing section AA,

$$\Sigma o = \frac{V}{ujd} = \frac{5570 \times \frac{126 \times 51}{144}}{140 \times 0.87 \times 24.5} = 83.5 \text{ in.}$$

For the steel crossing section BB, $\Sigma o = 83.5$ in.

The bond requirement is more severe. Use thirty-six $\frac{3}{4}$-in. round bars in each direction. This easily satisfies the fiber stress requirement. The steel crossing section BB is spaced uniformly over the 123 in. A.C.I. Article 1204g states that the steel crossing section AA shall be spaced uniformly over a band width of 123 in. This steel amounts to $\frac{2}{\frac{126}{123} + 1} = 0.99$ of the total area. The remainder is spaced uniformly over the two equal end strips of 1.5 in. In other words, our footing is so nearly square that a uniform spacing can be used over the entire width of 126 in. Use a standard semicircular hook at each end and 3 in. of concrete protection at the footing sides; then the long bars will be 11 ft. 8 in. long and the short bars 11 ft. 4 in. long (A.C.I. Art. 905).

283. Dowels. Short bars, or dowels, are placed at the column-footing junction to transfer the column load to the footing. They must have at least the area of the column steel in the column, in this case 9.36 sq. in. (six $1\frac{1}{4}$-in. square bars). A.C.I. Article 1103*c* states that these dowels must extend 32 diameters into the footing for a concrete strength $f'_c = 2500$ lb. per sq. in. In order to preserve dampproofing clearances at the base of the footing, it will be necessary to use $\frac{3}{4}$-in. bars. Use twenty-two $\frac{3}{4}$-in. round bars 48 in. long.

ILLUSTRATIVE PROBLEM 53

284. Design of Footing for Interior Column. Design a footing for the interior column of Problem 47 (Chapter 11). This basement column is of 6000-lb. concrete, 30 in. in diameter, and has an axial load of 1,112,000 lb. at its base. Use 2500-lb. concrete for the footing and an allowable soil pressure of 3 tons per square foot.

Use a square footing weighing 100,000 lb. The column section is assumed to be 27 in. square, of the same area as the 30-in. circle. By the methods of Problem 52 the following results are obtained,

Minimum area of footing	= 202 sq. ft.	Use 14 ft. 3 in. square
Depth for diagonal tension, d	= 33 in. and	$h = 38$ in.
Depth for bending, d	= 23 in.	Use $d = 33$ in.
Actual weight	= 96,500 lb.	
A_s	= 29.5 sq. in. and	$\Sigma o = 116.8$ in.

Use thirty 1-in. square bars in each direction, 16 ft. long, hooked.

285. Footing with Pedestal. The footing depths are quite shallow compared with those of former methods of design, so the saving of concrete by the use of pedestals is not so great as formerly. Let us investigate the possibility of using an unreinforced pedestal for this design.

A.C.I. Article 1207*a* states that the compressive stress on the gross area of a pedestal shall not exceed $f_c = 0.25f'_c = 625$ lb. per sq. in.

$$\text{Pedestal area} = \frac{1{,}112{,}000}{625} = 1780 \text{ sq. in.}$$

Use a pedestal 43 in. square, projecting 8 in. beyond the equivalent square column.

Depth for diagonal tension at section D, $d = 33$ in. (Fig. 179)

Depth for diagonal tension at section A, $d = 29$ in.

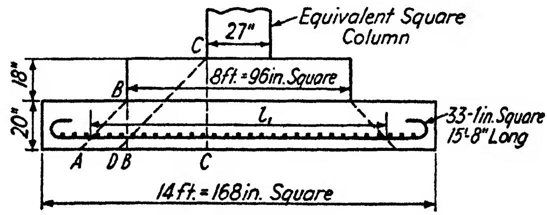


FIG. 179

In this case the saving of 4 in. of concrete over an area of 190 sq. ft. is probably not balanced in cost by the necessity of extra forms for the pedestal. This pedestal is unreinforced and it is not desirable here to use a smaller one reinforced as a column.

Let us also consider the possibility of using a larger pedestal to form the equivalent of a stepped footing. The footing depth at the edge of the column will have the values $d = 33$ in. and $h = 38$ in. computed above. Let us arbitrarily assume steps of 20 in. and 18 in., and estimate the weight as 64,000 lb. (Fig. 179).

$$\text{Area of base} = \frac{1{,}176{,}000}{6000} = 196 \text{ sq. ft.} \qquad \text{Use 14-ft. square.}$$

$$\text{Net pressure} = \frac{1{,}112{,}000}{196} = 5680 \text{ lb. per sq. ft.}$$

Depth for diagonal tension at critical section A, with $d = 15$ in. and l_1 as shown on Figure 179.

$$v = \frac{V}{bjd} = 75 = \frac{5680\left[196 - \dfrac{l_1^2}{144}\right]}{4l_1 \times 0.87 \times 15}$$

$l_1 = 126$ in.; therefore width of upper step $= 126 - 2 \times d = 96$ in.

$$\text{Weight of footing} = 63{,}500 \text{ lb.}$$

At section BB,

$$M = \frac{5680 \times 14 \times (36)^2}{2 \times 12} = 4{,}280{,}000 \text{ in.-lb.}$$

$$A_s = \frac{4{,}280{,}000}{20{,}000 \times 0.87 \times 15} = 16.4 \text{ sq. in.}$$

$$\Sigma o = \frac{V}{ujd} = \frac{5680 \times 14 \times 3}{140 \times 0.87 \times 15} = 130.6 \text{ in.}$$

At section CC,

$$M = \frac{5680 \times 14 \times (70.5)^2}{2 \times 12} = 16{,}500{,}000 \text{ in.-lb.}$$

$$A_s = \frac{16{,}500{,}000}{20{,}000 \times 0.87 \times 33} = 28.7 \text{ sq. in.}$$

$$\Sigma o = \frac{5680 \times 14 \times 70.5}{140 \times 0.87 \times 33 \times 12} = 116.4 \text{ in.}$$

To satisfy the area and perimeter requirements use thirty-three 1-in. square bars, spaced approximately 6 in. on centers. These bars will be hooked; they are 15 ft. 8 in. long.

A comparison of these results with constant-depth footing of Article 284 gives

	Constant-Depth	Stepped Footing
Area of footing, sq. ft.	203	196
Weight, lb.	96,500	63,500
Total depth, in.	38	38
Volume of concrete, cu. ft.	643	423
Weight of steel, lb.	3,270	3,520

A cost comparison for the stepped footing will compare the saving in concrete with the increase in steel and form costs. The necessary dowels must be placed in the top of the footing to lap up into the *circular* column.

ILLUSTRATIVE PROBLEM 54

286. Footing with Piles. Design the interior footing of Problem 53 if the load is supported on piles. Assume the piles to be 12 in. in diameter at the top and spaced 3 ft. apart, each pile carrying safely a load of 20 tons. Use $f'_c = 2500$ lb. per sq. in.

$$\begin{aligned}
&\text{Column load} && = 1{,}112{,}000 \text{ lb.}\\
&\text{Estimated footing weight} && = \underline{168{,}000} \text{ lb.}\\
&\quad\text{Total on piles} && = 1{,}280{,}000 \text{ lb.}\\
&\text{Number of piles} && = \frac{1{,}280{,}000}{40{,}000} = 32\\
&\quad\text{Net pile load} && = \frac{1{,}112{,}000}{32} = 34{,}750 \text{ lb.}
\end{aligned}$$

The piles will be arranged as shown in Figure 180.

Diagonal Tension. Assume the critical section to come somewhere near perimeter *ABCD*

$$v = \frac{V}{bjd} = \frac{34{,}750 \times 28}{4(27 + 2d) \times 0.87 \times d} = 75$$

$$2d^2 + 27d = 3728 \qquad \text{and} \qquad d = 36.9 \text{ in.}$$

Use $d = 37$ in. and $h = 37 + 3 + 6 = 46$ in.

Fiber Stress. Taking moments about section *EE*,

$$M_E = 34{,}750(4 \times 76.5 + 6 \times 40.5 + 6 \times 4.5) = 20{,}000{,}000 \text{ in.-lb.}$$

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{20{,}000{,}000}{196 \times 18 \times 12}} = 21.7 \text{ in.}$$

Use $d = 37$ in. and $h = 46$ in.

Actual Weight. Let us use the footing base shown in Figure 180, assuming that the concrete saved justifies the extra form cost of cutting off the corners.

$$\text{Weight} = 150\left[\frac{46}{12}\left(324 - \frac{4 \times (3.67)^2}{2}\right) - \frac{\pi}{4} \times \frac{1}{2} \times 32\right] = 169{,}000 \text{ lb.}$$

This is almost exactly the weight assumed.

Steel.

$$A_s = \frac{20{,}000{,}000}{20{,}000 \times 0.87 \times 37} = 31.1 \text{ sq. in.}$$

Bond will be checked at section *EE*.

$$\Sigma o = \frac{V}{ujd} = \frac{34{,}750 \times 16}{140 \times 0.87 \times 37} = 123 \text{ in.}$$

Use thirty-two 1-in. square bars in each direction. Most of these bars will be 19 ft. 8 in. long, but those in the outer $3\frac{1}{2}$ feet will be made varying lengths.

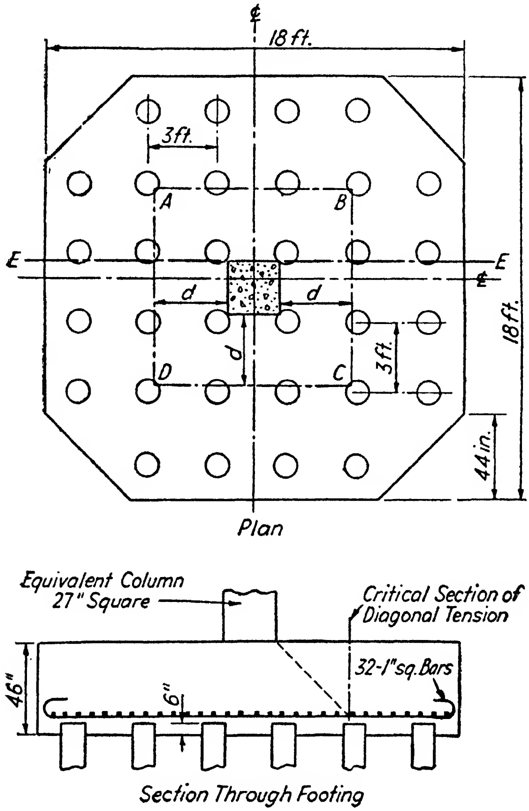


FIG. 180

COMBINED FOOTINGS

287. Combined Footings. In congested districts a building is frequently carried to the property line with the exterior columns at or near the line. It is not possible to center a simple footing on the exterior column axis. A combined footing carrying the exterior column and an interior column is often substituted. If the building fronts on two streets, the corner column is often carried on a combined footing with two exterior columns and at least one interior column. In certain cases the allowable soil pressures may be low enough so that the footings overlap and a continuous slab results. The ultimate expansion is a raft foundation which is an inverted floor system. Footings carrying more than two columns are statically indeterminate.

288. Two-Column Footings. The footing carrying one exterior and one interior column should be designed to meet the following requirements:

1. An area great enough to give the safe allowable soil pressure.
2. A shape such that a uniform net soil pressure is given.

In other words, the resultant of the loads must act through the center of gravity of the base. If the footing is of constant depth, the footing weight acts at the center of gravity of the base whatever shape is adopted. The resultant of the column loads must act there also.

Rectangular Base. If the only restriction is the projection of the footing beyond the exterior column, the footing base can be rectangular.

Trapezoidal Base. If the rectangular base overlaps the opposite footing, or if there is a restriction to the projection beyond the interior column, a footing base trapezoidal in shape will be necessary to fulfill the two requirements listed above.

Connected Footings. Occasionally it is advantageous to use separate footings connected by a beam. The eccentric loading of the exterior footing is balanced by a pressure of the interior column on the connecting beam.

If this beam, or strap, rests on undisturbed soil at the level of the footing bottoms, it may be considered part of the combined footing area. If reliance is placed on piles, or caissons, to carry the column loads, because of low allowable soil pressures, the strap may be regarded as a cantilever beam loaded at its end by the interior column. In such a case the strap often tapers in depth, and effort is made to have only loose fill under it.

ILLUSTRATIVE PROBLEM 55

289. Design of Rectangular Combined Footing. Design a combined footing to carry the exterior column stack of Problem 52 and the interior column stack of Problem 53. These columns are 29 ft. apart on centers. Assume that the outside of the footing cannot project more than 18 in. beyond the outer face of the exterior column. The allowable soil pressure is 3 tons per square foot. Use 2500-lb. concrete.

290. Dimension of Base.

$$\begin{aligned} \text{Exterior column load} &= 600{,}000 \text{ lb.} \\ \text{Interior column load} &= 1{,}112{,}000 \text{ lb.} \\ \text{Estimated footing weight} &= \underline{200{,}000} \text{ lb.} \\ & 1{,}912{,}000 \text{ lb.} \end{aligned}$$

$$\text{Area of base} = \frac{1{,}912{,}000}{6000} = 319 \text{ sq. ft.}$$

$$\text{Resultant of column loads} = 1{,}712{,}000 \text{ lb.}$$

Moment of column loads about center of exterior column:

$$M = 1{,}112{,}000 \times 29 = 1{,}712{,}000 x_r$$

$$x_r = 18.85 \text{ ft.}$$

The resultant acts $18.85 + 2.5 = 21.35$ ft. from the outer edge of the footing. This point is the center of gravity of the rectangular base.

$$\text{Length of footing} = 21.35 \times 2 = 42.7 \text{ ft.} \qquad \text{Use 42 ft. 9 in.}$$

$$\text{Width of footing} = \frac{319}{42.75} = 7.46 \text{ ft.} \qquad \text{Use 7 ft. 6 in.}$$

291. Depth by Bending Moment. Bending and shear will be caused in the footing by the net soil pressure and the column loads. The design of the longitudinal beam

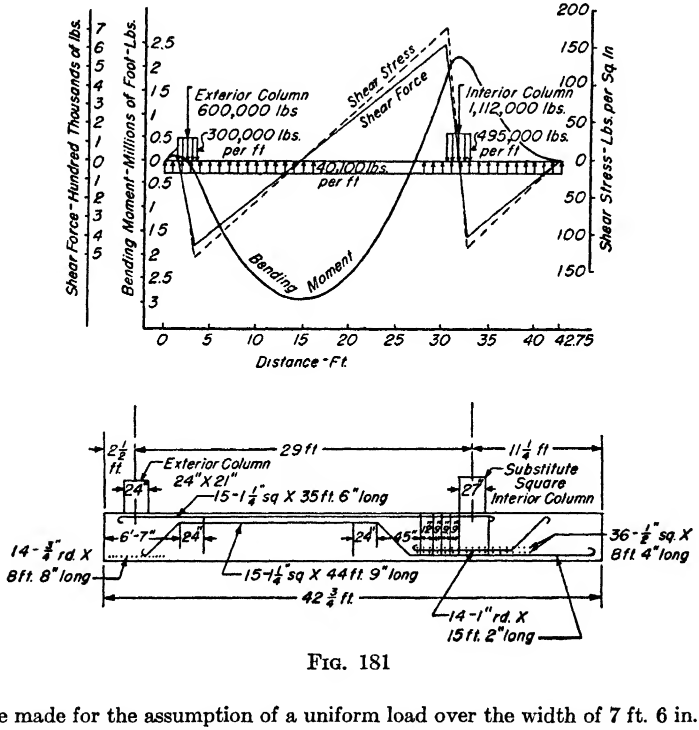


FIG. 181

will be made for the assumption of a uniform load over the width of 7 ft. 6 in. The column loads must later be spread transversely to justify this assumption.

$$\text{Net pressure } p = \frac{1{,}712{,}000}{320.6} = 5340 \text{ lb. per sq. ft.}$$

$$\text{Longitudinal beam load } w = 5340 \times 7.5 = 40{,}100 \text{ lb. per ft. length}$$

The loading, shear force, and bending moment diagrams for the longitudinal beam are given in Figure 181. The assumption is made that the column loads are uniformly distributed over the column width. The essential shear force and bending moment values for the longitudinal beam are listed below.

Distance from Exterior End ft.	Shear Force kips (1000 lb.)	Bending Moment ft.-kips
1.5	60.2	45.2
1.73	0	52.1
2.37	......	0
3.5	−460.0	−354.0
6.56	......	−1577.0
14.97	0	−2982.0
23.39	......	−1577.0
27.21	490.0	0
30.38	618.0	1775.0
31.54	0	2240.0
32.63	−403.7	2020.0
33.9	−355.0	1577.0
35.0	......	1200.0

Solving for the depth of the rectangular section to satisfy the maximum numerical moment,

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{2{,}982{,}000 \times 12}{196 \times 7.5 \times 12}} = 45 \text{ in.}$$

Use depth $h = d + 5 = 50$ in.

292. Shear. The maximum numerical shear force $V = 618{,}000$ lb.

$$v = \frac{V}{bjd} = \frac{618{,}000}{90 \times 0.87 \times 45} = 175 \text{ lb. per sq. in.}$$

This exceeds the allowable stress $v = 0.06f'_c = 150$ lb. per sq. in. for ordinary anchorage and is less than the allowable $v = 0.12f'_c = 300$ lb. per sq. in. for special anchorage. Use special anchorage.

The concrete can withstand a "shear equivalent of diagonal tension" $v = 0.03f'_c = 75$ lb. per sq. in. The combined footing is not a cantilever between columns, and diagonal tension cracks may occur close to a column. The critical section will be taken at the column face. The maximum shear equivalent $v = 175$ lb. per sq. in. necessitates the use of web reinforcement.

Use a total footing depth $h = 50$ in. The weight of the footing equals

$$W = \tfrac{50}{12} \times 320.6 \times 150 = 200{,}000 \text{ lb., as estimated.}$$

293. Steel. *Negative Steel.* This steel is placed in the top of the footing between the columns.

$$A_s = \frac{M}{f_s jd} = \frac{2{,}982{,}000 \times 12}{20{,}000 \times 0.87 \times 45} = 45.7 \text{ sq. in.}$$

This steel is spread over a width of 7 ft. 6 in. The concrete must be poured through it; therefore space it about 6 in. on centers. These bars will be in two rows.

Bond is checked at the sections of maximum rate of change of negative bending moment $\left(\frac{dM}{dx} = V\right)$. At the exterior end this occurs at the inside face of the column at 3.5 ft.; in the interior direction it occurs at the point of inflection, 27.2 ft. The perimeters needed at these two sections to satisfy an allowable bond stress $u = 0.075f'_c = 187.5$ lb. per sq. in. are

$$(\Sigma o)_{\text{ext.}} = \frac{V}{ujd} = \frac{460{,}000}{187.5 \times 0.87 \times 45} = 62.7 \text{ in.}$$

$$(\Sigma o)_{\text{int.}} = \frac{490{,}000}{187.5 \times 0.87 \times 45} = 66.7 \text{ in.}$$

Use thirty $1\frac{1}{4}$-in. square bars in two rows with $A_s = 46.8$ sq. in. with a spacing of 6 in. Bond is satisfied by 14 of these bars, so the lower row may be bent down according to fiber stress requirements. If the upper row of 15 bars remains, the moment that may be safely carried is

$$M = 15 \times 1.56 \times 20{,}000 \times 0.87 \times 46.5 \times \tfrac{1}{12} = 1{,}577{,}000 \text{ ft.-lb.}$$

From the bending moment diagram, the lower row may be bent down at sections 6.56 ft. and 23.39 ft. from the exterior end. The upper row will run by the points of inflection and be carried under the columns.

Positive Steel under interior column.

$$A_s = \frac{2{,}240{,}000 \times 12}{20{,}000 \times 0.87 \times 45} = 34.3 \text{ sq. in.}$$

Bond will be checked at the section having the greatest rate of change of bending moment ($V = 618{,}000$ lb.). This is two-way steel, as the transverse steel is also in the bottom, and the allowable $u = 0.056f'_c = 140$ lb. per sq. in.

$$\Sigma o = \frac{618{,}000}{140 \times 0.87 \times 45} = 112.5 \text{ in.}$$

Part of this steel can be supplied by the fifteen $1\frac{1}{4}$-in. square bars bent down from the lower row of the negative steel. Add fourteen 1-in. round bars as an upper row to satisfy both area and perimeter requirements. The fifteen $1\frac{1}{4}$-in. square bars can carry a moment of 1,577,000 ft.-lb. safely, so the 1-in. bars can be bent up at 33.9 ft., the bond stresses being also safe.

Exterior Column. The maximum bending moment is only 52,100 ft.-lb., so the concrete can undoubtedly carry the tensile stresses. Assume a depth of plain concrete $h = 48$ in.; the maximum tensile stress in the concrete will equal

$$f_t = \frac{My}{I} = \frac{52{,}100 \times 12 \times 6}{90 \times (48)^2} = 18 \text{ lb. per sq. in.}$$

The allowable tension $f_t = 0.03f'_c = 75$ lb. per sq. in. No tension steel is needed.

294. Diagonal Tension. The shear stress diagram is plotted in Figure 181. Above the value $v = 0.03f'_c = 75$ lb. per sq. in., diagonal tension steel must be provided. Since the column does not extend completely across the width of the footing, it will be assumed that diagonal tension cracks may start at the section at the face of the column. Diagonal tension steel will be provided by bent bars, where available, and by stirrups. At each section seven 2-rod stirrups of $\frac{5}{8}$-in. round bars will be hung on 14 of the 15 bars in the upper row of negative steel (Fig. 182).

$$\begin{aligned} \text{Maximum spacing } s &= \tfrac{48}{4} = 12 \text{ in., where } v > 150 \text{ lb. per sq. in.} \\ s &= 24 \text{ in. where } v < 150 \text{ lb. per sq. in.} \\ \text{Bent bars} \quad s &= 24 \text{ in., where } v > 150 \text{ lb. per sq. in.} \\ s &= 48 \text{ in., where } v < 150 \text{ lb. per sq. in.} \end{aligned}$$

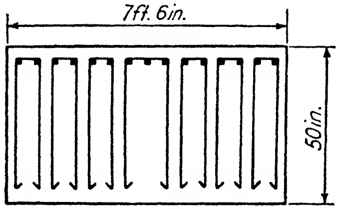


FIG. 182

The shear stress varies from $v = 130$ lb. per sq. in. to zero for sections from 3.5 ft. to 14.97 ft. It varies from 0 to 175 lb. per sq. in. from 14.97 ft. to 30.38 ft. Beyond the interior column the shear stress varies from 114 lb. per sq. in. to zero from 32.63 ft. to 42.75 ft. The shear stress exceeds 75 lb. per sq. in. from 3.5 to 8.36 ft.; 21.56 ft. to 30.38 ft.; and from 32.63 ft. to 36.15 ft. Use

$$v = v_c + v_s = 75 + \frac{14 \times 0.307 \times 20{,}000}{90 \times s \times 1} = 75 + \frac{955}{s}$$

The stirrup spacings can now be figured. The bent bars consist of fifteen $1\frac{1}{4}$-in. square bars, or, beyond the interior column, of fourteen 1-in. round bars. If either set is used at the maximum spacing of 48 in., the maximum shear stress each group can withstand is

$$v = 75 + \frac{15 \times 1.56 \times 20{,}000}{90 \times 48 \times 0.707} = 228 \text{ lb. per sq. in. for the } 1\tfrac{1}{4}\text{-in. bars}$$

$$v = 147 \text{ lb. per sq. in. for the 1-in. round bars.}$$

In other words, these bent bars can be used in any part of the span at a spacing of 48 in., *provided* the shear stress is less than 150 lb. per sq. in.

The cracks tend to appear at the top of the footing between the columns. Fifteen $1\frac{1}{4}$-in. bars can be bent down at 6.56 ft., or 37 in. from the inner face of the exterior column. No stirrups are needed in this length. Diagonal tension reinforcement must be used for 8.36 − 3.5 = 4.86 ft. = 59 in. Beyond 6.56 ft. one set of stirrups at 24 in. will suffice.

These fifteen $1\frac{1}{4}$-in. bars can also be bent down at 23.39 ft., which is 84 in. from the exterior face of the interior column. The shear stress reduces to 150 lb. per sq. in. at 27 in. from this face. For this distance use 3 stirrup sets at 9-in. spacing, then 1 spacing at 12-in. (39 in. out). We are now 45 in. from the bent bars and they can reinforce this length. From 21.56 ft. to 23.39 ft. (22 in.) use 1 stirrup set at 24 in.

Beyond the interior column the cracks will occur at the bottom of the footing. The fourteen 1-in. round bars can be bent up at 33.9 ft., or 15 in. from the column. Use 1 stirrup set close to the column face, then bend these bars up 15 in. out. They care for all cracks for 48 in., or 63 in. out. The steel arrangement is shown in Figure 181.

295. Transverse Beams. The column loads reach the footing on the narrow column area. The design of the longitudinal beam has assumed that these column loads are spread the full width of the footing. The design should justify this assumption. In some cases this is done by providing transverse steel the whole length of the footing. It seems desirable, however, to spread the load laterally at once. The

1940 Joint Committee Specifications require that the transverse steel shall "be placed uniformly within a band having a width not greater than the width of the column plus twice the effective depth of the footing."

This lateral transference of load might be accomplished by resting the column on an I-beam grillage. Such beams would be designed for a uniformly distributed (average) load, whose value would be the column load divided by the area of bearing between the I beams and the footing. In the same way a transverse concrete beam might be placed on the footing instead of the I beams. The transverse beam loading is not so evident if the concrete beam is placed within the footing. It is designed, however, as though it rested on and loaded the longitudinal beam. Now that the beam is within the footing it is difficult to justify its existence as an independent beam with no shear forces or bending couples on its sides. The design as an independent beam is on the safe side and should provide adequate strength and stiffness for the lateral transfer of the column loads.

296. Transverse Beam. *Exterior Column.* In order to place this beam symmetrically under the column it will be made 5 ft. wide. The net load w per foot of transverse length equals

$$w = \frac{600{,}000}{7.5} = 80{,}000 \text{ lb. per ft.}$$

The transverse beam comprises two cantilevers, each projecting 34.5 in. beyond the column faces. The maximum moment occurs at the column face and equals

$$M = \frac{wl^2}{2} = \frac{80{,}000 \times (34.5)^2}{2 \times 12} = 3{,}970{,}000 \text{ in.-lb.}$$

$$A_s = \frac{3{,}970{,}000}{20{,}000 \times 0.87 \times 45} = 5.07 \text{ sq. in.}$$

$$\Sigma o = \frac{V}{ujd} = \frac{80{,}000 \times 34.5}{187.5 \times 0.87 \times 45 \times 12} = 31.4 \text{ in.}$$

There is no longitudinal steel on the bottom of the footing at the exterior column, so this transverse steel is one-way steel with the allowable $u = 0.075f'_c$. Fourteen $\frac{3}{4}$-in. round bars in one row satisfy both fiber stress and bond and give a spacing about 4.5 in. on centers for the 60-in. width.

Interior Column.

$$w = \frac{1{,}112{,}000}{7.5} = 148{,}500 \text{ lb. per ft.}$$

$$M = \frac{148{,}500 \times (31.5)^2}{2 \times 12} = 6{,}140{,}000 \text{ in.-lb.}$$

$$A_s = \frac{6{,}140{,}000}{20{,}000 \times 0.87 \times 45} = 7.84 \text{ sq. in.}$$

$$\Sigma o = \frac{148{,}500 \times 31.5}{140 \times 0.87 \times 45 \times 12} = 71 \text{ in.}$$

This steel is two-way steel with an allowable $u = 140$ lb. per sq. in. The perimeter requirement is high compared to the area, and thirty-six $\frac{1}{2}$-in. square bars in two rows will be used which have a spacing of 6.5 in. on centers in the width of $2 \times 45 + 27 = 117$ in. Figure 181 shows this steel which is also hooked at its ends.

ILLUSTRATIVE PROBLEM 56

297. Design of Trapezoidal Footing. In Problem 55 let us assume in addition that the projection beyond the center line of the interior column cannot be more than 7 ft. Let the total length of the footing l = 38 ft. 6 in. Assuming again a weight of the footing of 200,000 lb., the area of the base equals 319 sq. ft. and its center of gravity again must be 21.35 ft. from the exterior edge.

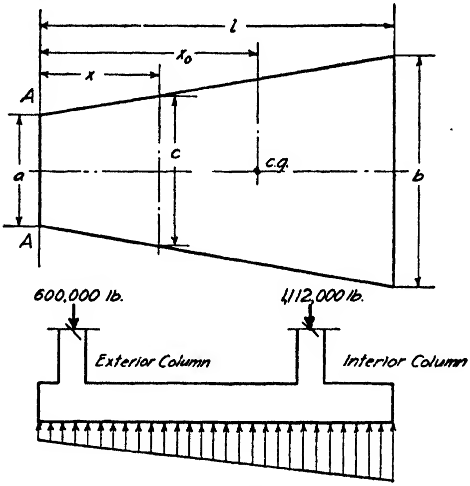


FIG. 183

By Figure 183,

$$\text{Area} = 319 = \left(\frac{a+b}{2}\right) l = \left(\frac{a+b}{2}\right) 38.5 \tag{240}$$

$$\text{Center of gravity } x_o = \frac{(al)\frac{l}{2} + \left[(b-a)\frac{l}{2}\right]\frac{2}{3}l}{al + (b-a)\frac{l}{2}}$$

$$x_o = \frac{\frac{l^2}{6}(a+2b)}{\frac{l}{2}(a+b)} = \frac{l\,(a+2b)}{3\,(a+b)} = 21.35 \tag{241}$$

From equation 240, $a + b = \dfrac{638}{38.5} = 16.58.$

Substituting in equation 241,

$$b = \frac{21.35 \times 16.58 \times 3}{38.5} - 16.58 = 11.0$$

$$a = 16.58 - 11.0 = 5.58$$

Use a = 5 ft. 7 in. and b = 11 ft.

The width c at a distance x from the small end equals

$$c = a + \frac{b - a}{l} x = 5.58 + 0.141x$$

$$\text{Net pressure } p = \frac{1,712,000}{319} = 5370 \text{ lb. per sq. ft.}$$

The net pressure *per foot length* is uniformly varying and equals

$$pc = 5370(5.58 + 0.141x) \text{ lb. per ft.}$$

The shear force and bending moment diagrams can be plotted as in the previous problem and the same procedure used for design. The longitudinal steel will be spread fanwise over the trapezoid and alternate bars will be cut off as allowed by moment or bond as the small end is approached. The diagonal tension stirrups will require a separate detail for each set as the supporting bars vary in spacing. The span of the transverse beams is approximated by using the center-line dimension.

ILLUSTRATIVE PROBLEM 57

298. Design of a Connected Footing. Design a connected footing to support the exterior and interior columns of Problem 55, using a 2500-lb. concrete and an allowable soil pressure of 3 tons per square foot. Assume that the strap beam bears on undisturbed soil.

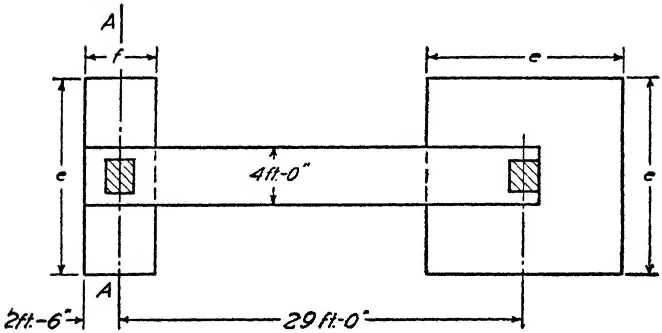


FIG. 184

Follow the procedure of Problem 55; the necessary area is 319 sq. ft. and its center of gravity should be 21.35 ft. from the outer edge of the footing. Assume the connecting beam to be 4 ft. wide, the interior footing to be e feet square and the exterior footing to be the same width e (Fig. 184).

$$\text{Area} = 319 = e^2 + ef + 4(31.5 - f - 0.5e)$$

$$e^2 + ef - 2e - 4f = 193$$

Moments about AA give

$$18.85 \times 319 = 29e^2 + ef\left(\frac{f}{2} - 2.5\right) + 4(31.5 - f - 0.5e)\left(f - 2.5 + \frac{31.5 - f - 0.5e}{2}\right)$$

$$4344 = 29.5e^2 + 0.5ef^2 - 2.5ef + 10f - 58e - 2f^2$$

Solving,

$$e = 13.156 \text{ ft.} \qquad \text{Use 13 ft. 2 in.}$$

$$f = 5.06 \text{ ft.} \qquad \text{Use 5 ft. 1 in.}$$

It is possible, of course, to obtain other dimensions by making the width of the exterior footing different from that of the interior.

$$\text{Net pressure } p = \frac{1,712,000}{319} = 5370 \text{ lb. per sq. ft.}$$

299. Design of the Connecting Beam. If the exterior footing and beam are taken as a unit, the loads are those shown in Figure 185 to the edge of the interior footing

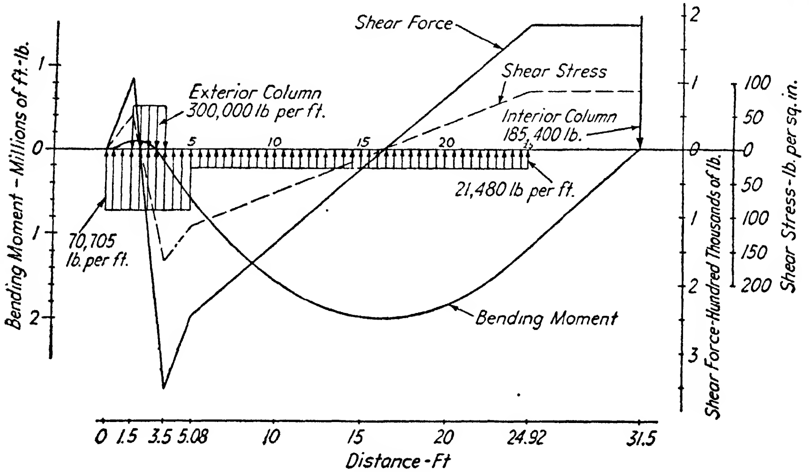


FIG. 185

(24.92 ft. from outside of exterior footing). This system is not in equilibrium, and the tendency of the exterior column to tip the footing-beam unit is balanced by the 185,400-lb. load applied by the interior column. This 185,400 lb. is the difference between the column load and the net upward pressure of the interior footing. The beam is assumed to penetrate the interior footing until under the interior column. The loads, shear force, and bending moment diagrams are plotted in Figure 185. Certain essential values are tabulated below.

Distance from Exterior End ft.	Shear Force kips (1000 lb.)	Bending Moment ft.-kips
1.5	106.06	79.5
1.96	0	104.0
3.5	−352.53	−166.7
5.08	−240.6	−636.2
16.28	0	−1982.6
24.92	185.4	−1181.2
31.5	185.4	0

Depth by Bending Moment.

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{1{,}982{,}600 \times 12}{196 \times 48}} = 50.3 \text{ in.}$$

Use $d = 51$ in. and $h = 51 + 5 = 56$ in.

Shear. Between footings, the beam is rectangular and the maximum shear force $V = 240{,}600$ lb. Within the exterior footing-beam combination the cross section will probably be shaped as in Figure 186. In this case the maximum shear stress

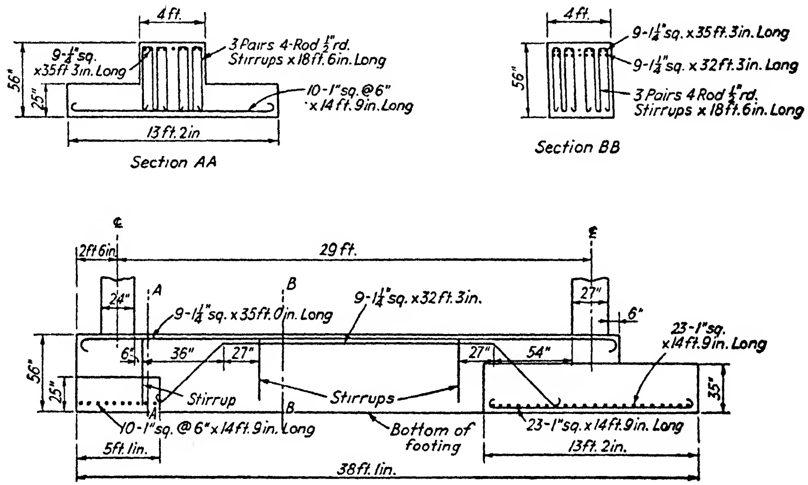


FIG. 186

will occur between the tension steel and the neutral axis (Art. 51). If the neutral axis lies in the projecting beam, the width b is 48 in., as in the connecting beam. Maximum shear occurs at the interior face of the exterior column (3.5 ft. in.).

$$\text{Maximum stress } v = \frac{V}{bjd} = \frac{352{,}530}{48 \times 0.87 \times 51} = 166 \text{ lb. per sq. in.}$$

Use special anchorage. Allowable stress $v = 0.12f'_c = 300$ lb. per sq. in. Safe.

Steel.

$$A_s = \frac{M}{f_s jd} = \frac{1{,}982{,}600 \times 12}{20{,}000 \times 0.87 \times 51} = 26.8 \text{ sq. in.}$$

For bond,

$$\Sigma o = \frac{V}{ujd} = \frac{352{,}530}{187.5 \times 0.87 \times 51} = 42.4 \text{ in.}$$

Use eighteen $1\frac{1}{4}$-in. square bars in two rows. The top row will be run from the outer end of the exterior footing through the interior column. The second row may be bent down when bending moment and bond requirements permit. The top row can carry a moment of $M = 20{,}000 \times 9 \times 1.56 \times 0.87 \times 52 = 12{,}680{,}000$ in.-lb. =

1,057,000 ft.-lb. This occurs at 7 ft. and 25.7 ft. from the exterior end. The top row is safe for bond wherever the shear force is less than

$$V = u(\Sigma o)jd = 187.5 \times 9 \times 5 \times 0.87 \times 52 = 382{,}000 \text{ lb.}$$

These 9 bars fulfill bond requirements anywhere in the span.

At the exterior column there is a positive bending moment of 104,000 ft.-lb. Assuming a plain concrete section 53 in. deep the concrete tensile stress equals

$$= \frac{My}{I} = \frac{104{,}000 \times 12 \times 6}{48 \times (53)^2} = 56 \text{ lb. per sq. in.}$$

Allowable tension equals $0.03f'_c = 75$ lb. per sq. in. No tensile steel is needed.

Diagonal Tension. From the shear stress diagram of Figure 185 the following data are listed:

Distance from Exterior End ft.	Shear Stress v	Distance from Exterior End ft.	Shear Stress v
1.5	50	8.83	75
3.5	166	23.72	75
4.0	150	24.92	87
5.08	113	31.5	87

Diagonal tension reinforcement will be needed from 3.5 ft. to 8.83 ft., and from 23.72 ft. to 31.5 ft.

Maximum spacing of stirrups = 27 in. $(v < 150)$

Maximum spacing of stirrups = 13.5 in. between 3.5 ft. and 4.0 ft.

Maximum spacing of bent bars = 54 in. $(v < 150)$

The bent bars can be used at their maximum spacing wherever the shear stress is less than

$$v = v_c + \frac{a_s f_s}{bs(\sin \alpha)} = 75 + \frac{9 \times 1.56 \times 20{,}000}{48 \times 54 \times 0.707} = 225 \text{ lb. per sq. in.}$$

However, when the stress v is greater than 150 lb. per sq. in., these bars can only be spaced at 27 in. The diagonal tension cracks will tend to occur at the top of the beam. The lower row of $1\frac{1}{4}$-in. bars can be bent down at 7 ft. This section is 36 in. from the section where $v = 150$ lb. per sq. in. From 3.5 ft. to 4 ft. stirrups must be used; also from 7 ft. to 8.83 ft. Try two four-rod $\frac{1}{2}$-in. round stirrups hung upon 8 of the 9 bars in the upper row. Their closest spacing will be

$$166 = 75 + \frac{8 \times 0.196 \times 20{,}000}{48 \times s \times 1}$$

or

$$s = 7.2 \text{ in.}$$

Possible spacings are tabulated below.

Spacing s in.	Shear Stress v lb. per sq. in.	Distance from Interior Face of Exterior Column in.
6	...	..
9	147	7
27	99	36
..	75	64

Use one spacing of 6 in., then the bent bars care for 36 in. (42 in. beyond column), then one spacing of 27 in.

Between 23.72 ft. and 31.5 ft. stirrups and bent bars will be used as diagonal tension steel. The bars can be bent down at 25.7 ft. and they prevent cracks for 54 in. (30.2 ft.) The exterior face of the interior column is located at 30.38 ft., so bend down the bars at 25.92 ft., which is 54 in. away, and use no stirrups. From 25.92 ft. to 23.72 ft. use one 27-in. stirrup spacing to pass the $v = 75$ section. The steel arrangement is shown in Figure 186.

300. Design of Exterior Footing. This footing will be a cantilever extending out 55 in. from each side of the beam. For a strip 1 ft. wide the critical section for diagonal tension will be d in. from the face of the beam.

$$d = \frac{V}{vbj} = \frac{5370 \times (55 - d)}{75 \times 12 \times 0.87 \times 12} \qquad \text{Solving, } d = 20 \text{ in.}$$

The maximum bending moment at the face of the beam equals

$$M = \frac{wl^2}{2} = \frac{5370 \times (55)^2}{2 \times 12} = 677{,}000 \text{ in.-lb.}$$

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{677{,}000}{196 \times 12}} = 17 \text{ in.}$$

Use $d = 20$ in. and $h = 20 + 5 = 25$ in.

Steel.

$$A_s = \frac{677{,}000}{20{,}000 \times 0.87 \times 20} = 1.95 \text{ in.}$$

$$\Sigma o = \frac{5370 \times 55}{187.5 \times 0.87 \times 20 \times 12} = 7.55 \text{ in.}$$

Use 1-in. square bars spaced at 6 in. on centers. These bars are hooked at the ends.

301. Design of Interior Footing. This footing is designed as a simple footing with a net soil pressure of 5370 lb. per sq. ft. This corresponds to the column load of 1,112,000 − 185,400 = 926,600 lb. The computations follow the procedure of Problem 53 and give results of

Size = 13 ft. 2 in. square

Depth for diagonal tension $d = 29.4$ in.

Depth for bending moment $d = 20.2$ in.

Use $d = 30$ in. and $h = 35$ in.

$A_s = 20.6$ sq. in. and $\Sigma o = 89.8$ in.

Use twenty-three 1-in. square bars (hooked) in each direction, 14 ft. 9 in. long.

The total weight of the footing 169,000 lb. The weight was originally assumed to be 200,000 lb. The design may be corrected for the reduced actual weight with some slight decrease in the dimensions of the two footings.

ILLUSTRATIVE PROBLEM 58

302. Design of a Strap Footing Connecting Caissons. Assume that the exterior and interior columns of Problem 55 are to be supported by caissons. This is necessary because the soil immediately below the basement has a low allowable bearing. Thirty feet below the base of the columns is a hardpan whose allowable bearing is 10 tons per square foot. The exterior column is so close to the property line that a caisson cannot be placed axially with it. Its center will be located 1 ft. inside the exterior column center line, this being also closely the middle point of the exterior footing of Problem 57. To offset the eccentricity of the exterior caisson a strap will be carried to the interior column. In this case it will be assumed that the strap does not bear on consolidated soil and it will be sloped with provision made to have loosened fill under it. After a preliminary survey it is decided to make the exterior caisson shaft 3 ft. in diameter and the interior one 4 ft. in diameter, with the strap 3 ft. wide. This strap will also extend under the projecting part of the exterior column.

The forces acting on the strap are shown in Figure 187*b* (final dimensions are used). Taking moments about the center line of the exterior caisson,

$$-600{,}000 \times 1 - (\tfrac{37}{12} \times 3 \times 3.5 \times 150) \times \tfrac{1}{4} + 450 \times 26.5 \times 14.75 + \tfrac{937}{2} \times 26.5 \times 10.33 + 28F_2 = 0$$

$$F_2 = 10{,}600 \text{ lb.}$$

This force F_2 is the net force at the interior column to produce equilibrium of the strap. It includes the supporting force for the strap and the portion of the column load diverted to the strap. The force F_1 exerted by the exterior caisson equals

$$F_1 = 600{,}000 + 10{,}600 + 29{,}200 = 639{,}800 \text{ lb.}$$

Exterior Caisson. The load at the base of this caisson equals

From above	F_1 =	639,800 lb.
Weight of caisson shaft (22.6 ft.)	=	24,000
Weight of caisson base (4.33 ft.)	=	12,200
Total	=	676,000 lb.

$$\text{Base area} = \frac{676{,}000}{20{,}000} = 33.8 \text{ sq. ft.}$$

The maximum width of this expanded base cannot exceed $2 \times 2.5 = 5$ ft. Use the oval shape of Figure 187*c*, which has a longitudinal axis of 7.83 ft. and width of 5 ft. On the longitudinal axis the slope of the base to shaft will be approximately 60° (Fig. 187*d*).

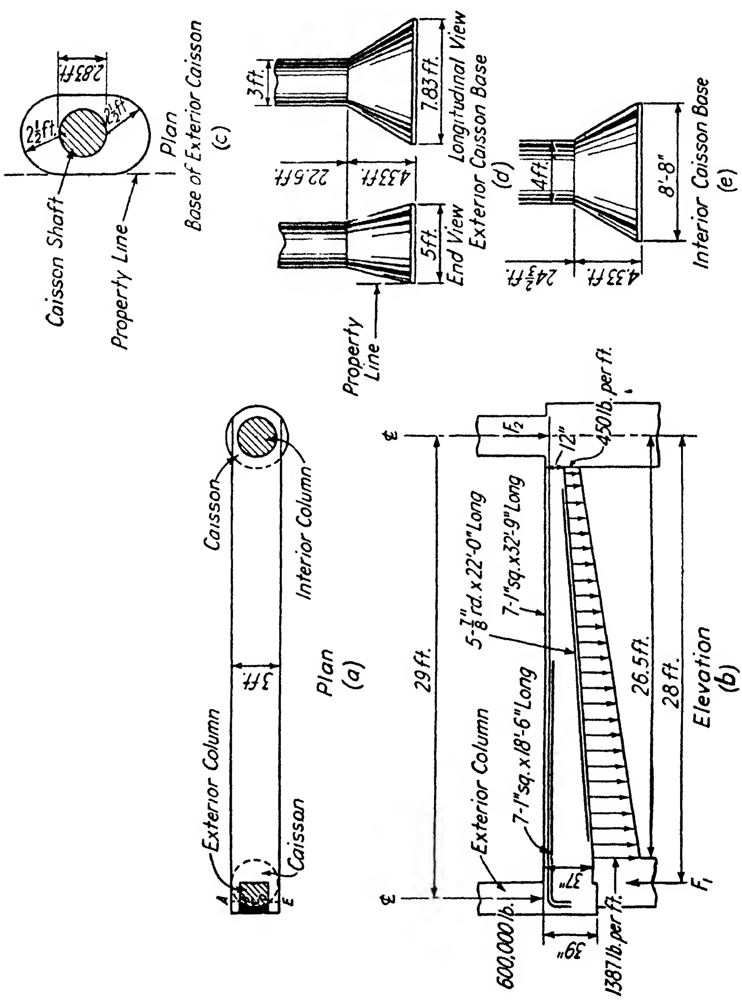


FIG. 187

The caisson shaft is loaded with a maximum force of 663,800 lb. Designed as a tied column ($f'_c = 2500$ lb. per sq. in.), it requires a steel ratio $p = 0.0127$ and area $A_s = 12.9$ sq. in. Use thirteen 1-in. square bars with suitable ties. The ratio $\frac{h}{t} = \frac{22.6 \times 12}{36} = 7.5$, so this is a short column.

Interior Caisson. The load at the column base equals

Interior column	=	1,112,000 lb.
Weight of caisson shaft ($24\frac{2}{3}$ ft.)	=	46,500
Weight of caisson base (4.33 ft.)	=	21,400
		1,179,900 lb.
F_2	=	10,600
Total	=	1,169,300 lb.

$$\text{Base area} = \frac{1{,}169{,}300}{20{,}000} = 58.5 \text{ sq. ft.}$$

Use base 8 ft. 8 in. diameter

The caisson shaft is designed as a tied column for a maximum load of 1,148,000 lb. If $f'_c = 2500$ lb. per sq. in., $p = 0.0114$, and $A_s = 20.7$ sq. in. Use twenty-one 1-in. square bars with suitable ties.

Strap. At the interior end the strap beam has a shear force $V = 10{,}600$ lb. Allowing $v = 50$ lb. per sq. in. for diagonal tension in concrete with no web steel (ordinary anchorage),

$$d = \frac{V}{vbj} = \frac{10{,}600}{50 \times 36 \times 0.87} = 6.8 \text{ in.}$$

Use $h = 7 + 5 = 12$ in.

At the center of the exterior caisson the moment in the strap by the previous moment equation, equals $M = 601{,}200$ ft.-lb.

$$\text{Depth } d = \sqrt{\frac{601{,}200 \times 12}{196 \times 36}} = 32 \text{ in.}$$

At the inner edge of the caisson the shear force $V = 34{,}940$ lb.

$$\text{Depth for diagonal tension, } d = \frac{34{,}940}{50 \times 36 \times 0.87} = 22.3 \text{ in.}$$

Use $d = 32$ in. and $h = 32 + 5 = 37$ in.

The exterior column overhangs the exterior caisson shaft as shown in Figure 187*a*. Assume the column load to be uniformly distributed; then the force on the blackened area outside the caisson equals 175,000 lb. This force tends to shear the strap along the line *ABCDE* ($22.4 + 2 \times 7.5 = 37.4$ in.). The allowable shear stress equals $v = 0.06f'_c = 150$ lb. per sq. in. The necessary depth d equals

$$d = \frac{V}{vbj} = \frac{175{,}000}{150 \times 37.4 \times 0.87} = 35.8 \text{ in.}$$

Use $h = 36 + 3 = 39$ in.

Strap Steel.

$$A_s = \frac{601{,}200 \times 12}{20{,}000 \times 0.87 \times 32} = 13.0 \text{ sq. in.}$$

For bond,

$$\Sigma o = \frac{175{,}000}{125 \times 0.87 \times 36} = 44.7 \text{ in.}$$

or

$$\Sigma o = \frac{34{,}940}{125 \times 0.87 \times 32} = 10.1 \text{ in.}$$

Use fourteen 1-in. square bars in two rows. This steel is placed in the top of the strap.

At the center of the strap, 14 ft. from the interior column, the moment equals 208,000 ft.-lb., which requires a depth $h = 23.8$ in. for concrete fiber stress; the actual depth is 24.5 in. With one row of steel, $d = 21$ in. and the necessary steel area $A_s = 6.83$ sq. in. The shear force at this section equals $V = 20{,}140$ lb., which gives a shear stress $v = 31$ lb. per sq. in. and steel perimeters $\Sigma o = 8.8$ in. The lower row of the steel can be cut off at this section.

When the live load varies on the floors above, the value of F_2 (Fig. 187*b*) will vary. If the loose fill under the strap consolidates, the case may arise where the strap is supported by the two caissons and tends to sag under its own weight. Using the average weight of 920 lb. per ft. as uniformly distributed,

$$M = \frac{wl^2}{8} = \frac{920 \times (28)^2}{8} \times 12 = 1{,}080{,}000 \text{ in.-lb.}$$

To avoid cracks at the bottom of the strap, use a steel area approximating

$$A_s = \frac{1{,}080{,}000}{20{,}000 \times 0.87 \times 21} = 2.96 \text{ sq. in.}$$

Use five $\frac{7}{8}$-in. round bars.

RETAINING WALLS

303. Lateral Soil Pressure. Retaining walls serve to hold back earth banks or slopes which would otherwise tend to slide. The thrusts on such walls are computed by theories developed in the texts on soil mechanics. Tests checking these theories are as yet few in number owing to the cost and difficulty of testing full-sized walls. The Rankine theory has been generally used by American engineers for design, but recent tests seem to indicate that the Coulomb theory is nearer actual conditions. The Coulomb theory will be used in this text.

The Rankine theory assumes that, if a small prism of earth is taken, bounded by planes parallel to the surface of the earth and planes parallel to the plane of contact of wall and earth, the pressure w on the upper plane (Fig. 188*a*) is equal to the weight of earth above the plane, and that the pressure p on the side plane is a conjugate stress, and parallel to the surface of the earth. It is assumed that the presence of the wall does not affect this analysis. Therefore, the pressure on the wall is uniformly varying.

In the Coulomb theory, the wall restrains the sliding of some mass of earth such as ABC (Fig. 188*b*). This mass weighs W pounds. On the plane of sliding AC the pressure F is inclined from the normal so that the maximum soil friction is used, the angle of internal friction being ϕ. Similarly the pressure E between the wall and earth develops the maximum friction on plane AB, and E is inclined the angle of friction z.

The mass must be in equilibrium under the three forces W, F, and E. Some plane AC produces the maximum pressure E on the wall, and it is for this case that the wall should be designed. The wall pressure equals

$$E = \frac{wh^2}{2} \frac{\sin^2(\theta - \phi)}{\sin^2 \theta \sin(\theta + z)\left(1 + \sqrt{\dfrac{\sin(\phi - \delta)\sin(\phi + z)}{\sin(\theta - \delta)\sin(\theta + z)}}\right)^2} \tag{242}$$

where w = unit weight of soil
h = vertical height of wall
δ = angle of the slope with the horizontal
θ = angle of the inner face with the horizontal.

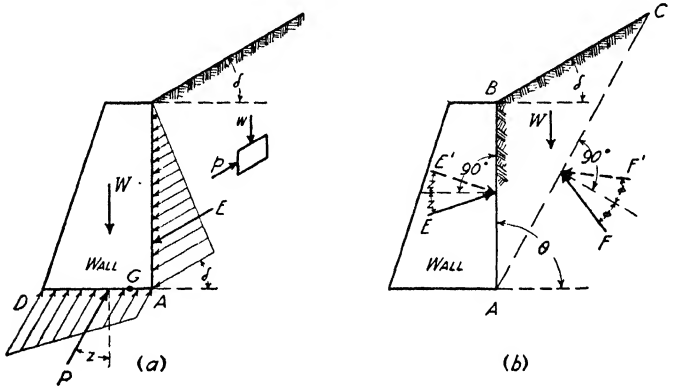


FIG. 188

Equation 242 is complicated in form, but has the advantage of being general. It may be used for any of the earth pressure theories that assume uniformly varying pressures if appropriate values of the angles ϕ and z are substituted.

The discussion above refers to the case known as *active* pressure of the wall and the bank. This implies that the wall moves away from the bank as it deforms. When an abutment is pushed into the soil by the thrust of an arch, or when a slope *creeps* or tends to landslip, *passive* pressure exists and the lateral soil pressure is greatly increased. Passive pressure implies that the wall and bank approach each other. In Figure 188*b* the wall pressure E' for passive pressure will act on the other side of the normal to plane AB, and the force on any sliding plane AC will act at F'. Equation 242 no longer holds for the value of E'. Most walls are designed for active pressure, and many wall failures are caused by the application of passive pressures to these designs. In this text active pressures will be assumed.

The Coulomb lateral pressures are uniformly varying. Texts on soil mechanics should be consulted for pressure distributions if the wall changes its slope, or if there is a *surcharge* load on the bank near the wall.

304. Gravity Walls. Many walls are built of plain concrete. Such walls are designed to be in equilibrium under the action of

W = weight of wall
E = earth pressure on a side face
P = resultant pressure on the wall base

Figure 188a shows such a case. The weight W acts at the center of gravity of the cross section. Either the Coulomb or Rankine theory gives a uniformly distributed lateral pressure E. The force P is also assumed to be uniformly varying, because the concrete base AD remains a plane surface and will produce uniformly varying deformations in the soil as the wall tends to tip. If the soil is elastic, the soil pressures will also be uniformly varying.

Applying the conditions of equilibrium, the magnitude and position of the pressure P can be determined. For safety it is necessary that

1. The maximum intensity of the normal soil pressure p_v on the base be within the allowable soil pressure.

2. The base be wholly in compression, and the neutral axis (zero pressure) be outside the base. The limiting case occurs when the neutral axis is at A. The resultant force P then acts $\frac{AD}{3}$ from point D. For the case shown in Figure 188a, P acts nearer the center. This requirement is summed up in the statement that the force P must act within the middle third of the base. If the neutral axis is within the base, say at G, the portion AG of the base will be lifted off the soil.

3. The force P does not make an angle with the normal to the base greater than the angle of friction z.

Gravity footings safely proportioned may or may not be more economical than reinforced cantilever footings, depending on concrete and excavation costs. Comparative designs should be made.

ILLUSTRATIVE PROBLEM 59

305. Design of a Gravity Wall. Design a trapezoidal gravity wall with vertical outer face to restrain a bank 22 ft. high. The surface slopes at 10°. The base of the footing is to be 4 ft. below the lower level (Fig. 189). The soil weighs 100 lb. per cu. ft.; its coefficient of internal friction ϕ = 36°; and the coefficient of friction of concrete on earth z = 30°. The maximum allowable soil pressure equals 3 tons per square foot.

The maximum pressures occur during construction before the earth consolidates. At that time the most severe conditions occur if the back fill is placed and the front fill is not. There will be a soil pressure E on the face CB but none on AE. After several trials the section shown is adopted.

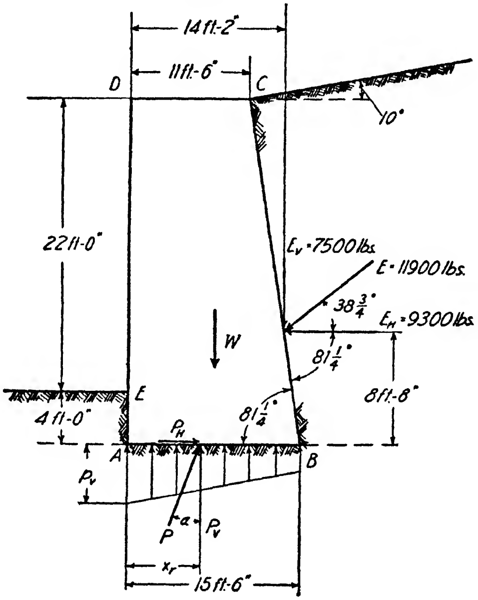


FIG. 189

Taking a foot length of the wall, by equation 242. $\phi = 36°$, $z = 30°$, $\theta = 98\frac{3}{4}°$, $\delta = 10°$:

$$E = \frac{wh^2}{2}\,\frac{\sin^2 62\frac{3}{4}°}{\cos^2 8\frac{3}{4}° \cos 38\frac{3}{4}°\left(1 + \sqrt{\dfrac{\sin 26° \sin 66°}{\sin 88\frac{3}{4}° \cos 38\frac{3}{4}°}}\right)^2}$$

$$E = 33{,}800 \times 0.353 = 11{,}900 \text{ lb. per ft. length}$$

Vertical component $\quad E_v = E \sin 38\frac{3}{4}° = 7500$ lb. per ft.

Horizontal component $E_H = E \cos 38\frac{3}{4}° = 9300$ lb. per ft.

Taking moments about A,

	Force		Moment
Weight of wall $11.5 \times 26 \times 1 \times 150 =$	$44{,}900 \times$	$5.75 =$	258,000 ft.-lb.
$4 \times 26 \times \frac{1}{2} \times 150 =$	$7{,}800 \times$	$12.83 =$	100,000
$E_v =$	$7{,}500 \times$	$14.17 =$	106,000
$P_v = \Sigma V =$	60,200		+464,000
$E_H =$	$9{,}300 \times$	$8.67 =$	−81,000
		$\Sigma M =$	+383,000 ft.-lb.

$$x_r = \frac{M}{P_v} = \frac{383{,}000}{60{,}200} = 6.37 \text{ ft.}$$

This is well within the third point $\frac{15.5}{3} = 5.17$ ft., and the base has compression over the full width. However, the maximum soil pressure controls the length of the base. Assuming the soil pressure on the rectangular base to be uniformly varying, the distance from the neutral axis to the center of pressure being x_1 and to the center of gravity being x_o:

$$x_1 - x_o = \frac{I_o}{Ax_o} \quad \text{or} \quad 1.38 = \frac{1 \times (15.5)^3}{12 \times 1 \times 15.5x_o}$$

$$x_o = \frac{(15.5)^2}{1.38 \times 12} = 14.45 \text{ ft.}$$

The neutral axis is 6.7 ft. beyond point B of the base. If a is the intensity of pressure 1 ft. from the neutral axis, the resultant force P_v equals $P_v = ax_oA$ or

$$a = \frac{60{,}200}{14.45 \times 1 \times 15.5} = 269 \text{ lb. per cu. ft.}$$

Maximum pressure p_v $= 269(15.5 + 6.7) = 5980$ lb. per sq. ft.

Maximum allowable p_v $= 6000$ lb. per sq. ft. Safe.

The friction P_H $= 9300$ lb. between base and soil

Maximum available friction $F = P_v \tan z$

$F = 60{,}200 \times 0.577 = 34{,}700$ lb. Safe.

When the front fill is placed and consolidation takes place the design will have an additional factor of safety.

306. Reinforced Cantilever Walls. Such walls consist of a reinforced footing upon which is placed a reinforced wall. The wall may be placed at any point on the footing consistent with economy or the peculiarities of the particular design (Fig. 190). In the usual retaining wall (Fig.

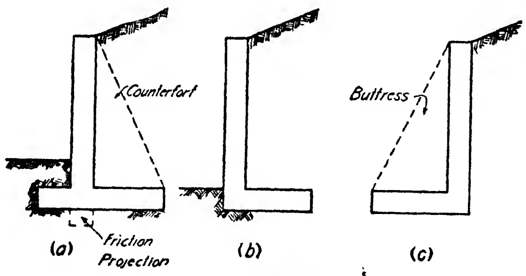


FIG. 190

(190*a*) the wall is designed as a cantilever fixed at the footing and loaded by the lateral soil pressure. The inner footing is designed as a second cantilever supported at the wall-footing junction and loaded with earth pressure above and below, while the exterior footing is designed as a third cantilever. The order of design procedure can be

1. Design of wall.
2. Selection of footing base by checking whole wall footing for
   a. Compression over whole base.
   b. Maximum soil pressure.
   c. Maximum friction on base.
3. Design of interior and exterior footing cantilevers.

Occasionally additional frictional resistance is provided by a projection at the bottom of the footing which assists by direct bearing on the soil (Fig. 190*a*). Experience has shown that mixes richer than 2500-lb. concrete show in the exposed surfaces more cracks due to shrinkage and frost deterioration.

307. Retaining Walls with Counterforts or Buttresses. Walls restraining banks of considerable height are often more economically designed if the wall is braced or supported. Supports placed within the earth bank are called *counterforts*; those on the outside are *buttresses*. Both wall and inner footing are designed as continuous slabs with longitudinal steel. The bending moments for continuous slabs are so much less than cantilever moments that thinner slabs and less steel can be used. There is, however, the additional cost of the counterfort or buttress to be considered.

ILLUSTRATIVE PROBLEM 60

308. Design of Cantilever Retaining Wall. Design a cantilever retaining wall to restrain the bank given in Problem 59. Use 2500-lb. concrete and assume the back fill is placed but the front fill is not.

309. Wall Design. *Thickness.* Assume a vertical inner face for the wall and a footing thickness not less than 20 in. The lateral earth pressure for a length of wall of 1 ft. and a depth of 24 ft. 4 in. equals

$$E = \frac{100(24.33)^2}{2} \times \frac{\sin^2 54^\circ}{\sin^2 90^\circ \cos 30^\circ \left(1 + \sqrt{\dfrac{\sin 26^\circ \sin 66^\circ}{\sin 80^\circ \cos 30^\circ}}\right)^2}$$

$$E = 29{,}550 \times 0.266 = 7850 \text{ lb. per ft. length}$$

$$E_v = 3925 \text{ lb. per ft.}$$

$$E_H = 6800 \text{ lb. per ft.}$$

Maximum moment at the base of wall (thickness $= t$):

$$M = 6800 \times \frac{24.33}{3} \times 12 - 3925 \times \frac{t}{2} = 661{,}000 - 1963t$$

Neglecting the minus term, the necessary thickness is figured approximately and then assumed $t = 20$ in.

$$M = 661{,}000 - 39{,}250 = 621{,}750 \text{ in.-lb.}$$

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{621{,}750}{196 \times 12}} = 16.3 \text{ in.}$$

With a minimum covering of 3 in. (A.C.I. Art. 507*a*),

$$\text{Minimum } t = 16.3 + 3.5 = 19.8 \text{ in.}$$

Use $t = 20$ in. and $d = 16.5$ in.

310. Shear and Diagonal Tension. At any section the shear force V equals the earth pressure E_H above this section. At the base,

$$\text{Maximum shear } v = \frac{V}{bjd} = \frac{6800}{12 \times 0.87 \times 16.5} = 40 \text{ lb. per sq. in.}$$

$$\text{Allowable shear } v = 0.06f'_c = 150 \text{ lb. per sq. in.}$$

Diagonal tension for a cantilever can be checked at a section d inches above the support (base). The designer does not wish to use diagonal tension steel in a continuous wall. The allowable stress for concrete is $v = 0.02f'_c = 50$ lb. per sq. in. The section at the base is safe, and therefore the critical section above it is safe. It will be noticed that the value of d varies with the depth h, while $V = E_H$ varies with h^2. Therefore, if the base section is safe all other sections are also.

311. Steel. The wall will taper from 20 in. at the base to 12 in. (without decorative cornice) at the top. With four rows of steel in the wall a thickness less than 12 in. is not justified. The batter will be given on the outside face. At any section,

$$E = \frac{100h^2}{2} \times 0.266 = 13.3h^2$$

$$E_H = 0.866E = 11.52h^2$$

$$E_v = 0.5E = 6.65h^2$$

$$M = 11.52h^2\left(\frac{12h}{3}\right) - 6.65h^2\left(\frac{t}{2}\right) = 46.1h^3 - 3.33h^2t \text{ in.-lb.}$$

Also

$$M = f_sA_sjd = 20{,}000 \times 0.87 \times A_sd = 17{,}400A_sd \text{ in.-lb.}$$

Therefore

$$A_s = \frac{46.1h^3 - 3.33h^2t}{17{,}400d}$$

Assuming $t = d + 3.5$, the values of the steel area A_s are plotted in Figure 191*b* by assuming values of h in feet and substituting the corresponding values of d and t in inches, it being true that $t = 12 + \frac{8}{24.3}h$. At the base,

$$A_s = \frac{621{,}750}{20{,}000 \times 0.87 \times 16.5} = 2.16 \text{ sq. in.}$$

Use 1-in. square bars spaced at 5.5 in. placed in the inner face of the wall.

Bond.

$$u = \frac{V}{\Sigma o j d} = \frac{6800}{2.18 \times 4 \times 0.87 \times 16.5} = 55 \text{ lb. per sq. in.}$$

$$\text{Allowable } u = 0.05f'_c = 125 \text{ lb. per sq. in.}$$

To Cut Off Bars. The plot of steel areas shows that the necessary area drops off very rapidly and it is economical to use bars of two or three lengths.

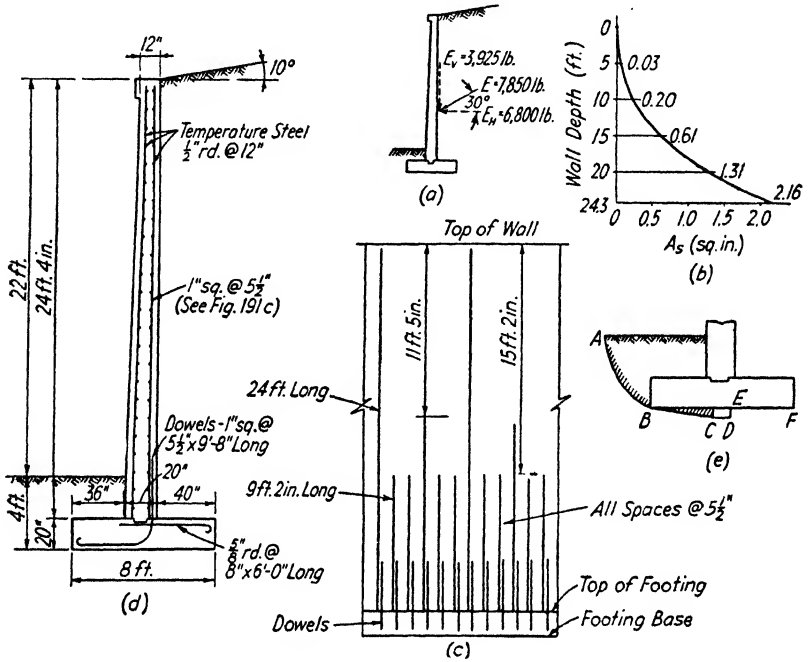


FIG. 191

Cut off 2 bars out of 3. The area remaining equals $\frac{2.18}{3} = 0.73$ sq. in. From Figure 191*b* this occurs at $h = 16$ ft. from the top. These bars will be anchored 10 diameters = 10 in. and will end about 15 ft. 2 in. from the top.

Cut off half of the remaining bars. $A_s = \frac{2.18}{6} = 0.36$ sq. in. This area is satisfied at $h = 12$ ft. 3 in. Anchorage of 10 in. ends the bar about 11 ft. 5 in. from the top. The remaining bars run to the top and are spaced 33 in. apart. Figure 191*c* shows this spacing for a longitudinal elevation.

Temperature Steel. The outside of the wall is exposed to the full temperature variation whereas the inside is somewhat insulated. It is customary to place temperature steel in the outer face equivalent to that for roof slabs ($p = 0.0025$) and half as much at the inner face. The outer face will be reinforced with both vertical and horizontal bars, but the inner face will have horizontal bars as temperature steel wired to the vertical tension steel. In each set, using an average $d = 12.5$ in., there is an amount

$$A_s = pbd = 0.00125 \times 12 \times 12.5 = 0.19 \text{ sq. in.}$$

Use $\frac{1}{2}$-in. round bars spaced at 12 in.

312. Location of Wall. Unless there are special restrictions at the site, the most economical retaining wall is the one with the least permissible width of the base footing. This least width is affected by the position of the wall upon the footing.

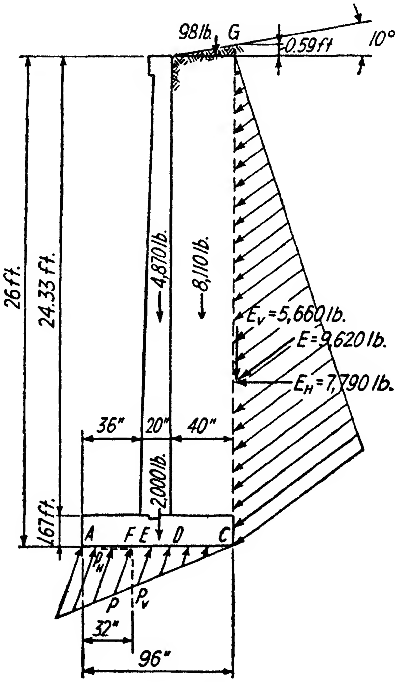


FIG. 192

One rule that has been proposed[3] assumes that greatest economy ensues when the weight of the wall plus the fill on the footing equals the weight of fill alone extending out to the point of application of the resultant pressure on the base (point F in Fig. 192). The least width of base is obtained when zero base pressure occurs at the edge of the footing (point C, Fig. 192). In this case the resultant pressure acts at $\frac{2}{3}l$ from point C, where l is the footing width. In applying this rule, a width $l = 8$ ft. is assumed after several trials. Using the average width of the wall,

$$\text{Weight of wall} = 150 \times 24.33 \times \left(\frac{12 + 20}{2 \times 12}\right) = 4870 \text{ lb.}$$

[3] "A Helpful Rule for Use in Designing Retaining Walls," by D. B. Hall, *Civil Engineering*, Vol. 6, No. 3, March 1936, p. 203.

Assuming the fill to extend out $c_1 l = 8c_1$ from point C,

$$\text{Weight of fill} = 100\left[24.33 \times 8c_1 + \frac{0.1763}{2} \times 64c^2_1\right]$$

$$= 564.2c^2_1 + 19{,}464c_1$$

$$\text{Wall} + \text{fill} = 564.2c^2_1 + 19{,}464c_1 + 4870 \tag{243}$$

If fill alone extends out to $\frac{2}{3}l = 5.33$ ft., its weight equals

$$100\left[(24.33 + 1.41c_1 - 0.94)5.33 + \frac{0.94}{2} \times 5.33\right] = 752c_1 + 12{,}725 \tag{244}$$

Equating equations 243 and 244, $c_1 = 0.41$ and $c_1 l = 40$ in. This places the wall near the center of the footing. Practice varies with the job conditions and with the judgment of the designer, but the wall is usually placed somewhere between $\frac{2}{3}l$ and $\frac{l}{2}$ from point C, unless a property line or river bank necessitates its being placed near one end of the footing.

313. Width of Footing Base. Check the assumed footing width of 8 ft. for

1. Earth pressure (compressive) on full width.
2. Maximum intensity of normal soil pressure on base.
3. Adequate frictional resistance to sliding of the retaining wall. The retaining wall plus the fill resting on the base will be taken as a rigid body.

The forces acting on this body are

1. Weight of wall and footing.
2. Weight of fill.
3. Pressure on plane CG (Fig. 192) due to earth bank to right of CG.
4. Pressure on the base.

The pressure on the plane CG is computed by use of equation 242 with $z = \phi$, since this is a case of undisturbed earth bearing on earth fill.

$$E = \frac{100 \times (26.59)^2}{2} \times 0.272 = 9620 \text{ lb.}$$

$$E_v = 5660 \text{ lb.} \qquad \text{and} \qquad E_H = 7790 \text{ lb.}$$

Taking moments about A at the toe of the footing (Fig. 192),

	Force lb.	in.		Moment in.-lb.
Wall (average)	4,870	× 48	=	233,760
Footing	2,000	× 48	=	96,000
Soil (24.33 ft. deep)	8,110	× 76	=	616,360
Soil (slope)	98	× 82.7	=	8,110
E_v =	5,660	× 96	=	543,360
$P_v = \Sigma V$ =	20,738 lb.	Sum	=	+1,497,590
$E_H = P_H$ =	7,790	× 107	=	−833,530
		ΣM_A	=	664,060

$$x_r = \frac{\Sigma M_A}{P_v} = \frac{664{,}060}{20{,}738} = 32 \text{ in.}$$

The resultant P of the pressure on the base acts at $\frac{l}{3}$ from point A. There is compression over the whole base and zero pressure occurs at point C.

Maximum Soil Pressure. The base pressure is assumed to be uniformly varying and the maximum normal pressure p occurs at A.

$$20{,}738 = p_A \times \frac{1 \times 8}{2}$$

$p_A = 5185$ lb. per sq. ft.; the allowable pressure equals 6000 lb. per sq. ft.

Friction on Base. The maximum available friction equals

$$P_v \tan z = 20{,}738 \times 0.577 = 11{,}950 \text{ lb.}$$

Actual friction $= E_H = 7790$ lb. This gives a factor of safety of 1.53 against sliding. Some designers consider the choice of the angle z open to such an error that they provide a factor of safety of 2. They would probably provide a projection, or key, at the base of the footing to increase sliding resistance. Failure in such a case, with the front fill also in place, would mean rupture along some such line ABC (Fig. 191*e*) plus the friction on the surfaces CD and EF. It is apparent that this resistance is greater than that figured above.

314. Design of Inner Cantilever Footing. The forces acting on the heel are

1. Weight of earth on inner footing.
2. Weight of footing.
3. Horizontal and vertical components of the soil pressure on the base.

The force P on the base makes an angle with the normal to the base:

$$\tan^{-1} \frac{P_H}{P_v} = \tan^{-1} \frac{7790}{20{,}738} = \tan^{-1} 0.376$$

The normal soil pressure on the base at one foot from C (Fig. 192) equals $p_1 = 648$ lb. per sq. ft. At D the normal pressure $p_D = 2160$ lb. per sq. ft. The horizontal component, or friction, varies from zero at C to $p'_D = 2160 \times 0.376 = 810$ lb. per sq. ft. at D. The moment about the center of the footing section at D equals

	Force lb.	in.		Moment in.-lb.
Earth (24.33 ft. deep)	8,110	× 20	=	162,200
Earth (slope)	98	× 26.7	=	2,620
Footing	833	× 20	=	16,670
Downward shear	+9,041			+181,490
Normal base pressure	−3,600	× 13.3	=	−48,000
Resultant shear	+5,441 lb.			
Tangential base pressure	1,350	× 10	=	−13,500
			$\Sigma M_D =$	+120,000

The required depth at section D for bending equals

$$d = \sqrt{\frac{120{,}000}{196 \times 12}} = 7.1 \text{ in.}$$

The minimum depth $h = 8 + 5 = 13$ in. Since the wall is 20 in. thick at its junction with the footing, we shall continue to use a 20-in. footing depth. The shear stress at section D equals

$$v = \frac{5441}{12 \times 0.87 \times 15} = 35 \text{ lb. per sq. in.}$$

The allowable shear equals $v = 0.06f'_c = 150$ lb. per sq. in. Diagonal tension is checked at a section $d = 15$ in. from section D for cantilever footings. The allowable diagonal tension equals $v = 0.03f'_c = 75$ lb. per sq. in., because all footing steel is hooked. This is safe since the shear stress at the critical section will be less than at section D.

Steel.

$$A_s = \frac{120{,}000}{20{,}000 \times 0.87 \times 15} = 0.46 \text{ sq. in. per ft.}$$

$$\Sigma o = \frac{5441}{187.5 \times 0.87 \times 15} = 2.22 \text{ in. per ft.}$$

Use $\frac{5}{8}$-in. round bars at 8-in. spacing.

315. Design of Outer Footing Cantilever. Before the outer fill is placed the forces acting on the outer part of the footing are

1. Weight of footing.
2. Horizontal and vertical components of the soil pressure on the base.

The normal components of the soil pressure are $p_A = 5185$ lb. per sq. ft. and $p_E = 3240$ lb. per sq. ft. The tangential components are $p_A = 1950$ lb. per sq. ft. and $p_E = 1220$ lb. per sq. ft. The bending moment about the center of the section at E equals

	Force lb.	in.	Moment in.-lb.
Normal pressure (uniform)	9,720 ×	18 =	175,000
(varying)	2,920 ×	24 =	70,100
Upward shear	= 12,640 lb.		+245,100
Footing	750 ×	18 =	−13,500
Resultant shear	= 11,890 lb.		
Tangential pressure	= 4,755 ×	10 =	−47,550
		ΣM_E =	+184,050

The necessary depth at section E to satisfy bending moment is

$$d = \sqrt{\frac{184{,}050}{196 \times 12}} = 8.84 \text{ in.}$$

Minimum depth $h = 9 + 5 = 14$ in. but we continue to use $h = 20$ in. The shear stress at section E equals

$$v = \frac{11{,}890}{12 \times 0.87 \times 15} = 77 \text{ lb. per sq. in.}$$

This is safe. At a section 15 in. toward A the shear force equals 7640 lb. and the shear stress $v = 49.5$ lb. per sq. in. This is safe, since the footing steel is hooked (special anchorage).

Steel.

$$A_s = \frac{184{,}050}{20{,}000 \times 0.87 \times 15} = 0.71 \text{ sq. in.}$$

$$\Sigma o = \frac{11{,}890}{187.5 \times 0.87 \times 15} = 4.86 \text{ in.}$$

The footing is poured first, then the wall, with a joint at their junction. This joint includes a key to add to the shear resistance. The wall steel is detailed in three lengths, of which the shortest extends 9 ft. above this junction. This is too great a distance to project above the footing before the wall forms are placed, so dowels will be used to supply the necessary 2.16 sq. in. of wall steel at the junction section, the wall steel being placed later in the forms and resting on the footing. These dowels run 40 diameters (40 in.) into the wall and must be anchored at least 40 in. in the footing. Let the dowels extend into the outer footing on the bottom as reinforcement. This gives $A_s = 2.18$ sq. in. and $\Sigma o = 8.72$ in. It would save steel to use two thirds of the dowels in this shape and run one third into a footing key, but this arrangement would add to the difficulty of installing the steel and inspecting its placement.

The steel is detailed in Fig. 191*d*. Provision should be made to drain the back of the wall, otherwise it would be necessary to provide for water pressure or ice pressure on the wall, in addition to soil pressure.

ILLUSTRATIVE PROBLEM 61

316. Design of Counterfort Retaining Wall. Design a counterfort retaining wall to restrain the earth bank given in Problem 59. Use $f'_c = 2500$ lb. per sq. in. and assume the back fill to be placed and front fill not yet placed.

317. Spacing of Counterforts. Assume a wall of minimum thickness of 12 in. and estimate the footing to be 20 in. thick.

Wall. The lateral soil pressure on the wall is the same as in Problem 60. Referring to Article 309 the intensity of inclined pressure equals $100 \times 0.266 = 26.6$ lb. per sq. ft. at a depth of 1 ft. The wall is now regarded as a continuous slab supported by the counterforts and also at the footing.

It is customary to design using strips 1 ft. high as independent rectangular beams supported by the counterforts. This is an approximation, as the bottom of the wall stiffens the lower strips. Also, the different strips have varying loads and, hence, deflect unequally, producing shear forces and bending moments along the sides of the strips. These forces and couples on the beam sides are neglected for commercial design, and we shall deal with the horizontal pressures only. Vertical construction joints will probably occur every 60 to 80 ft., so the slab will be designed for the end span of a continuous beam. The maximum numerical bending moment will occur at the first interior support of the 1-ft. strip just above the footing. The average horizontal pressure p_h on this strip equals

$$p_h = 0.866 \times 26.6 \times 23.83 = 550 \text{ lb. per sq. ft.}$$

$$M = \frac{pl^2}{10} = \frac{550 \times 12l^2}{10} = 660l^2 \text{ in.-lb.}$$

$$M = Kbd^2 = 196 \times 12 \times (8.5)^2 = 170{,}000 \text{ in.-lb.}$$

$$\text{Clear span } l = \sqrt{\frac{170{,}000}{660}} = 16 \text{ ft.}$$

Since it is not desirable to use diagonal tension steel in the wall, the shear stress will be kept within the allowable value for concrete of $v = 0.02f'_c$ for ordinary anchorage. At the first interior support the shear force $V = 0.575p_h l$

$$50 = \frac{0.575 \times 550 \times l}{12 \times 0.87 \times 8.5} \qquad l = 14 \text{ ft.}$$

If $v = 75$ lb. per sq. in., $l = 21$ ft.

Inner Footing. Preliminary trials of footing dimensions must usually be made. Assuming that such trials result in the dimensions of Figure 193 and that the point

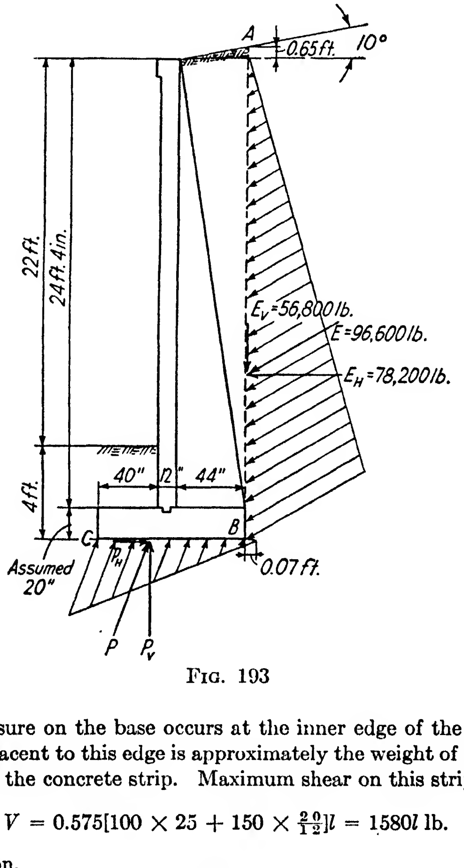


FIG. 193

of zero earth pressure on the base occurs at the inner edge of the footing, the load on a 1-ft. strip adjacent to this edge is approximately the weight of earth on the strip plus the weight of the concrete strip. Maximum shear on this strip equals

$$V = 0.575[100 \times 25 + 150 \times \tfrac{20}{12}]l = 1580l \text{ lb.}$$

for diagonal tension.

$$v = \frac{1580l}{12 \times 0.87 \times 14.5} = 10.4l$$

If $v = 50$, $l = 4.8$ ft. If $v = 75$, $l = 7.2$ ft.

Counterfort. The stresses in the counterfort will also be affected by the counterfort spacing; but, since the counterfort dimensions are dependent on computations yet to be made, no attempt will now be made to compute the minimum spacing.

Let us adopt tentatively a counterfort width of 18 in. with a clear spacing of

8.5 ft. and center-to-center spacing of 10 ft. Use special anchorage, but it may be necessary to increase the footing depth.

318. Location of Wall. Accepting again the criterion used in Problem 60 to locate the wall and assuming the footing to be 8.0 ft. wide with zero pressure at the inner edge, the weight of earth 10 ft. long and a width of $\frac{2}{3} \times 8.0 = 5\frac{1}{3}$ ft. from the inner edge equals

$$W_1 = 100\left[5.33 \times 10(24.33 + 0.1763x - 0.94) + 5.33 \times 10 \times \frac{0.94}{2}\right]$$
$$= 127{,}200 + 940x$$

where x is the width of fill on the inner footing.

For a 10-ft. length the weight of

$$\text{Wall} = 24.33 \times 10 \times 1 \times 150 = 36{,}500 \text{ lb.}$$

$$\text{Counterfort} = 24.33 \times 1.5 \times \frac{x}{2} \times 150 = 2735x \text{ lb.}$$

$$\text{Earth between counterfort} = 100 \times 8.5 \times \left(24.33x + \frac{0.1763x^2}{2}\right) = 20{,}683x + 75x^2$$

$$\text{Earth over counterfort} = 100 \times 1.5 \times (24.33 + 0.1763x)\frac{x}{2} = 1825x + 13.22x^2$$

$$W_2 = 36{,}500 + 25{,}243x + 88.22x^2$$

Equating $W_1 = W_2$, $x = 3.65$ ft. Let the inner footing project 44 in. beyond the wall and the outer footing project 40 in.

319. Stability of Wall Footing. Adopting these dimensions, take a 10-ft. length of the wall footing as a rigid body bounded by the plane AB in the earth bank. The earth pressure on this plane, for $z = \phi = 36°$, $\delta = 10°$, $\theta = 90°$, is (see Art. 313, Problem 60):

$$E = \frac{100 \times (26.65)^2}{2} \times 0.272 \times 10 = 96{,}600 \text{ lb.}$$

$$E_v = 56{,}800 \text{ lb.} \qquad \text{and} \qquad E_H = 78{,}200 \text{ lb.}$$

Taking moments about point C (Fig. 193),

		Force lb. × in.	Moment in.-lb.
Wall		= 36,500 × 46 =	1,679,000
Footing	$= 150 \times 8.0 \times \frac{20}{12} \times 10$	= 20,000 × 48 =	960,000
Counterfort	$= 150 \times 3.67 \times \frac{24.33}{2} \times 1.5$	= 10,050 × $66\frac{2}{3}$ =	670,000
Earth	$= 2{,}433 \times 3.67 \times 8.5$	= 75,950 × 74 =	5,620,000
	$2{,}433 \times \frac{3.67}{2} \times 1.5$	= 6,700 × $81\frac{1}{3}$ =	545,100
	$65 \times \frac{3.67}{2} \times 10$	= 1,190 × $81\frac{1}{3}$ =	97,000
	E_v =	56,800 × 96 =	5,452,800
	$P_v = V$ =	207,190 lb.	+15,023,900 in.-lb.
	$P_H = E_H = 78{,}200 \times \frac{26.65 \times 12}{3}$	=	−8,336,100
		ΣM =	+6,687,800 in.-lb.

$$x_r = \frac{\Sigma M}{P_v} = \frac{6{,}687{,}800}{207{,}190} = 32.3 \text{ in.}$$

The resultant of the soil pressure on the base acts 0.3 in. within the middle third, so there is compression over the whole base.

Maximum Soil Pressure. If the soil pressure on the base varies uniformly its resultant $R = ax_oA$, where

a = intensity 1 ft. from neutral axis
x_o = distance to center of gravity of base from neutral axis
A = area of base

This resultant acts at a distance $(x_r - x_o)$ from the center of gravity of the base equal to $x_r - x_o = \dfrac{I_o}{x_oA}$, where I_o is the moment of inertia of base area about its center of gravity. For this problem, the distance from the point of application of the resultant to the center of gravity equals $48 - 32.3 = 15.7$ in. Then

$$\frac{15.7}{12} = \frac{10 \times (8.0)^3}{12 \times 10 \times 8.0x_o}$$

$x_o = 4.07$ ft., or the neutral axis is 0.07 ft. beyond the inner edge of the footing. The maximum vertical intensity of pressure $p_{max.}$ equals

$$207{,}190 = a \times 4.07 \times 10 \times 8 \qquad \text{and} \qquad a = 635 \text{ lb. per cu. ft.}$$

$p_{max.} = 8.07a = 5130$ lb. per sq. ft. (whereas the allowable soil pressure is 6000 lb. per sq. ft.)

Sliding. The horizontal earth pressure tending to slide the footing away from the bank equals $E_H = 78{,}200$ lb. The friction available, if $z = 30°$ for earth on concrete, equals $P_v \tan z = 207{,}190 \times 0.577 = 119{,}600$ lb. This gives a factor of safety against sliding of 1.53, which is ample. Therefore adopt a footing 8 ft. wide, with the wall center placed 50 in. from the inner edge and 46 in. from the outer.

320. Depth of Footing. *Inner Footing.* Figure 194 shows the vertical pressures acting on the inner footing. If a strip 1 in. wide is taken at the inner edge, the net

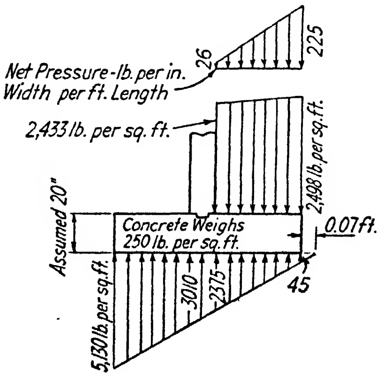


FIG. 194

downward pressure nearly equals $\frac{2708}{12}$ lb. per ft. length of the strip. The footing is a continuous beam spanning from counterfort to counterfort.

Since the moments and shears are numerically greater in the end span, design for this case.

$$M_N = \frac{wl^2}{10} = 225 \times (8.5)^2 \times \frac{12}{10} = 19{,}500 \text{ in.-lb.}$$

$$d = \sqrt{\frac{M}{Kb}} = \sqrt{\frac{19{,}500}{196 \times 1}} = 10 \text{ in.}$$

At the first interior support the maximum shear $V = 0.575 \times 225 \times 8.5 = 1100$ lb. Use special anchorage; then the necessary depth, if no web steel is used, equals

$$d = \frac{V}{vbj} = \frac{1100}{75 \times 1 \times 0.87} = 16.8 \text{ in.}$$

The depth $h = 16.8 + 5 = 21.8$ in. Use 22 in.

The *outer footing* is not supported by a buttress, so it will be designed as a cantilever extending out from the wall-footing junction. The forces acting on a strip 1 ft. wide are

1. Weight of footing (assumed 20 in.).
2. Inclined earth pressure on the base.

Taking moments about the center of the section at the wall junction,

	Force lb.	in.	Moment in.-lb.
Vertical Pressure			
Uniform $= (3{,}010 - 250) \times \frac{40}{12}$	$= 9{,}200$	$\times 20$	$= 184{,}000$
Varying $= (5{,}130 - 3{,}010) \times \frac{1}{2} \times \frac{40}{12}$	$= 3{,}535$	$\times \frac{2}{3} \times 40$	$= 94{,}270$
	$V = 12{,}735$ lb.		$+278{,}270$
Horizontal Pressure			
$\frac{78{,}200}{207{,}190}(5130 + 3010) \times \frac{1}{2} \times \frac{40}{12}$	$= 5{,}130$	$\times 10$	$= -51{,}300$
		$\Sigma M =$	$226{,}970$

For fiber stress:

$$d = \sqrt{\frac{226{,}970}{196 \times 12}} = 9.83 \text{ in.}$$

The inner footing will be 22 in. deep, with a value of $d = 17.0$ in. The outer-footing cantilever will be made the same and will be checked for diagonal tension at a distance of d from the wall, let us say 17.0 in. The shear force at this section equals

$$V = \left(\frac{5130 + 3915}{2} - 250\right) \times \frac{23}{12} = 8200 \text{ lb.}$$

$$v = \frac{8200}{12 \times 0.87 \times 17} = 47 \text{ lb. per sq. in.}$$

This is less than the allowable $v = 75$ lb. per sq. in., and the footing will be made a constant depth of 22 in. The previous computations will be only slightly changed if an extra 2 in. of footing concrete is substituted for earth.

321. Footing Steel. *Outer cantilever.*

$$A_s = \frac{226{,}970}{20{,}000 \times 0.87 \times 17.0} = 0.77 \text{ sq. in.}$$

$$\Sigma o = \frac{12{,}735}{187.5 \times 0.87 \times 17.0} = 4.60 \text{ in.}$$

Use $\frac{3}{4}$-in. round bars at 6-in. spacing. These bars will be bent up into the wall to act as dowels, or shear reinforcement at the junction of wall and footing. If the resistance of the concrete in the keyway is disregarded, the shear force of 6800 lb. per ft. at this junction produces a shear stress in this steel of $\frac{6800}{2 \times 0.442} = 7700$ lb. per sq. in. This is a safe steel shear.

Inner Footing. Let p = net pressure on a 1-in. strip (lb. per ft. per in. width)

$$\text{Exterior span—positive moment} = p\frac{l^2}{14} = \frac{(8.5)^2 \times 12}{14}p = 61.9p \text{ in.-lb.}$$

$$\text{Interior span—positive moment} = p\frac{l^2}{16} = 54.2p \text{ in.-lb.}$$

$$A_s = \frac{M}{20{,}000 \times 0.87 \times 17.0} = \frac{M}{296{,}000} \text{ sq. in. per in. width}$$

For the inch strip at the inner edge, exterior span:

$$A_s = \frac{61.9 \times 225}{296{,}000} = 0.047 \text{ sq. in. per in. width}$$

Use $\frac{5}{8}$-in. rounds at $6\frac{1}{2}$-in. spacing.

With the use of certain increments of spacing, there is tabulated below the locations where such spacing may be used, the net pressure being taken from Figure 194.

Exterior Span—Positive Steel—1-in. strip

Spacing in.	A_s sq. in.	Net Pressure lb. per ft.	Distance from Inner Edge
6.5	0.0473	226	0
8	0.0383	183	9
10	0.0307	147	18
12	0.0256	122	23

Interior Span—Positive Steel

7	0.0441	241	0
8	0.0383	209	3.5
9	0.0341	186	9
12	0.0256	139	19

In the interior spans, the negative moment equals $\frac{pl^2}{11}$ and will require an area of $\frac{16}{11} = 1.45$ of the positive steel. This can be supplied by alternating bars A and B as shown in Figure 195.

In the exterior span, the exterior support requires a negative area of $\frac{14}{24} = 0.583$ of the exterior positive steel; the interior support needs a negative area of $\frac{14}{10} = 1.4$

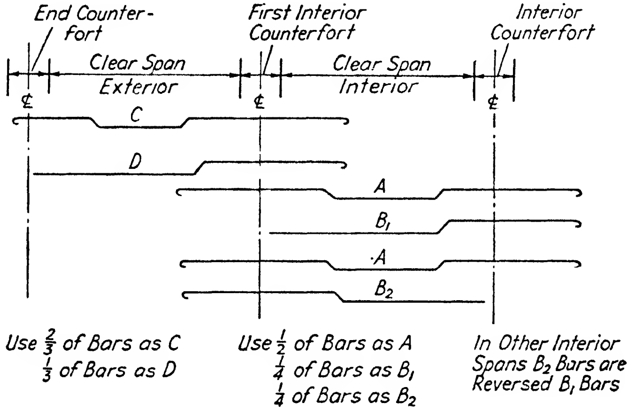


FIG. 195

of the exterior positive steel. These areas can be supplied by using 2 bars C for each bar D, if the negative steel from the interior spans is also included at the first interior support.

For the inch strip at the inner edge the steel arrangement of Figure 195 gives a steel area of 0.0804 sq. in. per in. width at the first interior support. This is equivalent to 0.262 bar. The shear force at the exterior face of this support is 1100 lb. per in. width.

$$\text{Bond stress } u = \frac{V}{ojd} = \frac{1100}{0.262 \times 1.96 \times 0.87 \times 17} = 145 \text{ lb. per sq. in.}$$

The allowable bond stress for special anchorage equals 188 lb. per sq. in. If this is safe, the bond stress is satisfactory elsewhere in the inner footing.

322. Wall Steel. *End Span.* Since the wall is regarded as a continuous beam, the positive moment in the end span will be $\frac{pl^2}{14}$, where p is the horizontal component of the lateral earth pressure. This pressure on the wall varies with the depth h.

$$p = 0.86 \times 26.6h = 23h \text{ lb. per sq. ft.} = 1.92h \text{ lb. per ft. per in. height}$$

$$M = \frac{1.92 \times (8.5)^2 \times 12}{14} h = 119h \text{ in.-lb.}$$

$$A_s = \frac{119h}{20{,}000 \times 0.87 \times 8.75} = \frac{h}{1280} \text{ sq. in.}$$

When 3 in. of dampproofing and $\frac{1}{2}$-in. round bars are used, the spacing of the bars may vary as shown below.

Spacing in.	Area sq. in. per in.	Maximum Depth h ft.	
		Exterior Span	Interior Span
10	0.0196	25.1	28.7
12	0.0163	20.9	23.9
15	0.0130	16.6	19.0
18	0.0109	14.0	15.9
21	0.0093	11.9	13.6
24	0.0081	10.4	11.9
27	0.0072	9.2	10.6
30	0.0065	8.3	9.5

The steel spacings for the interior spans can be figured also. An arrangement of bars similar to those of the inner footing will give the required steel for the negative bending at each support (Fig. 195).

The maximum bond stress will occur at the first interior support. For the inch strip at greatest depth (24.25 ft.), the shear force $V = 0.575 \times 8.5p = 9.37h = 228$ lb. The steel arrangement of Figure 195 shows a steel area of 0.0315 sq. in. per in. at this support, or 0.161 bars. The bond stress equals

$$u = \frac{228}{0.161 \times 1.57 \times 0.87 \times 8.75} = 118 \text{ lb. per sq. in.} \qquad \text{Safe.}$$

Since the steel area and perimeter vary with the depth h, and the shear force V does also, the maximum bond stress is constant for all critical depths.

Temperature Steel. The wall contains horizontal steel as main reinforcement; therefore, the temperature steel will be placed vertically. The outer face is the more exposed to temperature changes and shrinkage.

$$\text{Outer face } A_s = 0.0025 \times 12 \times 8.75 = 0.26 \text{ sq. in. per ft. height}$$

Use $\frac{1}{2}$-in. rounds spaced at 9 in. For the inner face use $\frac{1}{2}$-in. rounds spaced at 18 in.

323. Counterfort. The counterfort tends to fail in tension since the wall is pushed outward and the inner footing downward (Fig. 196). The rotation tendency of the wall is usually greater than that of the footing, but this unbalanced couple must lift the whole mass of earth on the footing. The tension failure will occur on the section AB. Higher sections, such as CD, are not so heavily stressed, since the moment of the forces acting on the wall varies with the square of the wall depth, whereas the thickness of section BC or AB varies with the wall depth.

By using the dimensions of the counterfort from Figure 196, the earth pressure on the counterfort can be found.

$$\phi = 36°, \qquad z = 30°, \qquad \delta = 10°, \qquad \theta = 98.6°$$

$$E = \frac{100 \times 1.5h^2}{2} \times 0.350 = 26.25h^2$$

This pressure makes an angle of 38.6° with the horizontal. Point B of Figure 196 is at a depth $h = 23.63$ ft. The earth pressure on the counterfort above point B is

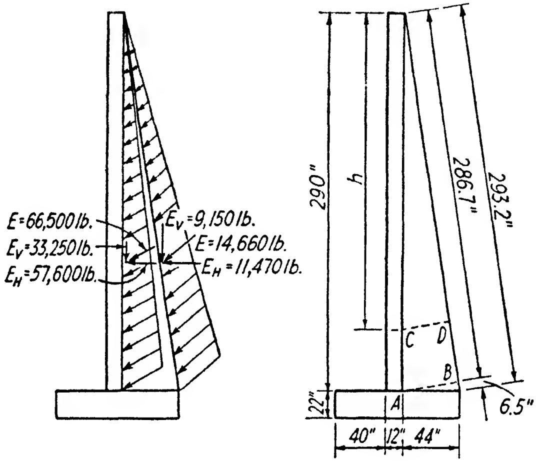


FIG. 196

$E = 26.25(23.63)^2 = 14{,}660$ lb. Its components equal $E_H = 11{,}470$ lb. and $E_V = 9150$ lb. Taking moments about A (Fig. 196) for a 10-ft. length of the wall:

$$\text{Wall } E_H = \frac{26.6}{2} \times 0.866 \times (24.17)^2 \times 8.5 = 57{,}000 \times \frac{24.17}{3} = \quad 459{,}000$$

$$\text{Counterfort} \qquad E_H = 11{,}470 \times \left(\frac{23.63}{3} + 0.54\right) = \quad \underline{96{,}600}$$

$$-555{,}600$$

$$E_V = 9{,}150 \times \tfrac{2}{3} \times 3.58 = \quad \underline{+21{,}900}$$

$$\Sigma M = -533{,}700 \text{ ft.-lb.}$$

The necessary depth of section AB equals

$$d = \sqrt{\frac{533{,}700 \times 12}{196 \times 18}} = 42.6 \text{ in.}$$

The section is 43.5 in. total depth. Allowing 3.5 in. to center of the steel, the actual $d = 40$ in. If the counterfort width is increased to 21 in., $M = 534{,}000$ ft.-lb., and $d = 39.4$ in., which is satisfactory. The change in width will only slightly affect the previous computations.

$$A_s = \frac{534{,}000 \times 12}{20{,}000 \times 0.87 \times 40} = 9.20 \text{ sq. in.}$$

Use six $1\frac{1}{4}$-in. square bars. Bond stresses are undoubtedly satisfactory. If it is desired to check them, it is best to return to the bond stress derivation (equation 27, Art. 54) to use

$$\text{Rate of change of bending moment } \frac{dM}{dh}, \quad \text{and} \quad \frac{dM}{dh} = u\Sigma ojd$$

The change of bending moment per inch of wall depth can be substituted satisfactorily. In this problem, $M = 37.84h^3$ ft.-lb. If $h = 24.17$ ft., $M = 534{,}000$ ft.-lb.; when $h = 24.08$ ft., $M = 528{,}400$ ft.-lb. The change of moment per inch

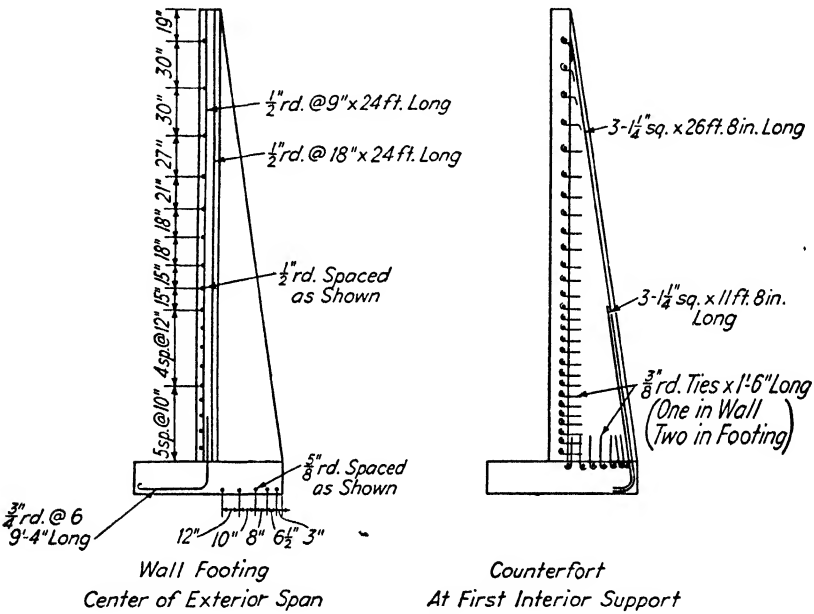


FIG. 197

depth equals $5600 \times 12 = 67{,}200$ in.-lb. and $u = 64$ lb. per sq. in.

The second row of steel can be cut off when the area required at some section CD is less than one half the area at section AB. Three $1\frac{1}{4}$-in. bars can care for a moment $M = 3 \times 1.56 \times 20{,}000 \times 0.87d = 81{,}500d$ in.-lb., where $d = 0.15 \times 12h - 3.5$. Equating the expressions for moment, $h = 16.9$ ft.

Ties. Since the wall and inner footing tend to pull away from their supporting counterfort, there must be tie bars to supply the tensile supporting forces.

Wall Ties. In the design of the wall (Art. 322) the shear force at the first interior support for the exterior span equals

$$V = 0.575lp = 0.575 \times 8.25 \times 23h = 109h \text{ lb. per ft. height}$$

where the clear span is now $l = 10.0 - 1.75 = 8.25$ ft.

If $\frac{3}{8}$-in. round horizontal bars are hooked over the wall steel at the counterfort, the tensile stress in these tie bars is

$$f_s = \frac{V}{ma_s} = \frac{109h}{0.11m} = 992\frac{h}{m}$$

where m = number of bars per foot height
a_s = area of the bar.

If the steel arrangement of Figure 195 and the corresponding maximum depth h are used, the greatest value of f_s = 11,700 lb. per sq. in. These bars must run at least 24 diameters = 9 in. into the counterfort. These $\frac{3}{8}$-in. ties will also serve at the other counterforts.

Footing Ties. At the first interior support the shear force on an inch strip equals

$$V = 0.575 \times 8.25(225 - 4.53x) = 1078 - 21.5x$$

where x = distance from inner edge (in.).

If two $\frac{3}{8}$-in. ties are hooked over each bar, one in each face,

$$f_s = \frac{V}{na_s} = 11{,}600 \text{ lb. per sq. in. (maximum)}$$

where n = number of ties per inch.

The complete steel layout is shown in Figure 197.

324. Cost Comparison. A comparative cost analysis is given in Table E for the three retaining walls just designed. It is assumed that the excavation charged to the wall is from outer face into the bank and that the excavation ends at the footing edge. The back fill is also charged to the wall but not the front fill. The gravity footing is assumed to be 2500-lb. concrete in order to give a wall which will weather well. The unit costs are

Excavation	4 cents per cubic foot
Back fill	2 cents per cubic foot
Concrete	
2500-lb., reinforced wall footings	37 cents per cubic foot
Steel	4 cents per pound
Forms	10 cents per square foot of side surface

The costs include no estimate for waste and are not necessarily typical for any particular locality. They do show the value of estimating the costs of comparative designs.

If the bank is higher, the counterfort wall is undoubtedly cheaper, If the depth is less, the cantilever wall will be cheaper. The designs are made for the end span. The interior spans will have less steel and be somewhat cheaper.

TABLE E. COST COMPARISON OF RETAINING WALLS

	Gravity		Exterior Span			
			Cantilever		Counterfort	
	Quantity per ft.	Cost	Quantity per ft.	Cost	Quantity per 10 ft.	Cost
Excavation, cu. ft.	405	16.20	132	5.28	1230	49.20
Fill, cu. ft.	53	1.06	83	1.64	822	16.44
Concrete: Wall			32.4		242	
Footing			13.3		147	
Counterfort			.....		78	
Total, cu. ft.	351	129.87	45.7	16.91	467	172.79
Forms: Wall			49.0		442	
Footing			3.3		37	
Counterfort			.....		132	
Total, sq. ft.	53	5.30	52.3	5.23	611	61.10
Steel: Wall			139		496	
Footing			81		360	
Counterfort			.....		635	
Total, lb.			220	8.80	1491	59.64
Cost						$359.17
Cost per foot length		$152.43		$37.86		$35.92

CHAPTER 13

THE REINFORCED CONCRETE STRUCTURE

The construction of a reinforced concrete structure involves the solution of engineering problems other than those of its design. A site is selected in view of business and transportation requirements. Soil conditions for some distance underground must be ascertained to determine allowable soil pressures and the type of foundation best suited to reduce settlement and to result in uniform settlement. The choice of wood-brick, steel frame, or reinforced concrete structure will be made on basis of cost, fire protection, and proposed use of the building. Column spacings and story heights will be influenced by necessary head room, machine clearances, and other pertinent considerations. Stairways, elevators, pipes, heating and ventilation all require space and often complicate the design.

The construction force must plan their plant to operate efficiently whether the location is in a city lot completely covered by the structure or in the country with ample room on all sides. In the midst of these decisions the designer starts work, with necessary revisions due to changes by the owner or architect, and is often forced to turn out plans under pressure to keep ahead of the field force.

In this chapter it is proposed to discuss some of the factors not yet mentioned, particularly those which affect the design of a structure as a whole.

FORMS

325. Forms. A requisite for proper construction of the design is that the members be made true to size without warping or distortion while the concrete is placed and hardens. The forms must be strong enough to support the plastic mass of concrete as well as the weight of the men, runways, buggies, etc. This entails a design of the forms for strength, and also for economy, since the form cost may be a quarter or a third of the total cost. The forms must be tight so that water and cement are not lost, and they must be so designed that they can be easily removed without injury, especially if they are to be used again. Above ground concrete surfaces should be left smooth enough to satisfy the functional and architectural requirements of the structure. In many structures forms can be used again economically in the floors above, either unchanged in size or cut down. This implies a stout form, careful stripping after use, cleaning, and possible repairs.

Forms are made of wood or steel. The steel form is usually limited to forms for circular columns and column capitals, and for slab and joist floors. Wood planks for slab and wall forms are usually tongued and grooved with a planed surface bearing against the concrete; often they are surfaced on all four sides. Square-edge planks are usually used for beam sides and column forms. Table F gives data for certain board and plank sizes.

TABLE F.* PROPERTIES OF BOARDS AND PLANKS, DRESSED FOUR SIDES (S4S)

Nominal Size in.	Dressed Size in.	Area sq. in.	Section Modulus on Edge cu. in.	Section Modulus on Flat cu. in.
1 x 4	$\frac{25}{32}$ x $3\frac{5}{8}$	2.83	1.71	0.37
1 x 6	$\frac{25}{32}$ x $5\frac{5}{8}$	4.39	4.12	0.57
1 x 8	$\frac{25}{32}$ x $7\frac{1}{2}$	5.86	7.32	0.76
2 x 4	$1\frac{5}{8}$ x $3\frac{5}{8}$	5.89	3.56	1.60
2 x 6	$1\frac{5}{8}$ x $5\frac{5}{8}$	9.14	8.57	2.48
2 x 8	$1\frac{5}{8}$ x $7\frac{1}{2}$	12.19	15.23	3.30
2 x 10	$1\frac{5}{8}$ x $9\frac{1}{2}$	15.44	24.44	4.18
2 x 12	$1\frac{5}{8}$ x $11\frac{1}{2}$	18.69	35.82	5.06
3 x 4	$2\frac{5}{8}$ x $3\frac{5}{8}$	9.52	5.75	4.16
3 x 6	$2\frac{5}{8}$ x $5\frac{5}{8}$	14.77	13.84	6.46
3 x 8	$2\frac{5}{8}$ x $7\frac{1}{2}$	19.69	24.61	8.61
3 x 10	$2\frac{5}{8}$ x $9\frac{1}{2}$	24.94	39.48	10.91
3 x 12	$2\frac{5}{8}$ x $11\frac{1}{2}$	30.19	57.86	13.21
4 x 4	$3\frac{5}{8}$ x $3\frac{5}{8}$	13.14	7.94	7.94
4 x 6	$3\frac{5}{8}$ x $5\frac{5}{8}$	20.39	19.12	12.32
4 x 8	$3\frac{5}{8}$ x $7\frac{1}{2}$	27.19	33.98	16.43
4 x 10	$3\frac{5}{8}$ x $9\frac{1}{2}$	34.44	54.53	20.81
4 x 12	$3\frac{5}{8}$ x $11\frac{1}{2}$	41.69	79.90	25.19

* Abstracted from "Wood Structural Design Data," National Lumber Manufacturer's Association.

Plywood panels are in common use to form extensive flat surfaces, such as floor slabs and walls. Panels may be obtained in thickness from $\frac{3}{16}$ to $1\frac{3}{16}$ in.; 48 in. wide by 96 in. long is a common size, but widths in even feet up to 8 ft. and lengths up to 16 ft. can be obtained on special order. The plywood panel gives smooth concrete surfaces and the joints are easily rubbed down when the formwork is well placed. In certain structures, where the added cost is justified, forms are faced on the inner side with special absorbent fiber boards. These remove some of the excess surface water and result in a more durable and impervious surface concrete. Ordinary form surfaces are brushed with mineral oil to prevent the concrete from adhering to the wood.

TABLE G.* ALLOWABLE STRESSES FOR WOOD FORMWORK

(POUNDS PER SQUARE INCH)

	Fiber Stress	Compression		Longitudinal Shear	Modulus of Elasticity
		$\parallel$ to grain	$\perp$ to grain		
Spruce and pine	1100	1000	250	110	1,100,000
Southern pine or Douglas fir	1900	1400	315	120	1,500,000
Plywood (Douglas fir)	2000	1500	325	120	1,600,000

* These allowable stresses are increased 20 per cent for wartime by the W.P.B. Specifications.

After the concrete has been poured a suitable time must elapse before the form is removed. This time varies with the type of member, probable immediate loads upon it, and with the curing requirements and temperature range. Columns and wall forms should remain in place at least 2 days, in cold weather at least 4 days, provided the beams and girders are still supported by shores. Slab forms and beam sides remain in place from 7 days to 2 weeks, the longer interval applying to long spans. Slabs are often supported by posts or shores after the forms are removed. Beam and girder bottoms are supported for 10 days to 3 weeks, depending on the load, span, and weather conditions. If test specimens of the concrete are taken as it is poured, forms can be removed when the concrete of the member in question has a strength in excess of the stresses used in design, or a 50 per cent excess of the dead and construction loads.

326. Wall Forms. The sheathing of wall forms (Fig. 198) is made of planks or plywood panels. These are held in place by vertical studs. Sometimes a sill is placed on the ground bearing against the lowest plank and the studs rest on this sill. Low walls are plumbed and held in place by braces nailed to the studs. Higher walls must be kept true by bolts, or ties, to prevent the bulging out of the sheathing as the concrete is poured; there must also be spacers to keep the two forms

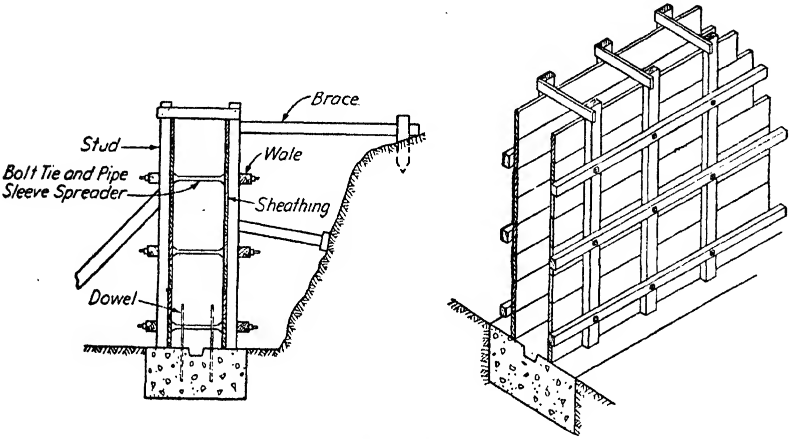


FIG. 198

the proper minimum distance apart. A combination of pipe sleeve and tie bolt is often used to perform both functions, and there are also many special types of ties in general use.

ILLUSTRATIVE PROBLEM 62

327. Design of a Wall Form. Design a wall form to resist a 5-ft. head of wet concrete exerting a lateral pressure of 140 lb. per sq. ft.

Sheathing. Assume 1-in. spruce or pine planks as continuous beams supported by the studs. Taking 1 in. of height of this planking and checking the interior spans,

$$M = \frac{wl^2}{11} = \frac{fbt^2}{6} = 5 \times \frac{140l^2}{12} \times \frac{12}{11} = \frac{1100}{6} \times 1 \times \left(\frac{25}{32}\right)^2$$

$$l = 1.32 \text{ ft.}$$

Use 1-in. plank with studs spaced at 16 in. The studs are supported by the tie-bolts and wales. Assuming the studs to be continuous beams of 2 x 4 section,

$$(5 \times 140 \times \tfrac{16}{12})l^2 \times \tfrac{12}{11} = 1100 \times 3.56$$

$$l = 1.96 \text{ ft.}$$

Use 2 x 4 studs with wales and bolts spaced at 22 in., except near the top of the wall where there cannot be a 5-ft. head of wet concrete. The wales are also continuous beams supported by the tie bolts on a stud span of 16 in., and spaced 22 in. apart. Assuming a 2 x 4 wale with an average pressure equal to the maximum of

$5 \times 140 = 700$ lb. per sq. ft., the force carried by the supporting ties equals $\frac{22}{12} \times 700 \times \frac{16}{12} = 1715$ lb. With a tensile strength of 20,000 lb. per sq. in., the necessary area is 0.09 sq. in. Use $\frac{3}{8}$-in. tie rods in a $\frac{7}{16}$-in. hole.

The wale is loaded only by the ties. As long as the level of the wet concrete is constant all ties have the same force and elongate the same amount, thereby causing no bending in the wale. Let us assume that a buggy load of concrete is dropped in opposite stud 5 to increase the depth of wet cement to 5 ft. at this stud, although the general level in the wall is 18 in. less. The pull in the tie in stud 5 is 1715 lb. and in the adjacent ties is 1200 lb. for 3.5 ft. of wet concrete. There will be bending in the wale due to the difference of 515 lb. Assuming a span of 32 in. between studs 4 and 6, we have a beam loaded with a concentrated force of 515 lb. on a span of 32 in. with ends fixed, or nearly so. The fixed-end moment for such a loading is $\frac{Wl}{8}$; let let us assume an end moment of $\frac{Wl}{10}$ with a corresponding positive moment of $\frac{3Wl}{20}$. The tie-hole reduces the wale width at stud 5 to $1\frac{5}{8} - \frac{7}{16} = 1.19$ in. The required section modulus $= \frac{3 \times 515 \times 32}{20 \times 1100} = 2.24$ (in.)^3. A 2 x 4 wale with a $\frac{7}{16}$-in. hole will give this section modulus.

328. Slab and Beam Forms. The slab and beam forms are supported by a system of shores stout enough to support the whole weight of the

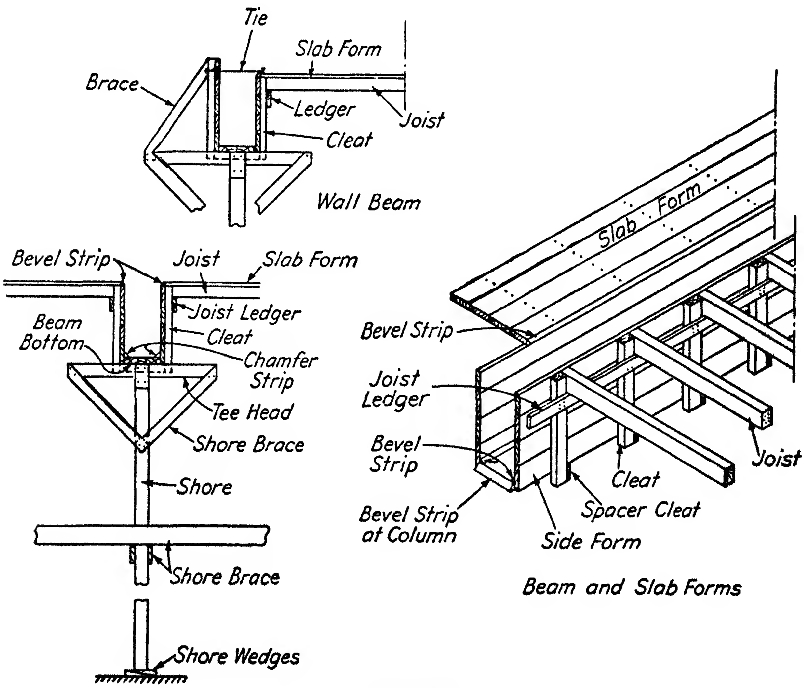


FIG. 199

floor. They should be so installed that the beam sides and slab forms can be removed while the beam bottoms are still shored. Posts can be replaced under the slab, if necessary. Figure 199 shows a possible system of slab and beam forms with supporting shores. There are other methods of supporting the beam sides by the use of bolts or clamps.

ILLUSTRATIVE PROBLEM 63

329. Design of Slab and Beam Forms. *Slab.* Design forms for the interior panels of the floor system of Figure 61 (Art. 121). The slab is 4 in. thick and the beam stem is 8 in. wide and 13 in. deep (Fig. 81, Art. 135). The beams are spaced 9 ft. 8 in. on centers. The joists under the slab form are supported by the joist ledger and can be nailed to the beam-side cleat. Their span can be taken as 8 ft. 4 in. The load per square foot of slab surface is taken as

Construction load of men and equipment	70
4-in. slab	50
Formwork	10
	130 lb. per sq. ft.

With a maximum moment of $\frac{wl^2}{8}$ the spacing of 2 x 8 Douglas fir joists is

$$130 \times b \times (8\tfrac{1}{3})^2 \times \tfrac{12}{8} = 1900 \times 15.23 \qquad b = 2.14 \text{ ft.}$$

Use 24 in. At this spacing of 24 in. the maximum deflection of the joists is about 0.33 in., and the allowable deflection $\frac{l}{360} = 0.28$ in., but after the construction loads move on, the deflection will be well under 0.28 in. No intermediate shores will be used. Maximum longitudinal shear $v = \frac{3V}{2bh} = 132$ lb. per sq. in., but this will again be reduced by the removal of the construction loads.

Using $\frac{1}{2}$-in. plywood panels for the slab form and a maximum moment of $\frac{wl^2}{11}$, the fiber stress figures

$$130 \times (2)^2 \times \tfrac{12}{11} = \tfrac{1}{6} \times 12 \times (\tfrac{1}{2})^2 \times f$$

$$f = 1140 \text{ lb. per sq. in.} \qquad \text{This is safe.}$$

Beam Forms. The load coming to the beam shores equals

Slab $9\frac{2}{3} \times 130$	= 1260
Beam stem (8×13)	= 110
Forms	30
	1400 lb. per ft. of beam length

The shore will be supported on a sill to spread its load on the slab below. It will be wedged in place on this sill. It will be braced in both directions by 1 x 4 or 1 x 6 planks placed high enough to allow passage underneath. This bracing is not sturdy enough to reduce the unsupported length. If the story height is 11 ft., the unsupported length equals 132 − 17 − 3 = 112 in. Assuming a 4 x 4 spruce post, the ratio $\frac{h}{d}$ = 30.8. This is a long column and the allowable stress is given by the Euler formula:

$$f = \frac{0.274Ed^2}{h^2} = 318 \text{ lb. per sq. in.}$$

The maximum load equals 318 × 13.14 = 4180 lb. and the load carried for a shore spacing l equals $1400l$ = 4180 or l = 3 ft.

The beam bottom will be a Douglas fir plank supported by the shores and stiffened by nailing to the beam sides. If the assistance of the beam side is disregarded, the necessary section modulus Z will be

$$1400 \times (3)^2 \times \tfrac{12}{11} = 1900Z \qquad Z = 7.23 \text{ (in.)}^3$$

A 3 x 8 plank is satisfactory. If the shore spacing is reduced to 2 ft., a 2 x 8 plank can be used.

Note. The beam stem is only $7\frac{1}{2}$ in. wide. Designers should compute the concrete section by using the true width of the beam bottom. Those concerns that use the nominal 8-in. width are reducing the factor of safety or assuming that the full live load is not applied until the concrete strength exceeds its 28-day value.

The beam sides will be made of 1-in. stock. Assuming 2 x 4 cleats spaced at 2 ft., the pressure of wet concrete on the lowest inch of the stem will be $140 \times \frac{17}{12}$ = 200 lb. per sq. ft. The maximum plank fiber stress in the lowest inch is

$$\frac{200}{12} \times (2)^2 \times \frac{12}{11} = \frac{1}{6} \times \left(\frac{25}{32}\right)^2 f \qquad f = 715 \text{ lb. per sq. in.}$$

It is not necessary to check the cleat dimensions. The joist ledger is supported by these cleats and is loaded by the joist which touches one side of the cleat. Assume a 2 x 4 ledger; the fiber stress will be very low and the shear stress v equals

$$v = \frac{3}{2}\frac{V}{bh} = \frac{3}{2} \times \frac{130 \times 8.33}{2 \times 5.89} \times \frac{22.38}{24} = 128 \text{ lb. per sq. in.}$$

where the supporting force of the joist equals $R = 130 \times 8.33 \times \frac{1}{2}$ and the maximum shear on the ledger equals $R \times \frac{22.38}{24}$. No allowance has been made for reduction of V by nailing the joist to the cleat. This load acts between joist and ledger as compression perpendicular to the grain. The compressive strength equals $\frac{542}{1\frac{5}{8} \times 1\frac{5}{8}}$ = 206 lb. per sq. in. This is safe, if the assistance given by nailing the joist to the cleat is again disregarded.

330. Column Forms. Rectangular column forms are made of wood. The forms on two opposite faces are made the exact width of the column (even plank widths) and the other two forms overlap for a tight fit. There are many systems of cleats, bolts, clamps, wedges, and the like in use. Figure 200 shows one using cleats, bolts, and wedges. The forms should be stout enough for use again on the floors above, if the story heights do not vary too much. The form can be cut down for width as the column width reduces. At the top of the form cuts are made big enough to admit the ends of the beam and girder forms. A cleanout opening should be left on two opposite faces at the bottoms to remove dirt, shavings, and loose ends after the steel is placed and before the concrete is poured.

Steel forms are usually employed for circular columns and those with capitals.

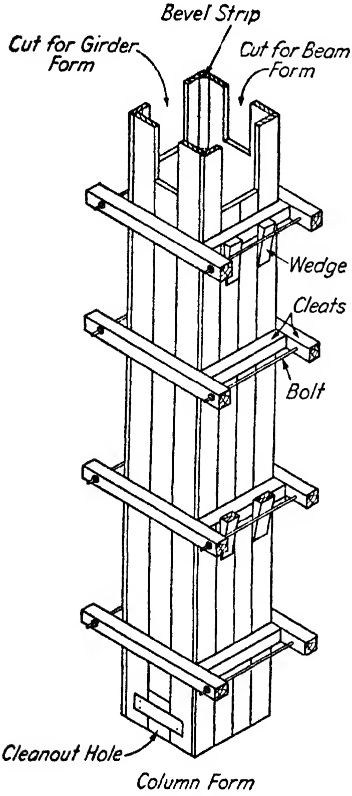


FIG. 200

ILLUSTRATIVE PROBLEM 64

331. Design of Column Forms. By assuming an average lateral pressure equal to 6 ft. of wet concrete at 140 lb. per sq. ft., and adopting $1\frac{1}{4}$-in. yellow pine planks, the spacing of cleats can be found.

$$M = \frac{wl^2}{11} = \frac{fbt^2}{6} =$$

$$\frac{6 \times 140}{12} l^2 \times \frac{12}{11} = \frac{1900}{6} \times 1 \times \left(\frac{17}{16}\right)^2$$

$l = 2.17$ ft. Use 26-in. spacing.

It is probable that a column may be filled more rapidly with wet concrete than a wall, so 6 ft. of wet concrete is used here, instead of the 5 ft. of Problem 62. This cleat spacing will be constant until within 6 ft. of the under side of the girder where column pouring will stop. Greater spacings can be used as the girder bottom is approached. Even the overlapping cleat, which is supported by the bolts, will be safe at this spacing for a column width up to 36 in., if the cleat is made 2 in. wide by 4 in. deep. Some designers prefer a 4 x 4 cleat since the area at the support is reduced by the bolt hole. The column form should be plumbed and braced after the steel assembly is installed.

PLANS AND DETAILS

332. Plans. The framing plan shows the location of columns, footings, beams, girders, and walls for the load-bearing members of a structure. If there are no unusual restrictions due to head room, machine clearances, or aisles, several different framing plans may be possible. These may involve different schemes of column arrangement as well as different beam and girder floor systems, or a comparison between beam and girder with flat slab or ribbed floors. The most economical system should be the designer's objective but this cannot be attained without comparative cost designs using approximate determinations of sizes plus costs and quantities from the estimating department.

Once the framing plan has been adopted, each individual member is designed and detailed, showing an elevation with the steel arrangement for beams and columns, plus typical cross sections. A plan view is given also for slab steel distribution. Drawings of forms and form layouts are made with schedules of lumber needed. Schedules of steel are also needed with a detail of each type of bar, showing bends and lengths. Extra reinforcement around holes, under machinery, and so on, must have special detail drawings.

Chairs, spacer bars, and collars are required to hold the steel in its proper position until the concrete is poured. These are not shown in the plans and the judgment of the steel foreman in the field usually determines where they shall be used. Splices of the column steel are shown for length on the elevations but the necessary offset to give clearance with the steel in the column above is left for the steel foreman.

The detail elevations of individual members do not completely show the complexity of the steel placement at the junctions of several members. Thus, a junction of an upper and lower column with an east-west continuous girder and a north-south continuous beam gives a delicate problem in passing the various reinforcements through the junction without interference. Such junctions should be studied carefully, even if special detail drawings are not supplied to the field force.

333. Construction Joints. It is seldom possible to pour a floor in one operation. The work must be laid out to complete what can be done in one day. Also, long structures need provision for the absorption of the expansion and contraction of temperature changes. This is accomplished by stopping the day's work at a construction joint, which is a section through beams and slabs. The portion completed the previous day will undergo part of its shrinkage before the next day's pouring starts. Compressive forces can be transmitted through a construction joint by the pressure on opposite sides. Tensile forces can only be carried by the steel passing through the joint, and vertical shear resistance is much reduced. For these reasons construction joints are usually made at sections of zero shear force, which is the section of maximum positive bending moment near mid-span. Elsewhere special joints with keyways should be used to increase the shear resistance. Sometimes special joints are inserted of metal or some plastic material to allow for the opening or closing due to temperature and shrinkage.

Column and wall joints are made at the upper end immediately under the floor beams. For the most part compressive stresses are transmitted through the joint, but a horizontal joint should be carefully cleaned to remove the laitance scum that settles on the surface of wet concretes. These columns and walls are usually poured several days before the floor above and shrinkage will be largely completed before the floor is poured.

334. Economy of Design. The choice of an economical concrete mix has been discussed for floor systems (Art. 66) and for columns (Table C, Art. 237). Sample cost comparisons have also been made for footings (Table D, Art. 275) and retaining walls (Table E, Art. 324). There are also certain general practices that tend to decrease costs.

Members poured in wood forms should have such sectional dimensions that stock widths can be used. There is an extra charge for ripping out the proper plank width in the mill.

Reinforcement is sold at a base price applying to $\frac{3}{4}$-in. round bars or larger. Bars smaller than $\frac{3}{4}$ in. are sold at an increasing price above the base; therefore $\frac{1}{4}$-in. round bars may cost 20 to 25 per cent more than the base price.

It is often advantageous to use the same stem depth for all beams on the same floor. The forms for beam sides and the supporting shores will be identical for all beams and it will be easier to run shafting and piping under the beams. If beams are supported by girders, the stem of the beam is usually shallower than that of the girder, but different depths also prevent interference of the bottom reinforcement.

STAIRWAYS AND SPECIAL BEAM SECTIONS

335. Angle Beams. Wall beams in a floor system frequently consist of a beam stem supporting a slab coming in on one side only. Beams at the outer edge of a balcony or those framing openings are also angle beams. A.C.I. Article 705 states that angle sections are designed similarly to tee beams, using a flange width $b = (b' + 6t)$, but not exceeding $b = \left(b' + \frac{l}{12}\right)$. Such a section violates the fundamental beam theory requirement that the section shall be symmetrical about the vertical plan of loading. It is also true that the external loads produce torsional as well as bending moments. In the past it has been customary to design wall beams for bending only, the torsional moments and the non-symmetrical section being disregarded. A method of approximating the torsional stresses has been given in Problems 25 and 26 (Chapter 8).

An endeavor is sometimes made to design the *transformed* section for bending so that the resultant tension and compression forces act in a plane more nearly vertical. This is accomplished by adding compression steel which is offset to the outside edge of the wall beam. The tension steel is offset toward the inside face. The object is to have the resultant of the compression forces in the concrete flange and the compression steel act vertically above the resultant tensile force.[1] The following problem illustrates such an attempt.

ILLUSTRATIVE PROBLEM 65

336. Wall Girder. The wall girder of Problem 25 has the following dimensions: $b' = 12$ in., $b = 36$ in., $h = 26$ in., $t = 4$ in., depth to positive steel $d = 22$ in., $f'_c = 2000$ lb. per sq. in. The positive bending moment $M_p = 880{,}800$ in.-lb. A trial computation is made with one 1-in. round bar as compression steel and four 1-in. round bars as tension steel, placed as shown in Figure 201. By A.C.I. Article 706

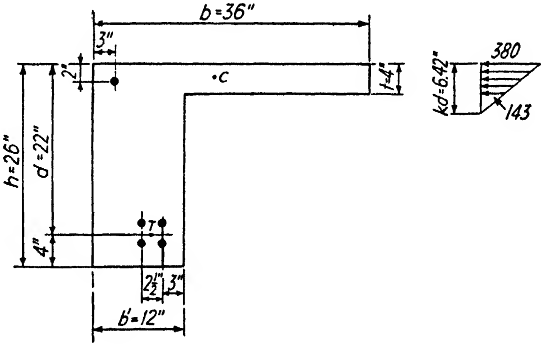


FIG. 201

the stress in the compression steel is taken as $2f'_s$. If the concrete is assumed to be in compression only over the flange depth, the center of gravity of the transformed area can be formed by taking moments of the areas about the top of the flange.

$$
\begin{array}{rcrcrcr}
36 \times 4 & = & 144 & \times & 2 & = & 288 \\
2 \times 14 \times 0.785 & = & 22 & \times & 2 & = & 44 \\
15 \times \ \ 4 \times 0.785 & = & \underline{47} & \times & 22 & = & \underline{1034} \\
A & = & 213 \text{ sq. in.} & & M & = & 1366 \text{ cu. in.}
\end{array}
$$

Center of gravity = 6.42 in. from top

[1] See *Reinforced Concrete Design,* Sutherland and Reese, p. 285.

The moment of inertia about the center of gravity equals

$$\begin{aligned}
\frac{36 \times (4)^3}{12} &= \quad 192 \\
36 \times 4 \times (4.42)^2 &= \;\; 2{,}817 \\
22 \times (4.42)^2 &= \quad 430 \\
47 \times (15.58)^2 &= \underline{11{,}438} \\
I &= 14{,}877 \text{ (in.)}^4
\end{aligned}$$

$$f = \frac{My}{I} = \frac{880{,}800}{14{,}877}y = 59.2y$$

$$f_c = 59.2 \times 6.42 = 380 \text{ lb. per sq. in. (maximum)}$$

$$f_c = 59.2 \times 2.42 = 143 \text{ lb. per sq. in. (minimum)}$$

$$2f'_s = 59.2 \times 15 \times 4.42 \times 2 = 7850 \text{ lb. per sq. in.}$$

$$f_s = 59.2 \times 15 \times 15.58 = 13{,}900 \text{ lb. per sq. in.}$$

Compression force in concrete:

Uniform $= 143 \times 36 \times 4 = 20{,}570$ lb., 2 in. down, 18 in. from outer face

Uniformly varying $= \frac{237}{2} \times 36 \times 4 = 17{,}060$ lb., 1.33 in. down, 18 in. from outerface

Force in compression steel:

$0.785 \times 7850 = 6{,}160$ lb., 2 in. down, 3 in. from outer face

Total compression force $= 43{,}800$ lb., acting 1.74 in. down

Force in tensile steel $= 4 \times 0.785 \times 13{,}900 = 43{,}800$ lb.

Taking moments about the outer face,

$$20{,}570 \times 18 + 17{,}060 \times 18 + 6160 \times 3 = 43{,}800x_c$$

$$x_c = 15.9 \text{ in.}$$

The resultant compression force C acts 15.9 in. from the outer face while the resultant tensile force T acts $12 - 4.25 = 7.75$ in. The plane of these two forces makes an angle with the vertical whose tan $= \dfrac{15.9 - 7.75}{22 - 1.74} = 0.402$, or an angle of 21.9°. If no compression steel were used and the tension steel were spread uniformly over the stem width in one row, this tangent would be tan $= \dfrac{18 - 6}{23 - 1.75} = 0.564$, or an angle of 29.4°. The compression stresses in the concrete are very low, as is to be expected with angle and tee beams under positive bending, so the compression steel is only used to help reduce the angle of inclination of the plane of the internal forces. By using the reinforcement of Figure 201 the moment of resistance at this section is found to have components of

Vertical bending moment $= 43{,}800 \times 20.26 = 880{,}800$ in.-lb.

Horizontal bending moment $= 43{,}800 \times 7.75 = 340{,}000$ in.-lb.

337. Stairways. A reinforced concrete stairway is often supported by beams on each side of the stair. Each thin riser and tread may be considered to be an angle beam supported by the side beams. The design of a single riser and tread as an angle beam requires the assumption that the adjacent units deflect the same amount as this one.

Other stairs are designed as inclined slabs whose thickness is the minimum thickness of the sawtooth section, the projecting riser and tread being an added dead load. The live and dead loads are inclined to the axis of the slab and give direct and bending stresses. However, this exact procedure is usually not followed for design. The slab is designed as though it were a horizontal slab with a span equal to the horizontal projection between the supporting beams. The vertical load is used and the computed thickness must be equal to or greater than the minimum *vertical* section through the slab. In case the stairway is supported at the far end of the landing, this landing is included as part of the substitute horizontal slab. Short steel bars should be placed at the supports for negative bending moments, in addition to the continuous positive steel running from support to support.

338. Holes and Openings. Openings of considerable size in a floor should be framed by beams, even in flat slab construction. Pipe holes, manholes, or small hatchways do not require such framing, if the hole does not interrupt much slab reinforcement. At the corners of rectangular holes local stress concentrations tend to produce cracks and it is advisable to place extra steel at these corners consisting of short bars placed at 45° with the axes of the rectangle.

Pipe holes may also be necessary in the stems of beams. If possible these should be placed in the upper part of the stem in order not to reduce the compression area for either positive or negative bending. In the center of the span the hole can be lower as the compression area is in the slab. Even though the hole is circular, local concentrations of stress will give a maximum shearing stress about three times the computed stress for a solid stem, so extra 45° and 135° steel is desirable, adjacent to the hole.

339. Wartime Allowable Stresses. The War Production Board's Emergency Specifications for Reinforced Concrete Design were based on the 1941 A.C.I. Code in general. The object of these specifications is to reduce the amount of steel used as reinforcement. Therefore, it is recommended that plain concrete be used wherever possible, even though the plain concrete design does not give the most economical footing or retaining wall. The reinforced beam section should be large enough so that no compression steel is needed. For column design, it is recommended that tied columns be used instead of spiral, that the longitudinal steel should not exceed 2 per cent, and that high-strength concrete be adopted.

To force the use of larger sections the maximum fiber stress in beams is reduced to $0.35f'_c$, instead of $0.45f'_c$. Shear and diagonal tension stresses are not changed but bond stresses are increased 10 per cent.

The tensile steel stress in the structural grade is increased 10 per cent to 20,000 lb. per sq. in., whereas for the other grades it is increased 20 per cent to 24,000 lb. per sq. in. The reduction of fiber stress in the concrete and increase in the steel will give larger sections and, also, less steel for the section chosen.

CHAPTER 14

THE STRUCTURE AS A RIGID FRAME

A principal advantage of reinforced concrete construction is the monolithic, continuous frame that is a result of good design. The previous chapters have considered the methods of design of the portions of such a frame that may be designated slabs, beams, columns, footings, and so on. Due regard has been given to the fact there is a certain restraint at the junction of slab with beam, a beam with column, a column with footing. For the most part conventional restraining moments have been assumed to act, but there are also many cases where the frame should be considered as a whole in order to estimate these restraining moments. Unequal spans of a continuous member or marked difference in the sizes of the supporting members are two of many possible cases not covered by conventional moment coefficients.

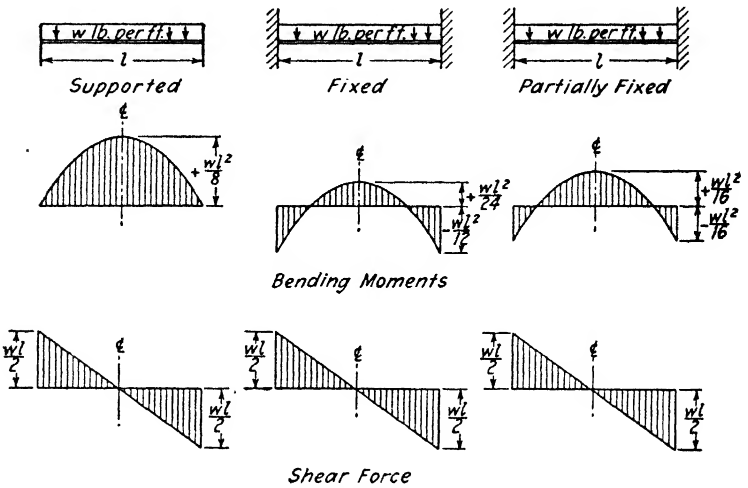
In the last 15 or 20 years great advances have been made in the analysis of statically indeterminate structural problems. These solutions may be divided into four general types, namely:

1. Continuous beams by the three-moment equation.
2. Slope-deflection method.
3. Moment-distribution method.
4. Strain-energy or work solutions.

These methods are all covered fully in modern texts of applied mechanics, or structures, and the basic equations will not be derived in this text. Their application to problems of reinforced concrete design will be discussed in this chapter.

340. Restraint of One-Span Beams. Let us first discuss a beam of a single span loaded with a uniformly distributed load of w pounds per foot (Fig. 202*a*). If the beam is supported at the ends, the bending moment diagram is a parabola with a maximum positive moment in the center of $M_p = \frac{wl^2}{8}$ and a moment of zero at the support. The shear force diagram varies in a straight line from $+\frac{wl}{2}$ to $-\frac{wl}{2}$ and is zero at the point of maximum positive bending moment.

If the same beam has fixed ends, the bending moment diagram is the same parabola, but the maximum positive bending moment is only $M_p = +\frac{wl^2}{24}$ and there is at the support a negative bending moment $M_n = -\frac{wl^2}{12}$ (Fig. 202*b*). The point of inflection ($M = 0$) occurs at $0.215l$. It will be noticed that the sum of the maximum negative and



(*a*) (*b*) (*c*)

FIG. 202

positive bending moments is $\frac{wl^2}{8}$. The parabola is identical with Figure 202*a* except that its base line ($M = 0$) has been shifted up two-thirds the ordinate $\frac{wl^2}{8}$. By fixing the ends the same load is carried with only two-thirds the maximum numerical bending moment and hence two-thirds the fiber stress that the simply supported beam has; or, better still, a smaller beam will carry the same load. The shear force diagram is unchanged.

The beam may also be partially fixed with a maximum negative bending moment somewhere between $\frac{wl^2}{12}$ and zero. Figure 202*c* has been computed for the case where the partial restraint gives a negative moment of $-\frac{wl^2}{16}$. The maximum positive moment at the center will figure $+\frac{wl^2}{16}$, and again the sum of the maximum positive and negative moments equals $\frac{wl^2}{8}$. The base line ($M = 0$) for the parabola has been shifted up half the ordinate $\frac{wl^2}{8}$. The shear force diagram is unchanged.

The general statement can be made that, as long as the beam is symmetrical for loading and restraint, the shear force diagram is unchanged and the bending moment diagram can be formed from a single curve by suitably shifting the base line for zero moment. This is utilized in Diagram 5 in the Appendix. The total height of 100 units is equivalent to $M = \frac{wl^2}{8}$. The total length of 100 units equals the span l. The base lines ($M = 0$) have been located for the moments specified by the A.C.I. Code for uniformly distributed moments.

CONTINUOUS BEAMS—THREE-MOMENT EQUATION

341. Three-Moment Equation. The solution for the moments at the supports of a continuous beam by the *three-moment equation* neglects the stiffness of the supporting members. A concrete beam *resting* on a support, such as a brick wall or steel column, fulfills this assumption but, if it is poured integrally with a concrete column or wall, the solution is an approximate one. The three-moment equation consists of the summation of expressions for the slopes of the beam to the left and right of a given support (such as support 2 in Fig. 203). This summation must equal zero. In equation 245 subscript 1 refers to properties of the beam to the left of the chosen support (called B) and subscript 2 refers to properties of the beam to the right of support B. Support A is the support immediately to the left of support B, and support C is immediately to the right. If the beam in span AB has a different moment of inertia than the beam in span BC, the three-moment equation is

$$\frac{1}{6EI_1}\left[\frac{-6EI_1v_A}{l} + (M_A + 2M_B)l_1 + \frac{6F_{AB}x_A}{l_1}\right] + \frac{1}{6EI_2}\left[\frac{-6EI_2v_C}{l_2} + (2M_B + M_C)l_2 + \frac{6F_{BC}x_C}{l_2}\right] = 0 \quad (245)$$

where E = modulus of elasticity of the material
I = moment of inertia of the beam
l = span of the beam

v = deflection of supports A or C relative to support B
M = moment at the support designated by the subscript
F = area of moment diagram of beam denoted by the subscript
x = distance from far support (A or C) to the center of gravity of this moment diagram.

All interior spans are assumed to be simply supported. Table H lists the moment area F and product $\frac{Fx}{l}$ for certain common beam loads.

If the two spans have the same moment of inertia, equation 245 becomes

$$-\frac{6EIv_A}{l_1} - \frac{6EIv_C}{l_2} + M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 + \frac{6F_{AB}x_A}{l_1} + \frac{6F_{BC}x_C}{l_2} = 0 \quad (246)$$

Equation 245, or 246, is called the *three-moment equation* because the expression deals with the bending moments at three adjacent supports with the deflection and loading terms known. The deflections v_A and v_C imply that the supporting columns at A and C shorten more than the column at B by the amounts v_A and v_C. These shortenings can be estimated by determining the strains in each column and multiplying by the column length to get the total shortening.

ILLUSTRATIVE PROBLEM 66

342. Shear and Bending Moment Diagrams for a Continuous Beam. Given a continuous beam of three spans (Fig. 203), fixed at support 1 and hinged at support 4.

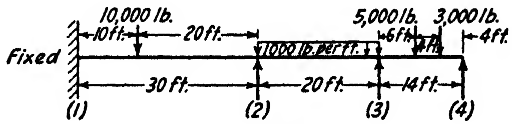


FIG. 203

The modulus of elasticity of the concrete will be taken as 2,000,000 lb. per sq. in., and the moment of inertia of the cross section as 14,000 (in.)^4 for each span. Column 1 shortens 0.05 in.; column 2, 0.25 in.; column 3, 0.20 in.; and column 4, 0.10 in.

Determine (*a*) the supporting forces; (*b*) the complete shear force diagram; (*c*) the complete bending moment diagram.

Taking the *origin B at support 1*. The end support is assumed to be fixed. Therefore, the change of slope i_1 equals zero. The deflection v_2 of support 2 relative to

1 is $v_2 = -0.20$ in. Using the expression for slope just to the right of support 1, which is the second parenthetical term of equation 245:

$$0 = \frac{-6\times2{,}000{,}000\times14{,}000(-0.20)}{30(12)^3} + (2M_1+M_2)30 + \frac{6\times10{,}000\times10\times20\times50}{6\times30}$$

or

$$0 = 2M_1 + M_2 + 132{,}700 \tag{247}$$

Origin B at support 2. There is a beam span each side of support 2. Relative to support 2, the deflection $v_1 = +0.20$ in. and $v_3 = +0.05$ in. Using the three-moment equation (246):

$$0 = \frac{-6\times2{,}000{,}000\times14{,}000}{(12)^3}\left(+\frac{0.20}{30}+\frac{0.05}{20}\right) + 30M_1 + 2M_2(30+20) + 20M_3 + \frac{6\times10{,}000\times10\times20\times40}{6\times30} + \frac{6\times1000(20)^3}{24}$$

$$0 = 3M_1 + 10M_2 + 2M_3 + 377{,}500 \tag{248}$$

Origin B at support 3. Relative to support 3 the deflection $v_2 = -0.05$ in. and $v_4 = +0.10$ in. Also note that $M_4 = 0$.

$$0 = \frac{-6\times2{,}000{,}000\times14{,}000}{(12)^3}\left(-\frac{0.05}{20}+\frac{0.10}{14}\right) + 20M_2 + 2M_3(20+14) + \frac{6\times1000(20)^3}{24} + \frac{6\times5000\times6\times8\times22}{6\times14} + \frac{6\times3000\times10\times4\times18}{6\times14}$$

$$0 = 5M_2 + 17M_3 + 520{,}000 \tag{249}$$

Solving the three simultaneous equations 247, 248, and 249,

M_1	M_2	M_3	
+2	+1		= −132,700
+3	+10	+2	= −377,500
	+5	+17	= −520,000

$M_1 = -59{,}000$ ft.-lb.; $M_2 = -14{,}800$ ft.-lb.; $M_3 = -26{,}250$ ft.-lb.; $M_4 = 0$

343. Shear Force and Bending Moment Diagrams. The loads and supporting couples and forces for each individual span are shown in Figure 204. When the supporting forces are computed the results are

$V_1 = 8{,}140$ lb. $\qquad V''_2 = 9{,}430$ lb. $\qquad V''_3 = 5{,}590$ lb.

$V'_2 = 1{,}860$ lb. $\qquad V'_3 = 10{,}570$ lb. $\qquad V_4 = 2{,}410$ lb.

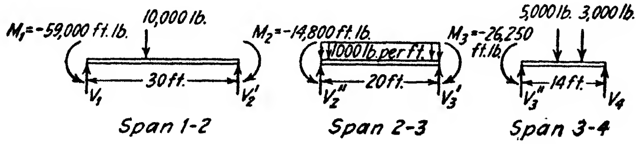
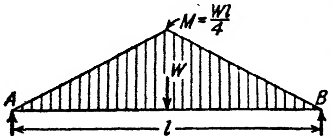
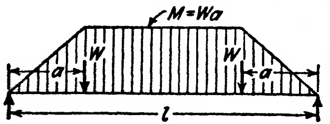
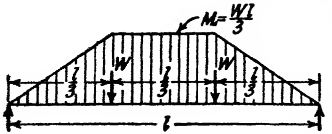
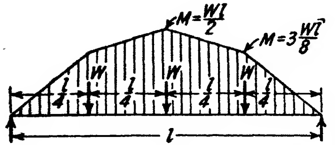
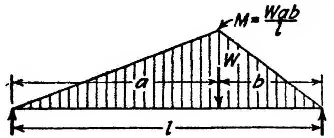
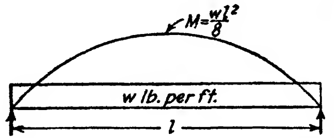
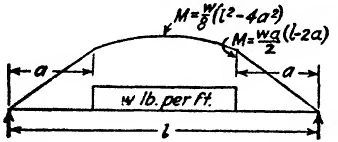
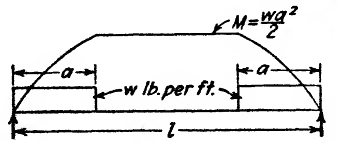


FIG. 204

TABLE H. BENDING MOMENT AREAS

Beam Load and Bending Moment Diagram	Moment Area F	Moment of F About End A $\frac{Fx_A}{l}$	Moment of F About End B $\frac{Fx_B}{l}$	Fixed End Moment C_{AB}	Fixed End Moment C_{BA}
	$\frac{Wl^2}{8}$	$\frac{Wl^2}{16}$	Same	$\frac{Wl}{8}$	Same
	$Wa(l - a)$	$\frac{Wa(l - c)}{2}$	Same	$\frac{Wa(l - a)}{l}$	Same
	$\frac{2}{9}Wl^2$	$\frac{Wl^2}{9}$	Same	$\frac{2}{9}Wl$	Same

	$\frac{5}{16}Wl^2$	$\frac{5}{32}Wl^2$	Same	$\frac{5}{16}Wl$	Same
	$\frac{Wab}{2}$	$\frac{Wab}{6l}(l + a)$	$\frac{Wab}{6l}(l + b)$	$\frac{Wab^2}{l^2}$	$\frac{Wa^2b}{l^2}$
	$\frac{Wl^2}{12}$ or $\frac{wl^3}{12}$	$\frac{Wl^2}{24}$ or $\frac{wl^3}{24}$	Same	$\frac{Wl}{12}$ or $\frac{wl^2}{12}$	Same
	$\frac{W}{12}(l^2 + 2al - 2a^2)$ or $\frac{w}{12}(l^3 - 6a^2l + 4a^3)$	$\frac{W}{24}(l^2 + 2al - 2a^2)$ or $\frac{w}{24}(l^3 - 6a^2l + 4a^3)$	Same	$\frac{W}{12l}(l^2 + 2al - 2a^2)$ or $\frac{w}{12l}(l^3 - 6a^2l + 4a^3)$	Same
	$\frac{Wa}{12}(3l - 2a)$ or $\frac{wa^2}{6}(3l - 2a)$	$\frac{Wa}{24}(3l - 2a)$ or $\frac{wa^2}{12}(3l - 2a)$	Same	$\frac{Wa}{12l}(3l - 2a)$ or $\frac{wa^2}{6l}(3l - 2a)$	Same

The complete shear force and bending moment diagrams are assembled in Figure 205. The supporting forces are

$$
\begin{aligned}
R_1 & & &= 8{,}140 \text{ lb.}\\
R_2 &= 1{,}860 + 9{,}430 &&= 11{,}290 \text{ lb.}\\
R_3 &= 10{,}570 + 5{,}590 &&= 16{,}160 \text{ lb.}\\
R_4 & & &= 2{,}410 \text{ lb.}\\
\text{Total upward forces} & & &= 38{,}000 \text{ lb.}\\
\text{Total load} & & &= 38{,}000 \text{ lb.}
\end{aligned}
$$

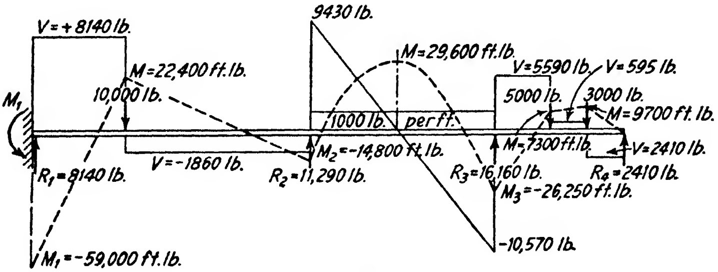


FIG. 205

With the complete shear force and bending moment diagram, it is possible to design the different spans for size and reinforcing steel. If the design does not give a moment of inertia approximating $I = 14{,}000$ (in.)^4, a corrected solution should be made. If the section varies in different spans, the solution should use equation 245 instead of equation 246.

344. Variation of Live Load. Most structures are designed for a maximum live load. It may well happen, however, that the structure is not fully loaded, and it may chance for certain spans that this irregular distribution of the live load gives greater numerical values of shear and bending moment. The following problem illustrates the variation in shear and bending moment diagrams produced by partial loading.

ILLUSTRATIVE PROBLEM 67

345. Continuous Beam—Live Load Varies. Given a beam of three spans fixed at the end supports (Fig. 206). The uniformly distributed load consists of a dead

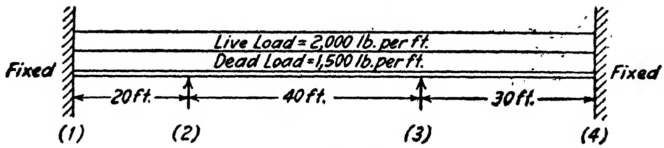


FIG. 206

load of 1500 lb. per ft. and a live load of 2000 lb. per ft. Assume the beam section to be constant in all spans and that the columns all shorten the same amount.

Determine the shear force and bending moment diagrams if the live load is added or omitted on complete spans.

Case I. Live Load on All Spans. At supports 1 and 4 the change of slope is zero. Using second part of equation 245 for support 1 and the first part for support 4,

$$0 = 2M_1 + M_2 + 350{,}000 \tag{250a}$$

$$0 = M_3 + 2M_4 + 787{,}500 \tag{250b}$$

Taking the origin B at support 2,

$$0 = 20M_1 + 2M_2(20 + 40) + 40M_3 + \frac{6 \times 3500(20)^3}{24} + \frac{6 \times 3500(40)^3}{24}$$

$$0 = M_1 + 6M_2 + 2M_3 + 3{,}150{,}000 \tag{251}$$

Taking the origin B at support 3,

$$0 = 40M_2 + 2M_3(40 + 30) + 30M_4 + \frac{6 \times 3500(40)^3}{24} + \frac{6 \times 3500 \times (30)^3}{24}$$

$$0 = 4M_2 + 14M_3 + 3M_4 + 7{,}962{,}500 \tag{252}$$

Solving these equations simultaneously gives

$M_1 = 18{,}700$ ft.-lb. $\qquad M_3 = -418{,}900$ ft.-lb.

$M_2 = -387{,}400$ ft.-lb. $\qquad M_4 = -184{,}200$ ft.-lb.

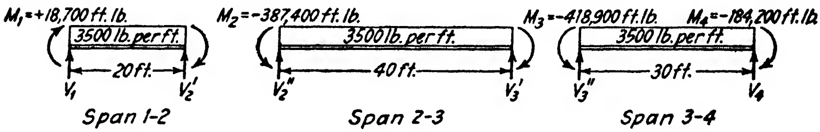


FIG. 207

346. Shear Force and Bending Moment Diagrams. Referring to the loads on the individual spans shown in Figure 207, the supporting forces are

$V_1 = 14{,}700$ lb. $\qquad V''_2 = 69{,}200$ lb. $\qquad V''_3 = 60{,}300$ lb.

$V'_2 = 55{,}300$ lb. $\qquad V'_3 = 70{,}800$ lb. $\qquad V_4 = 44{,}700$ lb.

The supporting forces equal

$$\begin{aligned}
R_1 & & &= 14{,}700 \text{ lb.}\\
R_2 &= 55{,}300 + 69{,}200 &&= 124{,}500\\
R_3 &= 70{,}800 + 60{,}300 &&= 131{,}100\\
R_4 & &&= \underline{44{,}700}\\
\text{Total upward supports} & &&= 315{,}000 \text{ lb.}\\
\text{Total load} & &&= 315{,}000 \text{ lb.}
\end{aligned}$$

The shear force and bending moment diagram is plotted on Figures 208 and 209.

347. Live-Load Variations. Similar solutions are made, assuming the live load to act as follows.

Case	Span	Case	Span
I	All	V	1–2
II	1–2 and 3–4	VI	2–3
III	2–3 and 3–4	VII	3–4
IV	1–2 and 2–3		

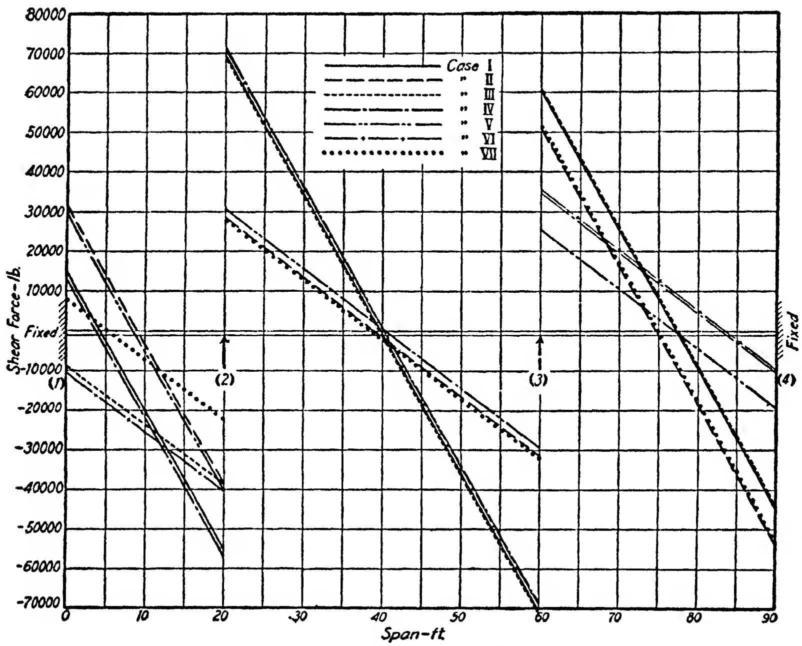


FIG. 208. Shear force diagrams.

The results are given in Table I and plotted on Figures 208 and 209. All the above results can also be obtained by the use of influence lines.

348. Discussion of Results. It is evident from Figures 208 and 209 that no one loading will give all the maximum values. In general, we may say for this problem that:

Maximum shear at end supports occurs when the end span has live load and the adjacent span does not.

Maximum shear at interior supports occurs when the two spans at the support have the live load and the adjacent one does not.

Maximum positive moment near center of *end spans* occurs when the end span carries the live load and the adjacent span does not.

Maximum positive moment near center of *interior spans* occurs when this span carries the live load and one adjacent span does also. The maximum is nearly reached, however, when this span carries the live load and the adjacent spans are without live load.

Maximum negative moment at an exterior support occurs when the end spans are loaded with live load and the adjacent interior span is not.

Maximum negative moment at interior supports occurs when the two spans meeting at the support carry live load and the adjacent span does not.

The bending moment diagrams overlap, and the positive moment diagrams for some of the lower curves have points of inflection near the support, thereby requiring some positive tension steel near the support. The accurate design of continuous

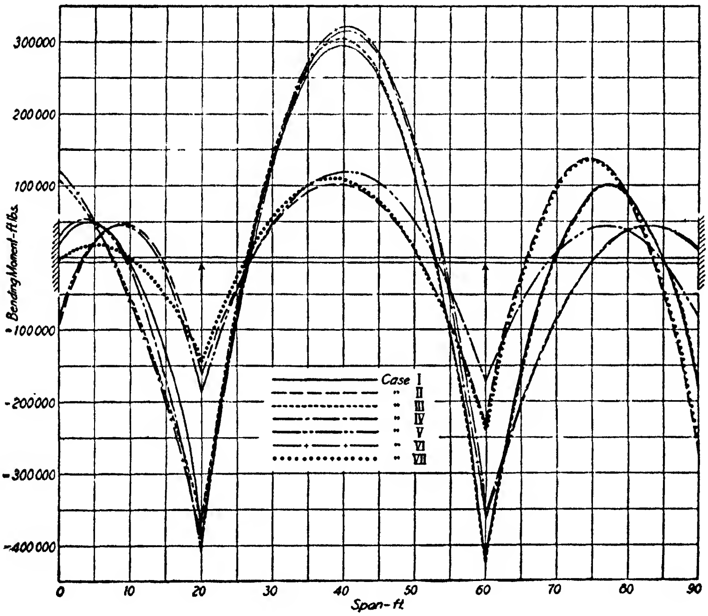
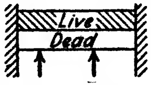
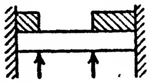
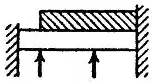
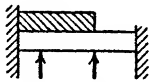
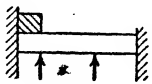
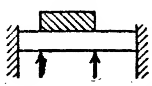
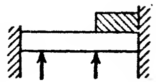


FIG. 209. Bending moment diagrams.

beams with variable live loads requires a repeated solution of possible loadings similar to the computations of this problem. Trial indicates the load positions giving maximum values.

349. Continuous Beams with Equal Spans. *Uniformly Distributed Loads.* The case of continuous beams of *equal spans* loaded with uniformly distributed loads has been carefully computed. The maximum positive and negative bending moments recommended in A.C.I. Article 701 (see Appendix) give the results, after modification for the probability of occurrence of some of the loadings. In general, the maximum positive bending moment in an interior span occurs when that span and each alternate span is loaded with the live and dead loads, and the

TABLE I. VARIATION OF LIVE LOAD—Problem 67

Case	I	II	III	IV	V	VI	VII
Loading							
M_1	+ 18,700	− 93,100	+108,600	+ 30,400	− 81,300	+120,500	− 2,800
M_2	−387,400	−164,800	−367,300	−410,800	−187,500	−390,600	−144,300
M_3	−418,900	−234,000	−425,000	−356,900	−172,500	−363,500	−240,300
M_4	−184,200	−276,700	−181,200	+ 9,900	− 82,500	+ 13,000	−273,500
R_1	14,700	31,400	− 8,800	12,900	29,700	− 10,600	7,900
R_2	124,100	66,900	107,300	128,400	70,700	+111,200	49,700
R_3	131,100	82,800	132,100	103,400	55,100	+104,400	83,800
R_4	44,700	53,900	44,400	10,300	19,500	+ 10,000	53,600
Total load	315,000	235,000	275,000	255,000	175,000	215,000	195,000
Maximum Shear Force							
Span 1–2							
at 1......	....	Max.					
at 2......	....	....	....	Max.			
Span 2–3							
at 2......	....	....	....	Max.			
at 3......	....	....	Max.				
Span 3–4							
at 3......	....	....	Max.				
at 4......	....	Max.					
Maximum Positive Moment							
Span 1–2...	....	Max. (center)	....	....	....	Max (at 1)	
2–3......	....	....	....	....	....	Max. (nr. center)	
3–4......	....	Max. (nr. center)	....	....	....	Max. (at 4)	
Maximum Negative Moment							
Support 1..	....	Max.					
2..	....	....	....	Max.			
3..	....	....	Max.				
4..	....	Max.					

adjacent spans and alternate spans are loaded with the dead load only. Similarly, the maximum negative moment at a support occurs when the two adjacent spans carry the live and dead loads; thereafter the alternate spans carry no live load.

There is obtained an upper and lower limit to possible bending moment curves for probable variations of the live load. Table J gives the spread of possible bending moment diagrams for the moment coefficients recommended by A.C.I. Article 701.

For distributed loads the maximum shear force in any one span will be $0.5wl$, if the negative moments at each support are equal. When the live load is removed from some spans this will not be true, and it is possible to realize live-load arrangements which, coupled with the uniform dead loads, will give a maximum shear force at a support of about $0.55wl$. This may occur in any span, but it has been commercial practice to design all interior spans for a maximum shear force of $0.5wl$ and let the possibility of some unusual live-load arrangement be cared for by employing a somewhat smaller allowable shear stress. The shear force at the first interior column for the exterior span is increased to $0.575wl$. There are two discontinuous shear force diagrams for this span, one varying from $0.5wl$ to $-0.5wl$, the other varying from $0.425wl$, at the exterior column to $-0.575wl$ at the first interior column.

350. Continuous Beams of Equal Span. *Concentrated Loads.* Beam and slab floor systems are frequently planned so that the girders running from column to column support beams which are intermediate between the columns. These intermediate beams are usually evenly spaced. The girder is designed for a small uniformly distributed load and the symmetrically placed concentrated loads brought to it by the intermediate beams. The concentrated loads will vary as the live load is shifted about on the floor, and analyses similar to those made for uniformly distributed loads must be made for these loads. The A.C.I. Code makes no recommendations for bending moment coefficients for these cases. The author proposes limiting moments for design.

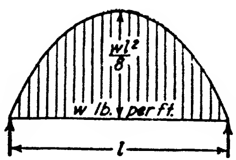
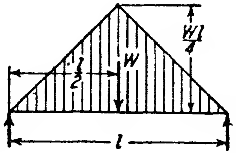
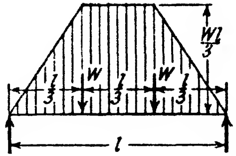
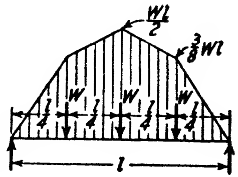
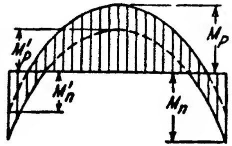
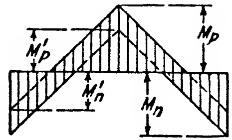
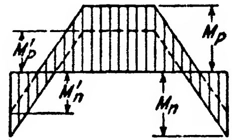
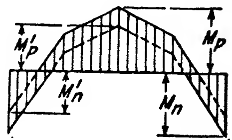
Table J gives the limiting values of the bending moment for symmetrically placed concentrated loads, covering the cases of one, two, and three intermediate beams.

351. Moving Loads. The preceding discussion applies to structures whose load is static, but may be shifted about from time to time. Other structures, such as bridges, receive moving loads, whose position is only temporarily static. The position of the load to give the maximum shear forces and bending moments for moving loads can be found by the use of influence lines. Any standard text on structures covers the construction of influence lines.

TABLE J. MOMENT COEFFICIENTS

Symmetrically Placed Loads

	Uniformly Distributed Loads	Girder Supporting		
		1 Intermediate Beam	2 Intermediate Beams	3 Intermediate Beams
Interior span				
Maximum positive M_p	$\frac{wl^2}{16}$	$\frac{Wl}{7}$	$\frac{3}{16} Wl$	$\frac{5}{16} Wl$
Maximum negative M_n	$\frac{wl^2}{11}$	$\frac{Wl}{8}$	$\frac{2}{9} Wl$	$\frac{5}{16} Wl$
Exterior span				
Maximum positive M_p	$\frac{wl^2}{14}$	$\frac{Wl}{9}$	$\frac{2}{11} Wl$	$\frac{4}{15} Wl$
Maximum negative M_n	$> \frac{wl^2}{24}$ (end) $\frac{wl^2}{10}$ (int.)	$\frac{3}{32} Wl$ (end) $\frac{Wl}{7}$ (int.)	$\frac{Wl}{6}$ (end) $\frac{Wl}{4}$ (int.)	$\frac{15}{64} Wl$ (end) $\frac{5}{14} Wl$ (int.)

Single span Ends supported				
Continuous beam Ends partially fixed				
	$M_p + M'_n = \frac{wl^2}{8}$ $M'_p + M_n = \frac{wl^2}{8}$	$M_p + M'_n = \frac{Wl}{4}$ $M'_p + M_n = \frac{Wl}{4}$	$M_p + M'_n = \frac{Wl}{3}$ $M'_p + M_n = \frac{Wl}{3}$	$M_p + M'_n = \frac{Wl}{2}$ $M'_p + M_n = \frac{Wl}{2}$

THE SLOPE-DEFLECTION METHOD

352. Solution by Slope-Deflection Methods. The discussion of the preceding articles has covered the solution of continuous beams. Except for the restraint of the end supports no consideration has been given to the rigidity of the supporting columns. There are methods of solution of indeterminate problems which do take into account all the members meeting at a joint. Such methods include work solutions and further use of slope-deflection relations. The slope-deflection method is well adapted for the analysis of structural frames and will be discussed in this chapter. The least-work solution is used in Chapter 15 in developing equations for the design of concrete arches.

353. Assumptions of the Slope-Deflection Method. Solutions for the end moments in members of a rigid frame by the method of slope-deflection, or its derivative, the method of moment distributions, are expedited by assumptions in regard to the deformations of these members. The assumptions are:

1. The members meeting at a joint all rotate through the same angle when an external couple is applied.
2. The change in length of columns and beams is small enough to be neglected.
3. Deformations due to shear forces may be neglected.
4. If the members of the frame all lie in the same plane, their relative stiffness can be measured by the ratios of $\frac{I}{l}$ of each member, where I is the moment of inertia and l the length.

Center line dimensions are commonly used for the length of each member. In the discussion of deflection of beams (Art. 163) the moment of inertia was taken as that of the concrete area for the depth d plus the moment of inertia of the positive tension steel. Rigid frame analyses are usually made before the reinforcement is computed. Therefore, it is customary to use the moment of inertia of the gross concrete section about its center of gravity, with no allowance for the steel. Since the solutions involve ratios of stiffnesses, this procedure is satisfactory when applied to all members.

The basic slope-deflection equation is obtained from the beam theory by use of moment-area methods. The conventions in regard to moments, slopes, and deflections differ from those used in the usual beam theory by algebraic methods. These conventions are:

1. Taking a member as a rigid body, the external end bending moment is *positive* if the moment-couple is clockwise.

2. The slope is *positive*, if the center line of the member rotates clockwise in regard to the unloaded position.

3. Deflections are *positive*, if the center line of the member at the joint considered is displaced by clockwise rotation.

Consider a beam whose left end is at A and right end at B. After loading the slope at A equals θ_A and at B is θ_B. The external moment at A is denoted M_{AB} and the moment at B by M_{BA}. The basic slope-deflection equations take the form

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - \frac{2F_{AB}}{l^2}(3x_B - l) \tag{253}$$

$$M_{BA} = 2EK(\theta_A + 2\theta_B - 3R) + \frac{2F_{AB}}{l^2}(2l - 3x_B) \tag{254}$$

where $K = \dfrac{I}{l}$ for the member and $R = \dfrac{v}{l}$.

In equation 253 v is the deflection of end B relative to A, and in equation 254 v is the deflection of A relative to B. In both equations v will have the same sign. These equations state that the moment at any section depends on

1. The modulus of elasticity of the material.

2. The stiffness $\dfrac{I}{l}$ of the span.

3. Certain terms which allow for the relative restraint and displacement of the sections A and B. For a beam fixed at the ends, whose supports settle the same amount, θ_A, θ_B, and R equal zero.

4. A term which allows for the loads between sections A and B. This term is computed as though the span were simply supported at A and B.

354. Load Term. If sections A and B be taken at the supports of a beam fixed at its ends, the terms θ_A, θ_B, and R equal zero in equations 253 and 254. Then the fixed-end moments at the supports equal

$$M_{AB} = \frac{-2F_{AB}}{l^2}(3x_B - l) = C_{AB}$$

$$M_{BA} = \frac{+2F_{AB}}{l^2}(2l - 3x_B) = C_{BA}$$

In other words, the load term of equations 253 and 254 is numerically equal to the fixed-end moment of a beam loaded as given. The *sign* of this term is determined by noting that the load tends to cause clockwise rotation about section A and is therefore balanced by an anti-clockwise or negative term C_{AB}. The load tends to cause anti-clockwise rotation about section B, and the balancing moment C_{BA} at the section is *positive.*

Examination of equations 253 and 254 in the light of this discussion permits a simplification to one statement. The bending moment at any section equals

$$M = 2EK(2\theta_{\text{section}} + \theta_{\text{distant section}} - 3R) \pm C_{\text{section}} \tag{255}$$

where C = fixed-end moment at the given section and the sign is determined by the statement above.

Table H (Art. 341) gives the fixed-end moments for certain common beam loadings.

ILLUSTRATIVE PROBLEM 68

355. Slope-Deflection Solution of a Rigid Frame. Determine the bending moment diagram for the continuous beam $ABCD$ of Figure 210 by the use of the slope-deflec-

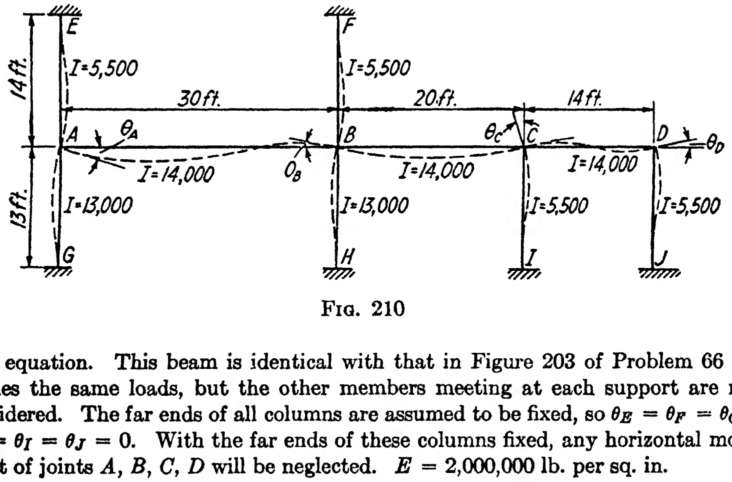


FIG. 210

tion equation. This beam is identical with that in Figure 203 of Problem 66 and carries the same loads, but the other members meeting at each support are now considered. The far ends of all columns are assumed to be fixed, so $\theta_E = \theta_F = \theta_G = \theta_H = \theta_I = \theta_J = 0$. With the far ends of these columns fixed, any horizontal movement of joints A, B, C, D will be neglected. E = 2,000,000 lb. per sq. in.

Equation 255 is used to express the moment at each end of a member in terms of the unknown joint rotations θ_A, θ_B, θ_C, and θ_D. The fixed-end moments can be obtained from Table H and are expressed in foot-pound units. Using the numerical value of E the deflection term $6EKR$ for the individual beam spans should also be expressed in foot-pound units.

From Figures 203 and 210:

$$K_{AE} = K_{BF} = \frac{5500}{14 \times 12} = 32.7 \text{ (in.)}^3 \qquad K_{CI} = K_{DJ} = 35.2$$

$$K_{AG} = K_{BH} = 83.3 \qquad K_{AB} = 38.9 \qquad K_{BC} = 58.3 \qquad K_{CD} = 83.3$$

By equation 255,

$$M_{AB} = 2E38.9\left[2\theta_A + \theta_B - \frac{3 \times (0.20)}{360 \times 12}\right] - \frac{10{,}000 \times 10 \times (20)^2}{(30)^2}$$

$$M_{BA} = 2E38.9\left[\theta_A + 2\theta_B - \frac{1}{7200}\right] + \frac{10{,}000 \times (10)^2 \times 20}{(30)^2}$$

$$M_{BC} = 2E58.3\left[2\theta_B + \theta_C - \frac{3(-0.05)}{240 \times 12}\right] - \frac{1000 \times (20)^2}{12}$$

$$M_{CB} = 2E58.3\left[\theta_B + 2\theta_C + \frac{1}{19{,}200}\right] + 33{,}330$$

$$M_{CD} = 2E83.3\left[2\theta_C + \theta_D - \frac{3(-0.10)}{168 \times 12}\right] - \frac{5000 \times 6 \times (8)^2}{(14)^2} - \frac{3000 \times 10 \times (4)^2}{(14)^2}$$

$$M_{DC} = 2E83.3\left[\theta_C + 2\theta_D + \frac{1}{6720}\right] + \frac{5000(6)^2 \times 8}{(14)^2} + \frac{3000 \times (10)^2 \times 4}{(14)^2}$$

$$M_{AE} = 2E32.7[2\theta_A] \qquad M_{BH} = 2E83.3[2\theta_B]$$

$$M_{BF} = 2E32.7[2\theta_B] \qquad M_{CI} = 2E35.2[2\theta_C]$$

$$M_{AG} = 2E83.3[2\theta_A] \qquad M_{DJ} = 2E35.2[2\theta_D]$$

Equilibrium equations can be written to evaluate the four unknown slopes. The end moments of each member act on the adjacent joint reversed in direction. Successively applying equilibrium to joints A, B, C, and D:

Joint A $\qquad M_{AE} + M_{AB} + M_{AG} = 0 \qquad (256)$

Joint B $\qquad M_{BA} + M_{BF} + M_{BC} + M_{BH} = 0 \qquad (257)$

Joint C $\qquad M_{CB} + M_{CD} + M_{CI} = 0 \qquad (258)$

Joint D $\qquad M_{DC} + M_{DJ} = 0 \qquad (259)$

These four equations are tabulated below.

$E\theta_A$	$E\theta_B$	$E\theta_C$	$E\theta_D$		Equation
619.7	77.8			= 66,050	256
77.8	852.9	116.6		= 20,570	257
	116.6	707.3	166.6	= −82,810	258
		166.6	474.1	= −63,050	259

Solving simultaneously,

$$E\theta_A = 103.04 \qquad E\theta_B = 28.19 \qquad E\theta_C = -98.56 \qquad E\theta_D = -98.355$$

Substituting in the moment equations above,

$M_{AB} = -47{,}800$ ft.-lb.	$M_{BA} = 13{,}000$ ft.-lb.
$M_{AE} = 13{,}500$ ft.-lb.	$M_{BC} = -26{,}100$ ft.-lb.
$M_{AG} = 34{,}300$ ft.-lb.	$M_{BF} = 3{,}700$ ft.-lb.
Sum $= 0$	$M_{BH} = 9{,}400$ ft.-lb.
	Sum $= 0$
$M_{CB} = 25{,}800$ ft.-lb.	$M_{DC} = 13{,}900$ ft.-lb.
$M_{CD} = -11{,}900$ ft.-lb.	$M_{DJ} = -13{,}900$ ft.-lb.
$M_{CI} = -13{,}900$ ft.-lb.	Sum $= 0$
Sum $= 0$	

The solution of Problem 66 by the three-moment equation is an approximation, as the stiffness of the supporting columns is neglected. In that problem, the support at 1 (or A) was assumed to be fixed and the other supports were assumed to be simply supported. Comparison of the results is given below.

	Slope-Deflection	Three-Moment
M_{AB}	−47,800	−59,000
M_{BA}	13,000	−14,800
M_{BC}	−26,100	
M_{CB}	25,800	−26,250
M_{CD}	−11,900	
M_{DC}	13,900	0

By the slope-deflection convention these end moments are all couples producing tension at the top of the beam and compression at the bottom. The signs of the moments of the three-moment solution also signify negative bending moments for all end moments. It is evident that the three-moment solution is a poor approximation for the actual structure.

When the complete moment and shear diagrams are computed as illustrated in Problem 66 the beam and column section can be checked for size. If changes seem advisable, the solution can be repeated for the new stiffness ratios. The dash lines of Figure 210 show, greatly exaggerated, the distortions of the frame due to the loads.

ILLUSTRATIVE PROBLEM 69

356. Statically Indeterminate Frame with Side-Sway. Given the frame shown in Figure 211. The beams AB and DE are loaded with a uniformly distributed load of 2000 lb. per ft., and beam CD has a load of 3000 lb. per ft. For this solution all columns and beams will be assumed to have the same modulus of elasticity E. Often columns have richer mixes and different moduli may be required. The ratio $K = \frac{I}{l}$ has been estimated by an approximate design for each beam and column, and this trial solution is made with the K values given in Figure 211.

Determine the bending moment and shear force diagrams for all members of the frame. The column footings are assumed to be large enough to fix the ends of the column stacks. Therefore: θ_F, θ_G, and θ_H equal zero.

If the frame were symmetrical, the columns would remain in the present vertical lines, but the given frame will have lateral displacements of the upper and lower

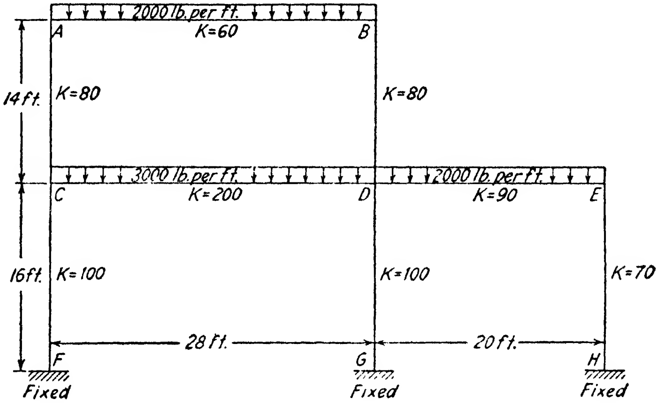


FIG. 211

columns. Assuming that the beams do not change in length, the two upper columns will displace the same amount, as will the three lower columns.

$$M_{AB} = 2E60(2\theta_A + \theta_B) - \frac{2000(28)^2}{12} \qquad M_{DE} = 2E90(2\theta_D + \theta_E) - \frac{2000(20)^2}{12}$$

$$M_{BA} = 2E60(\theta_A + 2\theta_B) + \frac{2000(28)^2}{12} \qquad M_{ED} = 2E90(\theta_D + 2\theta_E) + \frac{2000(20)^2}{12}$$

$$M_{AC} = 2E80(2\theta_A + \theta_C - 3R_1) \qquad M_{CF} = 2E100(2\theta_C - 3R_2)$$

$$M_{CA} = 2E80(\theta_A + 2\theta_C - 3R_1) \qquad M_{FC} = 2E100(\theta_C - 3R_2)$$

$$M_{BD} = 2E80(2\theta_B + \theta_D - 3R_1) \qquad M_{DG} = 2E100(2\theta_D - 3R_2)$$

$$M_{DB} = 2E80(\theta_B + 2\theta_D - 3R_1) \qquad M_{GD} = 2E100(\theta_D - 3R_2)$$

$$M_{CD} = 2E200(2\theta_C + \theta_D) - \frac{3000(28)^2}{12} \qquad M_{EH} = 2E70(2\theta_E - 3R_2)$$

$$M_{DC} = 2E200(\theta_C + 2\theta_D) + \frac{3000(28)^2}{12} \qquad M_{HE} = 2E70(\theta_E - 3R_2)$$

In the above equations there are seven unknown terms. It is necessary to have at least seven equations for the solution. Applying conditions of moment equilibrium to each joint gives

Joint A	$M_{AB} + M_{AC} = 0$	(260)
Joint B	$M_{BA} + M_{BD} = 0$	(261)
Joint C	$M_{CA} + M_{CD} + M_{CF} = 0$	(262)
Joint D	$M_{DB} + M_{DC} + M_{DE} + M_{DG} = 0$	(263)
Joint E	$M_{ED} + M_{EH} = 0$	(264)

Take the two columns AC and BD as the rigid body (Fig. 212). Take moments about joint D.

$$M_{AC} + M_{CA} + M_{BD} + M_{DB} = 0 \tag{265}$$

Take the three lower columns CF, DG, and EH as the rigid body. Take moments about joint D.

$$M_{CF} + M_{FC} + M_{DG} + M_{GD} + M_{EH} + M_{HE} - (H_{FC} + H_{GD} + H_{HE})16 = 0$$

Note that, if the whole frame be taken as a rigid body, $H_{FC} + H_{GD} + H_{HE}$ must equal zero. Then

$$M_{CF} + M_{FC} + M_{DG} + M_{GD} + M_{EH} + M_{HE} = 0 \tag{266}$$

There are now seven equations with seven unknowns. Table K lists the equations and gives the results of a solution. It is a long process to eliminate the unknowns

TABLE K

$E\theta_A$	$E\theta_B$	$E\theta_C$	$E\theta_D$	$E\theta_E$	$3ER_1$	$3ER_2$		Equation
560	120	160	....	...	−160	...	= +130,700	260
120	560	....	160	...	−160	...	= −130,700	261
160	...	1520	400	...	−160	−200	= +196,000	262
...	160	400	1880	180	−160	−200	= −129,300	263
...	...	....	180	640	...	−140	= −66,700	264
480	480	480	480	...	−640	...	= 0	265
...	...	600	600	420	...	−1080	= 0	266
Results 268.0	−258.4	+122.3	−59.76	−87.24	+54.03	+0.789		

successively. In part of the solution it was necessary to use seven significant figures. Substituting the values of $E\theta_A$, $E\theta_B$, etc., in the moment equations,

$$M_{AB} = 120(536.0 - 258.4) - 130{,}700 = -97{,}000 \text{ ft.-lb.}$$

$$M_{BA} = 120(268.0 - 516.8) + 130{,}700 = +100{,}800 \text{ ft.-lb.}$$

$$M_{AC} = 160(536.0 + 122.3 - 54.0) = +97{,}000 \text{ ft.-lb.}$$

$$M_{CA} = 160(268.0 + 244.6 - 54.0) = +73{,}500 \text{ ft.-lb.}$$

$$M_{BD} = 160(-516.8 - 59.8 - 54.0) = -100{,}800 \text{ ft.-lb.}$$

$$M_{DB} = 160(-258.4 - 119.5 - 54.0) = -69{,}100 \text{ ft.-lb.}$$

$$M_{CD} = 400(244.6 - 59.8) - 196{,}000 = -122{,}100 \text{ ft.-lb.}$$

$$M_{DC} = 400(122.3 - 119.5) + 196{,}000 = +197{,}100 \text{ ft.-lb.}$$

$$M_{DE} = 180(-119.5 - 87.2) - 66{,}700 = -103{,}900 \text{ ft.-lb.}$$

$$M_{ED} = 180(-59.8 - 174.5) + 66{,}700 = +24{,}500 \text{ ft.-lb.}$$

$$M_{CF} = 200(244.6 - 0.8) = +48{,}700 \text{ ft.-lb.}$$

$$M_{FC} = 200(122.3 - 0.8) = +24{,}300 \text{ ft.-lb.}$$

$$M_{DG} = 200(-119.5 - 0.8) = -24{,}100 \text{ ft.-lb.}$$

$$M_{GD} = 200(-59.8 - 0.8) = -12{,}100 \text{ ft.-lb.}$$

$$M_{EH} = 140(-174.5 - 0.8) = -24{,}500 \text{ ft.-lb.}$$

$$M_{HE} = 140(-87.2 - 0.8) = -12{,}300 \text{ ft.-lb.}$$

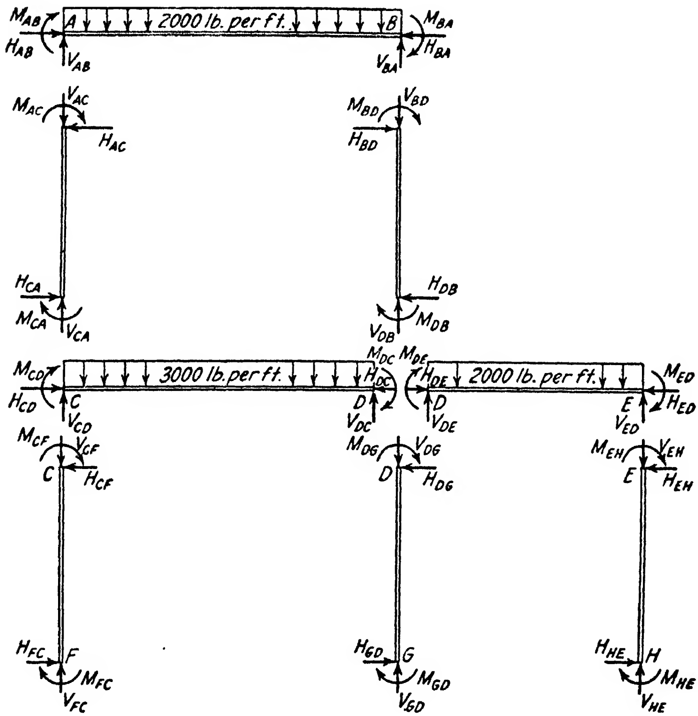


FIG. 212

These moments are used to compute the shear force and bending moment diagrams for the individual members. The computations for the supporting shears for the beams and columns are made as in Problem 66. Figure 212 represents the individual members, all moments being shown as positive. Those whose values are actually negative should be reversed in direction before the supporting shears are computed. It will be noticed that, if we take the upper floor frame *CABD*, the frame is in equilibrium at the base of the columns. In other words, $V_{CA} + V_{DB} = 2000 \times 28 = 56{,}000$ lb., and $H_{CA} + H_{DB} = 0$. Similarly, the whole frame is in equilibrium:

$$V_{FC} + V_{GD} + V_{HE} = 180{,}000 \text{ lb.}$$

$$H_{FC} + H_{GD} + H_{HE} = 0$$

ILLUSTRATIVE PROBLEM 70

357. Wind Pressure. Solve Problem 69 assuming the wind to blow from the left causing the pressures on the column stack FCA shown in Figure 213. Neglecting the beam loads in the computations, determine the shear force and bending moment diagrams for all members of the frame.

Assume again that the footings fix the column bases so that θ_F, θ_G, and θ_H equal zero. It is apparent that the column stacks will be pushed out of the vertical by the wind load. Assume that the axial loads in the beams do not change their lengths. Then the displacements of A and B are equal as are those of C, D, and E. Assume that A and B remain on the same level and that C, D, and E do also. The moment equations can now be written

Final Result ft.-lb.

$$
\begin{aligned}
M_{AB} &= 120E(2\theta_A + \theta_B) && = +2{,}842 \\
M_{BA} &= 120E(\theta_A + 2\theta_B) && = +3{,}983 \\
M_{AC} &= 160E(2\theta_A + \theta_C - 3R_1) + \frac{200(14)^2}{12} && = -2{,}842 \\
M_{CA} &= 160E(\theta_A + 2\theta_C - 3R_1) - \frac{200(14)^2}{12} && = -7{,}862 \\
M_{BD} &= 160E(2\theta_B + \theta_D - 3R_1) && = -3{,}984 \\
M_{DB} &= 160E(\theta_B + 2\theta_D - 3R_1) && = -4{,}912 \\
M_{CD} &= 400E(2\theta_C + \theta_D) && = +14{,}717 \\
M_{DC} &= 400E(\theta_C + 2\theta_D) && = +12{,}418 \\
M_{DE} &= 180E(2\theta_D + \theta_E) && = +5{,}914 \\
M_{ED} &= 180E(\theta_D + 2\theta_E) && = +7{,}276 \\
M_{CF} &= 200E(2\theta_C - 3R_2) + \frac{200(16)^2}{12} && = -6{,}855 \\
M_{FC} &= 200E(\theta_C - 3R_2) - \frac{200(16)^2}{12} && = -18{,}224 \\
M_{DG} &= 200E(2\theta_D - 3R_2) && = -13{,}420 \\
M_{GD} &= 200E(\theta_D - 3R_2) && = -15{,}107 \\
M_{EH} &= 140E(2\theta_E - 3R_2) && = -7{,}276 \\
M_{HE} &= 140E(\theta_E - 3R_2) && = -9{,}516
\end{aligned}
$$

The unknown terms comprise five slopes and two displacements, seven in all. A sketch similar to Figure 212 can be constructed and equations of equilibrium written:

For Joint A $\quad M_{AB} + M_{AC} = 0 \quad$ (267)

Joint B $\quad M_{BA} + M_{BD} = 0 \quad$ (268)

Joint C $\quad M_{CA} + M_{CD} + M_{CF} = 0 \quad$ (269)

Joint D $\quad M_{DB} + M_{DC} + M_{DE} + M_{DG} = 0 \quad$ (270)

Joint E $\quad M_{ED} + M_{EH} = 0 \quad$ (271)

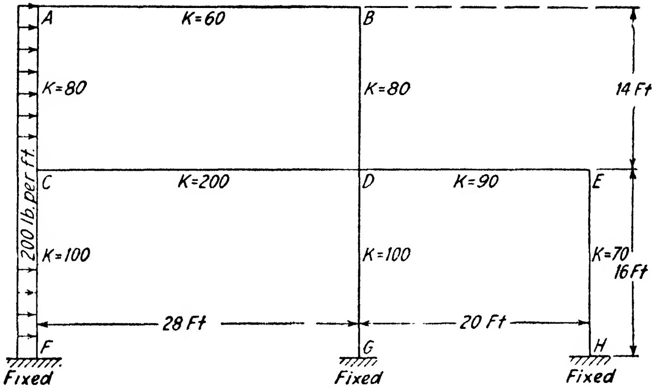


FIG. 213

Columns AC and BD. Moments about joint C.

$$M_{AC} + M_{CA} + M_{BD} + M_{DB} + 200 \times 14 \times 7 = 0 \tag{272}$$

Columns CF, DG, and EH. Moments about joint F.

$$M_{CF} + M_{FC} + M_{DG} + M_{GD} + M_{EH} + M_{HE} + 200 \times 16 \times 8 + 2800 \times 16 = 0 \tag{273}$$

Table L lists these equations and gives the solution for its unknowns. Substitution in the equations above gives the numerical results listed at the right of that page. From these values can be computed the shear force and bending moment diagrams for this case of the wind on the left side of the frame, and also the combined effect of floor load and wind. Analysis might also be made for wind on the right and the combined effect of floor load and wind on the right. The designer is then in a position to determine maximum shear forces and bending moments for the design. It may well result that the original moment of inertias will be modified enough to warrant a second analysis.

TABLE L

$E\theta_A$	$E\theta_B$	$E\theta_C$	$E\theta_D$	$E\theta_E$	$3ER_1$	$3ER_2$		Equation
560	120	160	....	...	−160	......	= −3,267	267
120	560	....	160	...	−160	......	= 0	268
160	...	1520	400	...	−160	−200	= −1,000	269
...	160	400	1880	180	−160	−200	= 0	270
...	...	....	180	640	.....	−140	= 0	271
480	480	480	480	...	−640	......	= −19,600	272
...	...	600	600	420	.....	−1080	= −70,400	273
Results +4.722	+14.239	+14.180	+8.432	+15.994	+61.805	+83.966		

358. Use of Slope-Deflection Method. A great variety of indeterminate structures can be solved by use of the slope-deflection method. The choice between this method and solution by other methods, such as virtual work or least work, depends on the relative labor of solution and the personal equation of the designer. As a result of his engineering experience he decides that one method is particularly suited to trussed structures, but, perhaps, that another method is better adapted for bents and frames with members in bending.

MODIFICATION OF SLOPE-DEFLECTION SOLUTION

359. Moment-Distribution Method. The slope-deflection equations have been used by Professor Hardy Cross to establish a method of determining the bending moment at the support of a member by successive approximations. Its advantages are that no equations are written out, and the solution of many simultaneous equations is avoided. The designer deals with numerical values at once. In order to avoid error or confusion the designer must decide on a standard procedure and representation.

360. General Principles. *A. An external moment (couple) applied at any joint is distributed among the members meeting at the joint in proportion to the stiffness constant* $\frac{I}{l} = K$ *of each member.*

Members Unloaded. Let A (Fig. 214a) be any joint in some structure at which, in this case, four members meet. The special case is assumed

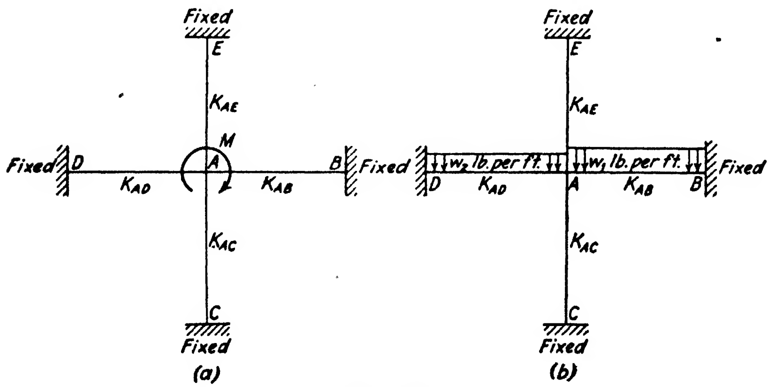


FIG. 214

of a fixed support at each of the far ends. Then $\theta_B = 0 = \theta_C = \theta_D = \theta_E$. Assume no load on the members and no deflection of the far ends relative to A. Some external couple M is applied to the joint A and tends to rotate the joint. Since there is equilibrium at A:

$$M_{AB} + M_{AC} + M_{AD} + M_{AE} + M = 0$$

but

$$M_{AB} = 2EK_{AB}(2\theta_A) = 4EK_{AB}\theta_A$$

$$M_{AC} = 4EK_{AC}\theta_A, \text{ etc.}$$

Then

$$\theta_A = -\frac{M}{4E(K_{AB} + K_{AC} + K_{AD} + K_{AE})} = -\frac{M}{4EK_T}$$

where K_T is the sum $(K_{AB} + K_{AC} + K_{AD} + K_{AE})$ and

$$M_{AB} = -M\frac{K_{AB}}{K_{AB} + K_{AC} + K_{AD} + K_{AE}} = -M\frac{K_{AB}}{K_T}$$

$$M_{AC} = -M\frac{K_{AC}}{K_{AB} + K_{AC} + K_{AD} + K_{AE}} = -M\frac{K_{AC}}{K_T}, \text{ etc.}$$

Therefore, the moments in the members at A are proportional to the respective stiffness constant $\frac{I}{l} = K$ of each member.

Member Loaded. Instead of applying the external couple M, let the member AB be loaded. In that case, by equation 255 (Art. 354),

$$M_{AB} = 4EK_{AB}\theta_A \pm C$$

$$M_{AC} = 4EK_{AC}\theta_A, \text{ etc.}$$

For equilibrium of joint A

$$M_{AB} + M_{AC} + M_{AD} + M_{AE} = 0$$

$$\theta_A = \mp\frac{C}{4E(K_{AB} + K_{AC} + K_{AD} + K_{AE})} = \mp\frac{C}{4EK_T}$$

$$M_{AB} = \mp C\left(1 - \frac{K_{AB}}{K_T}\right)$$

and

$$M_{AC} = \mp C\frac{K_{AC}}{K_T}, \text{ etc.}$$

Therefore, the following principle is true.

B. The bending moment M in the section of a member at any joint is resisted by the members meeting at this joint in proportion to the stiffness constant K of each member.

It has been assumed that the far ends of the members are fixed. Then for the unloaded members by equation 255

$$M_{CA} = 2EK_{AC}\theta_A$$

$$M_{DA} = 2EK_{DA}\theta_A, \text{ etc.}$$

These moments are one half the values of M_{AC}, M_{AD}, etc.

C. The bending moments at the fixed far end of unloaded members, which meet at a joint A tending to rotate, is one-half the resisting moment in the member at joint A.

361. Procedure. Given joint A (Fig. 214*b*) at which four members meet, with two loaded in this case.

1. Assume joint A and all far ends to be fixed. The fixed-end moments for the beam AB are

$$M_{AB} = +\frac{w_1l_1^2}{12} \quad \text{and} \quad M_{BA} = -\frac{w_1l_1^2}{12}$$

For AD:

$$M_{AD} = -\frac{w_2l_2^2}{12} \quad \text{and} \quad M_{DA} = +\frac{w_2l_2^2}{12}$$

The signs are governed by the fact that the joints are the rigid bodies and not the beams. The moments M_{AB} and M_{AD} are opposite in sign. If they are not equal in magnitude, there is an unbalanced couple M at joint A which tends to rotate the joint.

$$M = M_{AB} + M_{AD} = +\frac{w_1l_1^2}{12} - \frac{w_2l_2^2}{12}$$

2. Release joint A. By the discussion of principles A and B in the previous paragraph, the unbalanced couple M is resisted by the members in proportion to their respective stiffness constants. Then, under the new conditions,

$$M_{AB} = +\frac{w_1l_1^2}{12} - M\frac{K_{AB}}{K_T}$$

$$M_{AC} = -M\frac{K_{AC}}{K_T}$$

$$M_{AD} = -\frac{w_2l_2^2}{12} - M\frac{K_{AD}}{K_T}$$

$$M_{AE} = -M\frac{K_{AE}}{K_T}$$

3. The moments at the fixed far ends will be modified by a moment half the distributed values $\left(-M \frac{K_{AB}}{K_T}, \text{etc.}\right)$ given above.

4. Pass now to one of the far end joints which is fixed as was joint A in procedure 1. Take this joint and apply the above procedure to it, passing again to an adjacent joint until the transferred moments at the far ends become too small for consideration.

ILLUSTRATIVE PROBLEM 71

362. Solution of Continuous Beams. Solve Problem 67 for the bending moments at each support by the moment-distribution method. This problem included an analysis of the effect of adding or removing the live loads for certain spans and resulted in seven solutions by the three-moment equation. For the moment-distribution method the effect of a load of 1000 lb. per ft. on each single span will be computed and then these results combined for any of the seven combinations desired. This same procedure might have been adopted for the three-moment solution.

The three spans have the same moment of inertia I. The stiffness constant

$$K_{AB} = \frac{I}{l} = \frac{I}{20} = 0.05I$$

$$K_{BC} = \frac{I}{40} = 0.025I$$

$$K_{CD} = \frac{I}{30} = 0.033I$$

Let us deal first with a distributed load of 1000 lb. per ft. in span AB. The fixed-end moments equal

$$M_{AB} = \frac{wl^2}{12} = \frac{1000(20)^2}{12} = 33{,}300 \text{ ft.-lb.}$$

In Figure 215 these moments are listed in units of 1000 lb.-ft., or kip-ft., with the signs of direction governed by the fact that the rigid body is successively each joint. Joint A is fixed by the problem statement and agrees with the assumption. There is no need to release this joint. At joint B when the joint is released there is an unbalanced moment of -33.3. This is resisted by the beam AB and BC so that

$$M_{BA} = +33.3 \times \frac{K_{AB}}{K_T} = 33.3 \times \frac{0.050I}{0.075I} = 33.3 \times \frac{2}{3} = +22.2$$

$$M_{BC} = +33.3 \times \frac{K_{BC}}{K_T} = 33.3 \times \frac{0.025I}{0.075I} = 33.3 \times \frac{1}{3} = +11.1$$

The ratios of K are constant for each joint and independent of I. They are listed at each joint on the sketch at the top of Figure 215. At the fixed ends the distribution is made as though there were a beam beyond the joint of $K = \infty$. It absorbs all unbalanced effects. If there were a hinged end, the moment at that joint must be zero and all unbalanced moments are taken by the beam coming to the joint. In other words, we act as though there were a beam beyond the joint of $K = 0$.

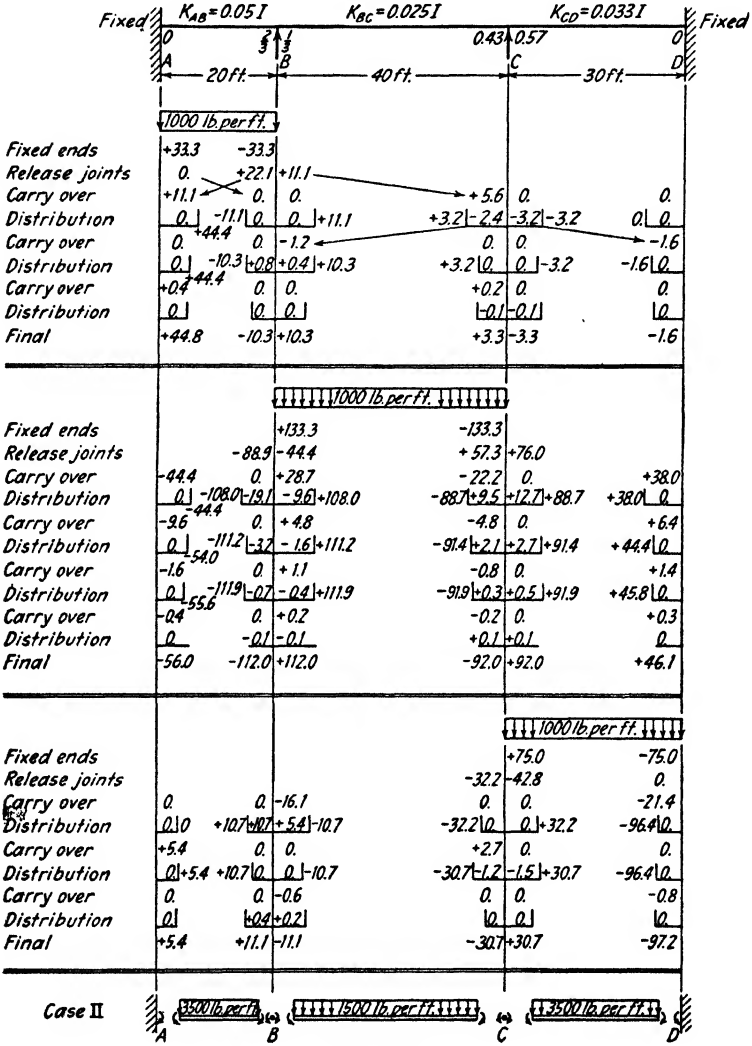


FIG. 215

The distributed resisting moments are listed in Figure 215. It has been assumed that joints A and B were released in turn. As each joint is released the adjacent joint (assumed fixed) receives a moment half the value distributed to that member. This "carry-over" moment is next listed. At joint A it is $0.5 \times 22.2 = +11.1$; at joint B for beam AB it is zero, for beam BC it is zero; at joint C for beam BC it is

$0.5 \times 11.1 = +5.6$, for beam CD it is zero; at joint D it is zero. These carry-over moments are now unbalanced moments at the joints, and they are now distributed according to the K ratio. The unbalanced moment at joint A of $+11.1$ is absorbed by the fixed end. At joints B and D the unbalanced moment is zero. At joint C the unbalanced moment is $+5.6$. It is distributed so that

$$M_{CB} = 0.43 \times 5.6 = -2.4$$

$$M_{CD} = 0.57 \times 5.6 = -3.2$$

The procedure is now finished and the first approximation can be summed up.

$$M_{AB} = +33.3 + 11.1 = 44.4$$

$$M_{BA} = -33.3 + 22.2 = -11.1$$

$$M_{BC} = +11.1$$

$$M_{CB} = +5.6 - 2.4 = +3.2$$

$$M_{CD} = -3.2$$

$$M_{DC} = 0$$

In Figure 215 a line is drawn at the end of the approximation and the totals listed beside the column of figures. It should be true that $M_{BA} = M_{BC}$ and $M_{CB} = M_{CD}$. The first approximation is not sufficient as the carry-over moments are appreciable. The carry-over moments are again listed and the resultant moments for the second approximation. Joint C is now balanced while joint B is unbalanced. The third approximation is then completed. It is apparent that the fourth is not required as the carry-over moments have disappeared.

The loads in the spans BC and CD are handled separately. The fixed-end moments equal

$$M_{BC} = \frac{wl^2}{12} = \frac{1000 \times (40)^2}{12} = 133{,}300 \text{ ft.-lb.}$$

$$M_{CD} = 75{,}000 \text{ ft.-lb.}$$

The computations in Figure 215 for these spans should be plain. The general conclusion from these three solutions is that three approximations give very closely the accurate moment. From the data in Figure 215 it is possible to handle any of the seven loadings of Problem 67. For instance, if Case II is taken, with a load of 3500 lb. per ft. in spans AB and CD, and a load of 1500 lb. per ft. in span BC, the bending moments at the supports are

$$M_{AB} = +44.8 \times 3500 - 56.0 \times 1500 + 5.4 \times 3500 = +91{,}900 \text{ ft.-lb.}$$

$$M_{BA} = -10.3 \times 3500 - 112.0 \times 1500 + 11.1 \times 3500 = -165{,}200 \text{ ft.-lb.}$$

$$M_{BC} = +165{,}200 \text{ ft.-lb.}$$

$$M_{CB} = +3.3 \times 3500 - 92.0 \times 1500 - 30.7 \times 3500 = -233{,}900 \text{ ft.-lb.}$$

$$M_{CD} = +233{,}900 \text{ ft.-lb.}$$

$$M_{DC} = -1.6 \times 3500 + 46.1 \times 1500 - 97.2 \times 3500 = -276{,}600 \text{ ft.-lb.}$$

These values check very closely those tabulated for Case II on Table I, Article 347. The moments on joint and beam are sketched at the bottom of Figure 215. The shear force and bending moment diagrams for the individual spans can now be computed as in Problem 67.

A comparison of the solution of this problem by the three-moment equation, by the slope-deflection equation, or by moment-distribution will enable the designer to decide which one is the easiest for him to use with continuous beams.

363. Deflection of Supports. The moment-distribution method has just been applied to a problem whose supports have no relative displacement. If the method of solution is compared with the slope-deflection equation for this case:

$$M_{AB} = 2EK(2\theta_A + \theta_B) \pm \text{(fixed-end moment)}$$

it will be seen that Professor Cross starts with the fixed-end moment and successively approximates the joint rotations.

When there is displacement of the supports, the slope-deflection equation becomes

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) \pm \text{(fixed-end moment)}$$

or

$$M_{AB} = 2EK(2\theta_A + \theta_B) + (-6EKR \pm \text{fixed-end moment})$$

Therefore, the moment-distribution method can be used if one starts with the fixed-end moments and the moment term due to deflection, correcting successively for joint rotations.

ILLUSTRATIVE PROBLEM 72

364. Continuous Beam with Deflecting Supports. Solve the continuous beam of Problem 66 for the bending moments at the supports by use of the moment-distribution method. Figure 216 shows the beam with its loads.

$$K_{AB} = \frac{I}{l} = \frac{14{,}000}{30 \times 12} = 38.9 \text{ (in.)}^3$$

$$K_{AC} = \frac{14{,}000}{20 \times 12} = 58.3 \text{ (in.)}^3$$

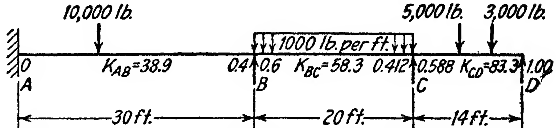
$$K_{CD} = \frac{14{,}000}{14 \times 12} = 83.3 \text{ (in.)}^3$$

Ordinarily the designer takes the K values as a round number, but in this problem we will use the above values in order to check the two previous solutions.

The deflection term equals:

$$+6EKR = \frac{6EIv}{l^2} = \frac{6 \times 2{,}000{,}000 \times 14{,}000}{(12)^3}\frac{v}{l^2} = 97{,}225{,}000\frac{v}{l^2} \text{ ft.-lb.}$$

where v is in inches and l in feet. The sign is positive as the rigid body is the joint not the beam.



	A			B	B			C	C			D
Deflection moment	+21.6			+21.6	-12.2			-12.2	-49.6			-49.6
Fixed end moment	+44.4			-22.2	+33.3			-33.3	+12.2			-13.4
Total moment	+66.0			- 0.6	+21.1			-45.5	-37.4			-63.0
Release joints	0.			- 8.2	-12.3			+34.2	+48.7			+63.0
Carry over	- 4.1			0.	+17.1			- 6.2	+31.5			+24.4
Distribution	0.	+61.9	-15.6	-6.8	-10.3	+15.6	-27.9	-10.4	-14.9	+27.9	0	-24.4
Carry over	-3.4			0.	- 5.2			- 5.2	-12.2			-7.5
Distribution	0.	+58.5	-13.5	+2.1	+3.1	+13.5	-25.9	+7.2	+10.2	+25.9	0	+7.5
Carry over	+1.1			0.	+3.6			+1.6	+3.8			+5.1
Distribution	0.	+59.6	-14.9	-1.4	-2.2	+14.9	-26.5	-2.2	-3.2	+26.5	0	-5.1
Carry over	-0.7			0.	-1.1			-1.1	-2.6			-1.6
Distribution	0.	+58.9	-14.5	+0.4	+0.7	+14.5	-26.1	+1.5	+2.2	+26.1	0	+1.6
Carry over	+0.2			0.	+0.8			+0.4	+0.8			+1.1
Distribution	0.			-0.3	-0.5			-0.5	-0.7			-1.1
Final	+59.1			-14.8	+14.8			-26.2	+26.2			0

FIG. 216

Deflection terms are

$$M_{AB} = \frac{+97{,}225{,}000 \times 0.20}{(30)^2} = +21{,}600 \text{ ft.-lb.}$$

$$M_{BC} = -\frac{97{,}225{,}000 \times 0.05}{(20)^2} = -12{,}200 \text{ ft.-lb.}$$

$$M_{CD} = -\frac{97{,}225{,}000 \times 0.10}{(14)^2} = -49{,}600 \text{ ft.-lb.}$$

The fixed-end moments on the joints are

$$M_{AB} = \frac{10{,}000(20)^2 \times 10}{(30)^2} = +44{,}400 \text{ ft.-lb.}$$

$$M_{BA} = \frac{10{,}000(10)^2 \times 20}{(30)^2} = -22{,}200 \text{ ft.-lb.}$$

$$M_{BC} = \frac{1000(20)^2}{12} = +33{,}300 \text{ ft.-lb.}$$

$$M_{CB} = -33{,}300 \text{ ft.-lb.}$$

$$M_{CD} = \frac{5000(8)^2 \times 6}{(14)^2} + \frac{3000(4)^2 \times 10}{(14)^2} = +12{,}200 \text{ ft. lb.}$$

$$M_{DC} = \frac{5000(6)^2 \times 8}{(14)^2} + \frac{3000(10)^2 \times 4}{(14)^2} = -13{,}400 \text{ ft.-lb.}$$

These values are listed and totaled at the top of the columns in Figure 216. It will be noticed that these totals should be identical with the numerical terms of the slope-deflection equations in Article 355. The release of joints, "carry over," and renewed distributions are identical in method with the previous problem. Note that the ratio of distribution at each joint is listed at the joint in the sketch at the top of Figure 216. At the first release there is no unbalanced moment at joint A since it remains fixed; at joint B the total unbalanced moment of $+20.5$ is balanced by the distribution

$$M_{BA} = 0.4 \times 20.5 = -8.2$$

$$M_{BC} = 0.6 \times 20.5 = -12.3$$

At joint C the total unbalanced moment of -82.9 is balanced by the distribution

$$M_{BC} = 0.412 \times 82.9 = +34.2$$

$$M_{CD} = 0.588 \times 82.9 = +48.7$$

At joint D the total unbalanced moment of -63.0 must be resisted by the beam CD, since the joint is hinged.

The carry-over moments are shown by arrows, and the second distribution occurs. There have been five approximations in this case, and the final moments check very closely the solutions by three-moment equation.

ILLUSTRATIVE PROBLEM 73

365. Moment-Distribution Applied to a Frame. Solve Problem 69 by use of the moment-distribution method. The frame is non-symmetrical, and the previous solution has shown that there is side-sway of the columns. Without the previous solution the designer would suspect that there is side-sway but does not know the amount. A possible solution by moment-distribution is to neglect the side-sway and make an independent computation for its effect. In other words, the moment M at any supporting joint equals

$$M = M_1 + M_2$$

where

$$M_1 = 2EK(2\theta_A + \theta_B) \pm C$$

and

$$M_2 = -6EK\frac{v}{l} = -6Ev\frac{I}{l^2}$$

The couple M_2 must be balanced by equal shear forces H acting at the top and bottom of each column. M_2 equals the shear H times the length, or

$$Hl = M_2 = 6E\frac{I}{l}\frac{v}{l} \qquad \text{and} \qquad v = \frac{M_2}{6E\dfrac{I}{l^2}}$$

To give an equal deflection v of the columns, these moments M_2 must be proportional to the $\frac{I}{l^2}$ of each column, or the shears H are proportional to $\frac{I}{l^3}$.

The problem can be solved by the moment-distribution method, if correction for this component couple M_2 be made to the column moments at the end of each approximation. Table M gives the solution for the frame with the beams loaded. The first approximation is completed for the component couple M_1 without regard to the column displacements.

TABLE M. MOMENT DISTRIBUTION. PROBLEM 73

	Joint A		Joint B		Joint C			Joint D				Joint E		F	G	H
Moment	AC	AB	BA	BD	CA	CD	CF	DC	DB	DG	DE	ED	EH	FC	GD	HE
K ratio	0.572	0.428	0.428	0.572	0.211	0.526	0.263	0.425	0.170	0.213	0.192	0.562	0.438	0	0	0
Fixed-end moment		+130.7	-130.7			+196.0		-196.0			+66.7	-66.7				
Distribution	-74.8	-55.9	+55.9	+74.8	-41.4	-103.0	-51.6	+55.0	+22.0	+27.5	+24.8	+37.5	+29.2			
Carry-over	-20.7	+28.0	-28.0	+11.0	-37.4	+27.5	0	-51.5	+37.4	0	+18.8	+12.4	0	-25.8	+13.8	+14.6
Distribution	-4.2	-3.1	+7.3	+9.7	+2.1	+5.2	+2.6	-2.0	-0.8	-1.0	-0.9	-7.0	-5.4	0	0	0
First Approximation	-99.7	+99.7	-95.5	+95.5	-76.7	+125.7	-49.0	-194.5	+58.6	+26.5	+109.4	-23.8	+23.8	-25.8	+13.8	+14.6
Carry-over	+1.1	+3.7	-1.6	-0.4	-2.1	-1.0	0	+2.6	+4.9	0	-3.5	-0.5	0	+1.3	-0.5	-2.7
Deflection correction	+5.6			+5.6	+5.6		-0.7		+5.6	-0.7			-0.5	-0.7	-0.7	-0.5
Distribution	-6.0	-4.4	-1.5	-2.1	-0.4	-0.9	-0.5	-3.8	-1.5	-1.9	-1.7	+0.6	+0.4	0	0	0
Second Approximation	-99.0	+99.0	-98.6	+98.6	-73.6	+123.8	-50.2	-195.7	+67.6	+23.9	+104.2	-23.7	+23.7	-25.2	+12.6	+11.4
Carry-over	-0.2	-0.8	-2.2	-0.8	-3.0	-1.9	0	-0.5	-1.1	0	+0.3	-0.9	0	-0.3	-1.0	+0.2
Deflection correction	+1.6			+1.6	+1.6		+0.7		+1.6	+0.7			+0.5	+0.7	+0.7	+0.5
Distribution	-0.3	-0.3	+0.6	+0.8	+0.5	+1.4	+0.7	-0.4	-0.2	-0.2	-0.2	+0.2	+0.2	0	0	0
Third Approximation	-97.9	+97.9	-100.2	+100.2	-74.5	+123.3	-48.8	-196.6	+67.9	+24.4	+104.3	-24.4	+24.4	-24.8	+12.3	+12.1
Carry-over	+0.3	+0.3	-0.2	-0.1	-0.2	-0.2	0	+0.7	+0.4	0	+0.1	-0.1	0	+0.4	-0.1	+0.1
Deflection correction	+1.1			+1.1	+1.1		+0.1		+1.1	+0.1			0	+0.1	+0.1	0
Distribution	-1.0	-0.7	-0.3	-0.5	-0.2	-0.4	-0.2	-1.0	-0.4	-0.5	-0.5	+0.1	0	0	0	0
Fourth Approximation	-97.5	+97.5	-100.7	+100.7	-73.8	+122.7	-48.9	-196.9	+69.0	+24.0	+103.9	-24.4	+24.4	-24.3	+12.3	+12.2

The two columns in the upper story are the same length and have the same moment of inertia. It has just been stated that, if the columns have the same displacement, the moment in each column will be proportional to the $\frac{I}{l^2}$ of the column, and the shear will be proportional to the $\frac{I}{l^3}$. In this case, the columns must have the same shears but opposite in direction in order to give equilibrium. Then

$$H_{CA} + H_{DB} = 0$$

$$H_{CA} \times 14 = M_{AC} + M_{CA}$$

$$H_{DB} \times 14 = M_{BD} + M_{DB}$$

Then

$$M_{AC} + M_{CA} + M_{BD} + M_{DB} = 0$$

This is one of the conditions used in the slope-deflection solution of Problem 69. At the end of the first approximation the moments sum up

$$\Sigma M_1 = -99.7 - 76.7 + 95.5 + 58.6 = -22.3$$

This unbalanced moment ΣM_1 must be balanced by a moment $\Sigma M_2 = +22.3$ distributed to the columns in proportion to their $\frac{I}{l^2} = \frac{K}{l}$, half being applied at each end to reproduce a fixed-end solution. The columns are alike, so

$$M_{AC} = M_{CA} = M_{BD} = M_{DB} = +\frac{22.3}{4} = +5.6$$

In the lower story it must be true that

$$M_{CF} + M_{FC} + M_{DG} + M_{GD} + M_{EH} + M_{HE} = 0$$

After the first approximation this equation runs

$$\Sigma M_1 = -49.0 - 25.8 + 26.5 + 13.8 + 23.8 + 14.6 = +3.9$$

This is balanced by an applied couple of $\Sigma M_2 = -3.9$ distributed to the columns in proportion to their $\frac{I}{l^2}$, in order to keep their displacements equal. These columns are all the same length so the ratio $\frac{I}{l^2} = \frac{K}{l}$ will be proportional to the K values. Then

$$M_{CF} = M_{FC} = M_{DG} = M_{GD} = -\frac{1}{2}\left(3.9 \times \frac{100}{270}\right) = -0.7$$

$$M_{EH} = M_{HE} = -\frac{1}{2}\left(3.9 \times \frac{70}{270}\right) = -0.5$$

In Table M the carry-over moments are listed first for the second approximation. Then the moments ΣM_2 are listed for the columns. The unbalanced moments at the joints are now distributed as usual. For instance, at joint A, the unbalanced moment equals $(+1.1 + 5.6 + 3.7) = +10.4$. This is distributed.

$$M_{AB} = 0.428 \times 10.4 = -4.4$$

$$M_{AC} = 0.572 \times 10.4 = -6.0$$

At the end of the second approximation the upper column moments sum up:

$$\Sigma M_1 = -99.0 - 73.6 + 98.6 + 67.6 = -6.4$$

This is divided up so that

$$M_{AC} = M_{CA} = M_{BD} = M_{DB} = +\frac{6.4}{4} = +1.6$$

The lower column moments sum up:

$$\Sigma M_1 = -50.2 - 25.2 + 23.9 + 12.6 + 23.7 + 11.4 = -3.8$$

This is balanced so that

$$M_{CF} = M_{FC} = M_{BD} = M_{DB} = \frac{3.8}{2} \times \frac{100}{270} = +0.7$$

$$M_{EH} = M_{HE} = \frac{3.8}{2} \times \frac{70}{270} = +0.5$$

Four approximations are listed with the unbalanced moments tending to reduce to zero. The results of the fourth approximation check very closely those obtained by the slope-deflection solution. The student can decide by similar comparisons which method gives him accurate results more rapidly.

366. Wind Loads. This method can be used for Problem 70 with the same frame subjected to wind pressure on the left side. In that case the upper columns appear as in Figure 217*a*. Note that $V_{AC} = V_{CA}$, $V_{BD} = V_{DB}$, and $H_{AC} = H_{BD} = H_{DB}$. Take moments about joint C with both columns as the rigid body:

$$M_{AC} + M_{CA} + M_{BD} + M_{DB} + 200 \times 14 \times 7 = 0$$

The correction applied at the end of each approximation must bring the sum of the column moments to $-19{,}600$ ft.-lb. $= -19.6$ ft.-kips.

Similarly the three lower columns as rigid bodies have the external forces shown in Figure 217*b*. Take the rigid body as whole frame. Then $V_{FC} + V_{GD} + V_{HE} = 0$ and $H_{FC} + H_{GD} + H_{HE} = 200(14 + 16) = 6000$ lb. acting to the left. Also $H_{CF} + H_{DG} + H_{EH} = 2800$ lb. acting to the right.

Take individual columns as rigid bodies. Then

$$H_{CF} + H_{FC} = 200 \times 16 = 3200 \text{ lb.} \qquad V_{CF} = V_{FC}$$

$$H_{DG} = H_{GD} \qquad V_{DG} = V_{GD}$$

$$H_{EH} = H_{HE} \qquad V_{EH} = V_{HE}$$

With the three columns as a rigid body, take moments about joint F:
$$M_{CF} + M_{FC} + M_{DG} + M_{GD} + M_{EH} + M_{HE} + 200 \times 16 \times 8 + 2800 \times 16 = 0$$

These column moments must be successively corrected to sum to −70,400 ft.-lb. = −70.4 ft.-kips. The student can construct the combined moment-distribution solution for this case.

Since the appearance of the moment-distribution method many variations in the conventions or method of operation have been devised, as

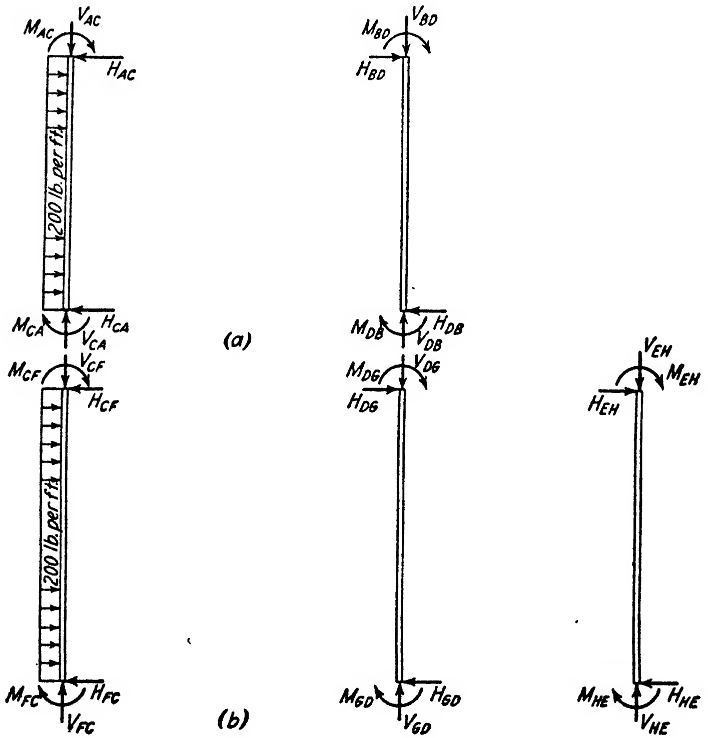


FIG. 217

well as short cuts to simplify its solution for the designer continually using this method.

367. Choice of Method. The general opinion of structural engineers seems to favor some form of slope-deflection solution for frames subjected to bending loads. The designer has welcomed the comparative ease of execution of the moment-distribution method. Other variations of the slope-deflection solutions have found favor.

The alternative solutions by use of virtual work, or least work, are described in this chapter and the next. They have their fields of usefulness particularly with trussed structures and arches.

The discussion in this chapter should enable the designer to attack the analysis of frames of unequal spans or unsymmetrical loading, for which he can find no previous solution.

MEMBERS OF VARYING SECTION

368. Moment Distribution Applied to Members with Variable Moment of Inertia. Any solution by means of equations for slope and deflection becomes more difficult if the moment of inertia of the member varies through its span. Haunched beams, columns with capitals, and even the transformed area of the ordinary reinforced concrete beams are illustrations of the common occurrence of such members. The flat slab floor system with a drop panel and column capitals is an extreme example. If a flat slab system is continuous and the spans are not "approximately equal," the 1941 A.C.I. Code recommends that the bending moments and shear forces be determined by an analysis of the structure as a continuous frame (A.C.I. Art. 1002). Prior to the adoption of this article Mr. R. L. Bertin studied one possible method of solution using moment-distribution. This survey has not been published, except for a short explanatory article in the July 1939 *Journal of the Boston Society of Civil Engineers*; but a summary of the method of solution will be given here to illustrate the application of moment-distribution to a flat slab continuous frame.

369. $\dfrac{M}{EI}$ Diagram for Flat Slab Construction. The continuous frame consists of the slab for the width of a bay taken continuously longitudinally, or transversely across the building, including the upper and lower columns in the bay width.

Slab. A section taken near the center of the span will be a rectangle whose width is the width of the bay and whose depth is that of the slab. Nearer the column the cross section is tee shaped as the drop panel is cut. Still nearer the column center line the cross section becomes larger as it includes portions of the column capital, and finally the section also cuts through the column. In this region the moment of inertia increases tremendously as compared with that at center span. If a single span of this slab is supported at the columns and an external couple M is applied at end A, the moment at any section varies uniformly from a value of M at A to zero at the far support B (Figure 218c). However, the $\dfrac{M}{EI}$ diagram will be exceedingly variable owing to the variation of I along the span. Figure 218*a* shows the general appearance of such a diagram. It is proposed to substitute the more regular area $CDEF$ for the full line $\frac{M}{EI}$ area, the distances xl being chosen so that the two areas are equal and their moments about the ends are equal (centers of gravity coincide). Mr. Bertin recommended that $x = b - (b - a)\frac{I}{I_1}$ (Fig. 218b).

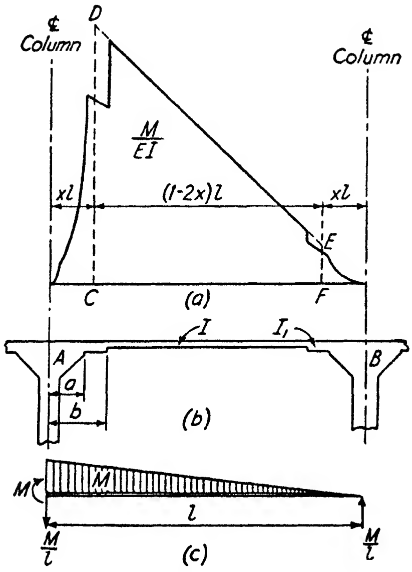


FIG. 218

If the couple M is applied at support A,

$$\text{Slope } \theta_A = \frac{Ml}{3EI}(1 - 3x + 3x^2 - 2x^3) = \frac{Ml}{3EI} \times f_4 \text{ (nearly)} \quad (274)$$

where $f_4 = 1 - 2.6x$.

$$\text{Slope } \theta_B = \frac{Ml}{6EI}(1 - 6x^2 + 4x^3) = \frac{Ml}{6EI} \times f_5 \text{ (nearly)} \quad (275)$$

where $f_5 = 1$ from $x = 0$ to $x = 0.05$

$f_5 = 1.066 - 1.33x$ from $x = 0.05$ to $x = 0.2$.

If the far end B is fixed, the "carry-over" factor for a distributed couple M at A equals

$$\text{Carry-over factor} = \frac{1}{2}\frac{f_5}{f_4} \quad (276)$$

The representative stiffness K_s for the slab equals

$$K_s = \frac{I}{l}\left(\frac{3f_4}{4f_4^2 - f_5^2}\right) \tag{277}$$

If a uniformly distributed load w pounds per foot acts on the slab, the fixed-end moment is

$$M = \frac{wl^2}{4}\left(\frac{f_5}{2f_4 + f_5}\right) \tag{278}$$

When $x = 0$, the fixed-end moment $= \dfrac{wl^2}{12}$, the carry-over factor $= \dfrac{1}{2}$, and the stiffness $K = \dfrac{I}{l}$, as commonly used for beams of constant moment of inertia.

Columns. The moment of inertia of the column will be constant until the column capital is reached. It then increases very rapidly and before the center line of the slab is reached, it is an exceedingly large value. As in the slab, the $\dfrac{M}{EI}$ values in the region of capital and slab are very small. It is proposed to substitute an $\dfrac{M}{EI}$ area, where I is the moment of inertia of the column proper. This diagram will be cut off at a distance yh below the slab center line, somewhere in the column capital. In practice yh is usually approximated by taking it equal to the distance to the bottom of the capital.

If a couple M is distributed at joint A to the lower end of the upper column,

$$\text{Slope } \theta_A = \frac{Mh}{3EI}(1 - y^3) = \frac{Mh}{3EI} \times f_3 \text{ (nearly)} \tag{279}$$

where $f_3 = 1$.

$$\text{Slope at top} = \frac{Mh}{6EI}(1 - 3y^2 + 2y^3) = \frac{Mh}{6EI} \times f_2 \text{ (nearly)} \tag{280}$$

where $f_2 = 1 - 0.5y$.

If a couple M is distributed at joint A to the upper end of the lower column,

$$\text{Slope } \theta_A = \frac{Mh}{3EI}(1 - y)^3 = \frac{Mh}{3EI} \times f_1 \text{ (nearly)} \tag{281}$$

where $f_1 = 1 - 2.5y$.

$$\text{Slope at bottom} = \frac{Mh}{6EI}(1 - 3y^2 + 2y^3) = \frac{Mh}{6EI} \times f_2 \text{ (nearly)} \tag{282}$$

When the far end of the upper column is fixed, the carry-over factor equals $\left(\frac{1}{2} \times \frac{f_2}{f_1}\right)$. When the lower end of the lower column is fixed, the carry-over factor is $\left(\frac{1}{2} \times \frac{f_2}{f_3}\right)$. (283)

Representative stiffness of upper column:

$$K_U = \frac{I}{h}\left(\frac{3f_1}{4f_1f_3 - f_2^2}\right) \tag{284}$$

Representative stiffness of lower column:

$$K_L = \frac{I}{h}\left(\frac{3f_3}{4f_1f_3 - f_2^2}\right) \tag{285}$$

The carry-over factor becomes $\frac{1}{2}$ and the stiffness $K = \frac{I}{h}$, if $y = 0$ for a column of constant moment of inertia.

ILLUSTRATIVE PROBLEM 74

370. Flat Slab Floor—Variable Span. Determine the moments in the continuous frame of Figure 219 due to a uniform load $w = 10{,}000$ lb. per ft. on span HI. The

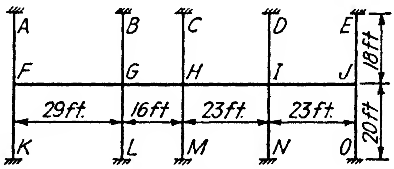


FIG. 219

dimensions of the slab and columns are given in Tables N and O. The bay is 23 ft. wide. Table P gives the moment distribution of the load on span HI, the far ends of the columns being assumed fixed.

$$\text{Fixed-end moment} = \frac{10 \times (23)^2}{4}\left(\frac{0.879}{2 \times 0.636 + 0.879}\right) = 540 \text{ ft.-kips}$$

The moment diagram for span HI due to this load is plotted in Figure 220. The positive moment of 159 ft.-kips can be used for comparison with other maximum positive moments in this span caused by other combinations of live and dead loads on any of the four spans. The negative moments have been obtained by using center line dimensions for the span; in other words these moments occur at the center of the columns. In beam and girder construction it is customary to use the negative moment at the edge of the column as the maximum value occurring in the actual beam cross section. For the flat slab the critical section may be one through the column, or colum capital, or drop panel. By reference to the moment coefficients long in use for fla

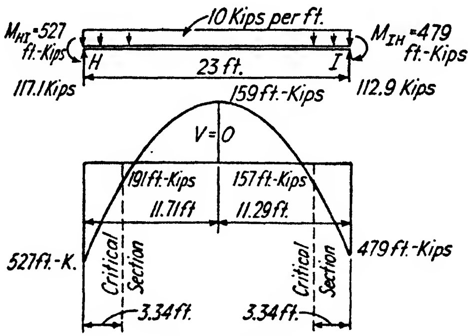


FIG. 220

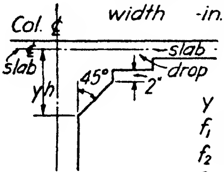
slabs of equal spans, A.C.I. Article 1002 recommends that the critical section for nega tive moment be taken at a distance $m = (0.073l + 0.57a)$. This distance $m = 3.34$ ft for span HI. The negative moments at these critical sections are shown in Figure 220. There has been a marked reduction from the center-line values. A.C.I. Article 100

TABLE N

SLAB	FG	GH	HI	IJ
Slab thickness-in	11	9	9	9
Drop panel, thickness-in				
left end	14	14	12	12
right end	14	12	12	12
dimension b-in	63	45	45	45
dimension a-in.				
left end	32	35	35	35
right end	35	35	35	32
I of slab -(in.)^4	30,650	16,750	16,750	16,750
I_1 at drop	43,310	37,250	26,410	26,410
dimension x-in.				
left end	0.118	0.201	0.140	0.140
right end	0.124	0.201	0.140	0.133
constant f_4				
left end	0.693	0.477	0.636	0.636
right end	0.678	0.477	0.636	0.654
constant f_5				
left end	0.909	0.798	0.879	0.879
right end	0.901	0.798	0.879	0.889
Stiffness K				
left end	167	458	137	137
right end	174	458	137	129
Carry-over Factor				
left to right	0.656	0.837	0.690	0.690
right to left	0.665	0.837	0.690	0.680

also recommends that the numerical sum of the maximum positive bending moment and the average of the two maximum negative moments, 333 ft.-kips in this case, shall not be less than $\frac{(10 \times 23) \times 23}{10}\left[1 - \frac{4}{3} \times (0.127)\right]^2 = 365$ ft.-kips. Of course,

TABLE O

UPPER COLUMNS	FA	GB	HC	ID	JE
Size, diameter-in		26	26	26	
thickness-in.	24				24
width -in.	36				36
					
y	0.150	0.150	0.150	0.150	0.150
f_1	0.625	0.625	0.625	0.625	0.625
f_2	0.925	0.925	0.925	0.925	0.925
f_3	1.0	1.0	1.0	1.0	1.0
I of column -(in)^4	41,500	22,400	22,400	22,400	41,500
Stiffness K	219	118	118	118	219
Carry-over factor bottom to top	0.740	0.740	0.740	0.740	0.740
LOWER COLUMNS	FK	GL	HM	IN	JO
Size, diameter-in.		30	30	30	
thickness-in.	24				24
width -in.	36				36
y	0.127	0.127	0.127	0.127	0.127
f_1	0.682	0.682	0.682	0.682	0.682
f_2	0.936	0.936	0.936	0.936	0.936
f_3	1.0	1.0	1.0	1.0	1.0
I of column-(in)^4	41,500	39,800	39,800	39,800	41,500
Stiffness K	280	268	268	268	280
Carry-over factor top to bottom	0.468	0.468	0.468	0.468	0.468

this single loading does not give either the maximum positive or the maximum negative moments in span HI for the dead load in all spans and all possible live-load combinations. The analysis so far is part of a solution similar to Problem 71.

THREE-DIMENSIONAL FRAMES

371. Resistance to Joint Rotation by Members Meeting from Three Directions. The preceding discussion of continuous frames has assumed that the members of the frame were located in a single plane. In many structures there are beams or girders framing into the column-beam joints perpendicular to this plane. If the loads tend to rotate the joint in the assumed plane, these members also resist such rotation and this resistance depends on the torsional stiffness of the perpendicular mem-

TABLE P

	Joint F			Joint G				Joint H				Joint I				Joint J		
Member	FA	FK	FG	GF	GB	GL	GH	HG	HC	HM	HI	IH	ID	IN	IJ	JI	JE	JO
$\frac{K}{\Sigma K}$	.329	.420	.251	.171	.116	.263	.450	.467	.120	.273	.140	.208	.178	.406	.208	.206	.348	.446
Fixed-end moment											+540	-540						
Distribution								-252	-65	-147	-76	+112	+96	+220	+112			
Carry-over							-211				+77	-52				+77		
Distribution				+36	+25	+55	+95	-36	-9	-21	-11	+11	+9	+21	+11	-16	-27	-34
Carry-over			+24				-30	+80			+8	-8			-11	+8		
Distribution	-8	-10	-6	+5	+3	+8	+14	-41	-11	-24	-12	+4	+3	+8	+4	-2	-3	-3
Carry-over			+3	-4			-34	+12			+3	-8			-1	+3		
Distribution	-1	-1	-1	+7.	+4	+10	+17	-7	-2	-4	-2	+2	+1	+4	+2	-1	-1	-1
ft.-kips Final Moment	-9	-11	+20	+44	+32	+73	-149	-244	-87	-196	+527	-479	+109	+253	+117	+69	-31	-38

bers. The discussion that follows will determine whether it is essential to consider the frame as a two-dimensional or three-dimensional frame.

372. Bending and Torsional Stiffness. When all members are in bending their relative stiffnesses are given by the comparative values of their $\frac{I}{l}$ ratios. When some of the members are subjected to torsion a more precise definition of stiffness must be adopted.

Bending stiffness of a straight member of constant moment of inertia may be defined as the couple applied at one end of a member, which

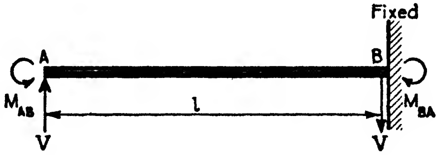


FIG. 221

produces *unit rotation* of the joint at that end, the other end being fixed (Fig. 221). By the slope-deflection equation, assuming A and B to remain at the same level,

$$\text{Bending stiffness} = K_B = M_{AB} = \frac{4EI\theta_A}{l} = \frac{4EI}{l} \tag{286}$$

$$\text{"Carry-over" factor} = \frac{1}{2}, \text{ since } M_{BA} = \frac{2EI\theta_A}{l} = \frac{M_{AB}}{2} \tag{287}$$

Torsional stiffness of the same member may be defined as the couple applied at one end of a member to give *unit rotation* of the joint at that end, if the other end is fixed (Fig. 222).

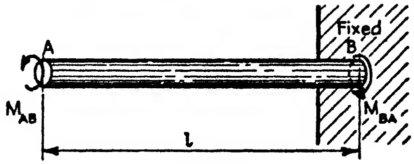


FIG. 222

$$\text{Torsional stiffness} = K_T = M_{AB} = \frac{GJ\theta_A}{l} = \frac{GJ}{l} \tag{288}$$

$$\text{"Carry-over" factor} = -1, \text{ since } M_{BA} = -M_{AB} \tag{288a}$$

373. Comparison of Bending and Torsional Stiffness. *Circular Section.* For circular sections of radius R,

$$\text{Bending stiffness} = K_B = \frac{4EI}{l} = \frac{\pi R^4 E}{l} \tag{289}$$

The relation between the torsional moment T and the angle of rotation θ equals $T = \dfrac{GI\theta}{l}$, where I is the polar moment of inertia.

$$\text{Torsional stiffness} = K_T = \frac{GI}{l} = \frac{\pi R^4 G}{2l} = \frac{3}{14}\frac{\pi R^4 E}{l} \tag{290}$$

The last term of equation 290 is obtained by the relation

$$G = \frac{E}{2(1+\nu)} = \frac{3}{7}E \tag{291}$$

where ν = Poisson's ratio = $\frac{1}{6}$ for concrete. Then

$$K_T = 0.214K_B \tag{292}$$

Rectangular Sections.

$$\text{Bending stiffness} = \frac{4EI}{l} = \frac{bh^3E}{3l} \tag{293}$$

The relation between the couple T and angle of rotation θ equals $T = \dfrac{GJ\theta}{l}$, where $J = k_1b^3h$.[1] k_1 depends on the ratio $\dfrac{h}{b}$ and can be obtained from Figure 107 (Art. 172).

$$\text{Torsion stiffness} = K_T = \frac{GJ}{l} = \frac{3}{7}\frac{k_1b^3hE}{l} \tag{294}$$

BENDING AND TORSIONAL STIFFNESS OF RECTANGULAR SECTIONS

Ratio $\dfrac{h}{b}$	$K_B = C_1\dfrac{b^4E}{l}$	$K_T = C_2\dfrac{b^4E}{l}$	Ratio $\dfrac{K_T}{K_B}$
	C_1	C_2	
(Square) 1.0	0.333	0.060	0.180
1.5	1.125	0.126	0.112
2.0	2.67	0.196	0.073

Tee Section. The bending stiffness $= \dfrac{4EI}{l}$. The moment of inertia is obtained using the gross section, where the flange width b is a conservative value (Art. 170). The center of gravity of the section is determined and the moment of inertia about the center of gravity is computed.

[1] Timoshenko, *Theory of Elasticity*, p. 249.

The torsional stiffness of tee sections has been determined by test and by the soap bubble analogy.[2] Approximate formulae for stiffness are suggested by Trayer and March (Fig. 223):

$$J = \mu_f \frac{bt^3}{16} + \mu_w \frac{(h - t)(b')^3}{16} + \alpha D^4 \tag{295}$$

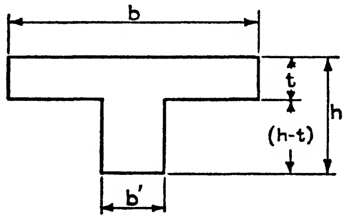


FIG. 223

where b and t are width and thickness of the flange

b' and $(h - t)$ are width and depth of the web

D = diameter of the largest circle that can be inscribed at the junction of flange and web

μ_f = flange constant for the ratio $\left(\frac{b}{t}\right)$ (See Fig. 224.)

μ_w = web constant for the ratio $\frac{2(h - t)}{b'}$.

$$\alpha = 0.148 \frac{\text{width of narrow rectangle}}{\text{width of wide rectangle}} = 0.148 \frac{t}{b'} \text{ (usually).}$$

Equation 295 assumes there is no fillet at the junction of flange and web.

$$\text{Torsional stiffness} = K_T = \frac{GJ}{l}$$

A single example will be taken for the tee section (Fig. 225). The center of gravity of the entire section is 9.67 in. from the top. The moment of inertia about the center of gravity I = 50,900 (in.)^4.

$$K_B = \frac{4EI}{l} = \frac{4 \times 50{,}900E}{l} = 203{,}600 \frac{E}{l}$$

For torsional stiffness,

$$\frac{b}{t} = \frac{60}{6} = 10 \qquad \text{and} \qquad \frac{2(h - t)}{b'} = \frac{2 \times 24}{12} = 4$$

[2] Nat. Advisory Committee for Aeronautics, Report 334, Trayer and March.

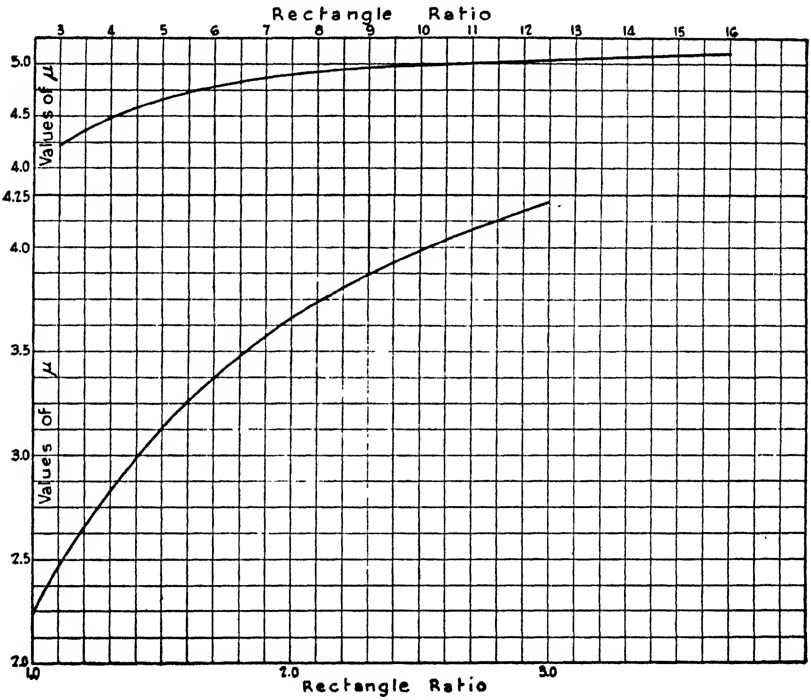


FIG. 224

$$J = \frac{5.00 \times 60 \times 216}{16} + \frac{4.49 \times 24 \times 1728}{16} + 0.148 \times \frac{6}{12}(12)^4$$

$$J = 4050 + 11{,}635 + 1535 = 17{,}220 \text{ (in.)}^4$$

$$K_T = \frac{GJ}{l} = \frac{3}{7}\frac{E}{l}(17{,}220) = 7380\frac{E}{l} \qquad \text{and} \qquad K_T = 0.036K_B$$

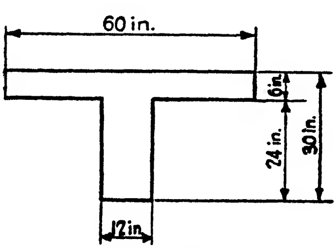


FIG. 225

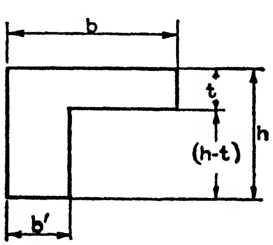


FIG. 226

Angle Section. The bending stiffness $= \frac{4EI}{l}$. The moment of inertia is again obtained about the center of gravity of the gross section (Fig. 226).

The torsional stiffness can be obtained by using

$$J = \mu_f \frac{(b - b')t^3}{16} + \mu_w \frac{h(b')^3}{16} + \alpha D^4 \tag{296}$$

where $(b - b')$ and t are width and depth of overhanging flange
b' and h are width and total height of web
D = diameter of largest circle that can be inscribed at the junction of flange and web
μ_f = flange constant for ratio $\dfrac{2(b - b')}{t}$ (See Fig. 224.)
μ_w = web constant for ratio $\dfrac{h}{b'}$
$\alpha = 0.07 \dfrac{\text{width of narrow rectangle}}{\text{width of wide rectangle}} = 0.07 \dfrac{t}{b'}$ (usually).

$$\text{Torsional stiffness} = K_T = \frac{GJ}{l}$$

An example of the determination of these stiffnesses is given in which the dimensions of Figure 227 are used. The center of gravity of the

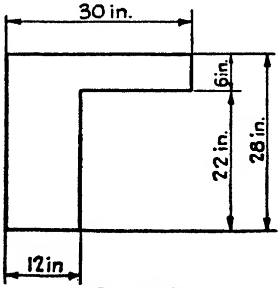


FIG. 227

section is 11.32 in. from the top. The moment of inertia about the center of gravity $I = 32{,}190$ (in.)^4.

$$K_B = \frac{4EI}{l} = 128{,}800 \frac{E}{l}$$

For torsional stiffness,

$$\frac{2(b - b')}{t} = \frac{2 \times 18}{6} = 6 \quad \text{and} \quad \frac{h}{b'} = \frac{28}{12} = 2.33$$

$$J = \frac{4.77 \times 18 \times 216}{16} + \frac{3.88 \times 28 \times 1728}{16} + 0.07 \times \frac{6}{12} (12)^4$$

$$J = 1160 + 11{,}730 + 730 = 13.620 \text{ in.}^4$$

$$K_T = \frac{GJ}{l} = \frac{3}{7}\frac{E}{l}(13{,}620) = 5840\frac{E}{l} \qquad \text{and} \qquad K_T = 0.045K_B$$

It is apparent that the circular section is by far the most efficient torsional section with a torsional stiffness of 0.214 of the bending stiffness. The usual concrete tee or angle beam is not an efficient shape to absorb torsional moments at a joint, as its torsional stiffness is usually less than 5 per cent of its bending stiffness. At a given joint such members in torsion will give little assistance to the perpendicular members in bending. Nevertheless, in order to cover the exceptional case of unusual loading, or a great divergence in section dimensions, the analysis of continuous space frames will be considered.

374. Moment Distribution Applied to Space Frames. If an unbalanced couple in the plane $EFBC$ is applied to the joint A (Fig. 228), all the members meeting at A resist rotation of the joint in this plane. The total stiffness resistance ΣK will be the sum of the bending stiffness of members, AB, AC, AE, and AF plus the sum of the torsional stiffnesses of the members AD and AG. If an unbalanced couple is applied to the joint A in the plane $DFGC$, the members AC, AD, AF, and AG resist in bending, and AB and AE are in torsion.

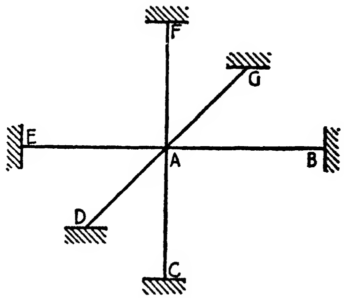


FIG. 228

When an unbalanced couple has been distributed to these members in proportion to their stiffnesses, the usual procedure of carry-over moments can be employed to bring unbalanced couples to the far ends (fixed ends for this operation) as with frames in a single plane.

As an example, assume that the beams of Figure 228 are all tee beams with a bending stiffness $K_B = 100$ and torsional stiffness $K_T = 4$. The columns AF and AC are square with a bending $K_B = 100$. If an unbalanced couple $M = 10{,}000$ is applied at joint A in the plane $EFBC$, the resisting moments at the end A of each member can be determined by moment distribution. $\Sigma K = 408$ and $M_{AE} = M_{AB} = M_{AF} = M_{AC} = \frac{100}{408} \times 10{,}000 = -2451$. The torsional moments in AD and AG equal $M_{AD} = M_{AG} = \frac{4}{408} \times 10{,}000 = -98$. The carry-over factor to the far end of members AE, AF, AB, AC is *plus one half*, whereas the carry-over factor for the torsional members AD and AG is *minus one.*

If the solution is made by dealing only with the members in plane $EFBC$, the distribution gives $M_{AE} = M_{AG} = M_{AF} = M_{AC} = -2500$ and $M_{AJ} = M_{AG} = 0$. The moments in the members in this plane are some 2 per cent greater than by the space frame solution, but the small torsional moments in AD and AG are neglected. It is so much easier to solve two-dimensional frames that space rame solutions are rarely made.

CURVED BEAMS

375. Curved Beams. Beams whose axes are curved in the horizontal plane will have both bending and torsional moments acting on the cross sections. The cases of bow girders and balcony beams will be taken as illustrations; it will be assumed that the load is uniformly distributed and the sections are uniform.

The ends of the girder will be assumed to be restrained by moments shown as vectors in Figure 229. The solution will be made by the

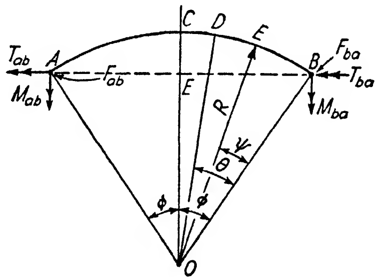


FIG. 229

method of virtual work,[3] temperature stresses being neglected. The basic equations are

$$1\delta_a + W_a = \delta_{ao} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} \qquad (297a)$$

$$1\delta_b + W_b = \delta_{bo} + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} \qquad (297b)$$

$$1\delta_c + W_c = \delta_{co} + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} \qquad (297c)$$

where X_a, X_b, X_c are numerical values of the redundants. δ_a, δ_b, δ_c are their displacements.

W_a, etc., equal the work done by the reactions when $X_a = 1$, etc., and δ_{ao}, etc., equal the work done by the forces developed under conditions $X_a = 1$, etc., when distorted under conditions $X = 0$.

[3] *Theory of Statically Indeterminate Structures*, Fife and Wilbur.

$$
\begin{aligned}
\delta_{aa} &= \Sigma \int \frac{M_a^2}{EI}\,ds + \Sigma \int \frac{T_a^2}{GJ}\,ds \\
\delta_{ab} &= \Sigma \int \frac{M_a M_b}{EI}\,ds + \Sigma \int \frac{T_a T_b}{GJ}\,ds \\
\delta_{ac} &= \Sigma \int \frac{M_a M_c}{EI}\,ds + \Sigma \int \frac{T_a T_c}{GJ}\,ds \\
\delta_{bb} &= \Sigma \int \frac{M_b^2}{EI}\,ds + \Sigma \int \frac{T_b^2}{GJ}\,ds \\
\delta_{bc} &= \Sigma \int \frac{M_b M_c}{EI} + \Sigma \int \frac{T_b T_c}{GJ}\,ds \\
\delta_{cc} &= \Sigma \int \frac{M_c^2}{EI}\,ds + \Sigma \int \frac{T_c^2}{GJ}\,ds \\
\delta_{ao} &= \Sigma \int \frac{M_a M_o}{EI}\,ds + \Sigma \int \frac{T_a T_o}{GJ}\,ds \\
\delta_{bo} &= \Sigma \int \frac{M_b M_o}{EI}\,ds + \Sigma \int \frac{T_b T_o}{GJ}\,ds \\
\delta_{co} &= \Sigma \int \frac{M_c M_o}{EI}\,ds + \Sigma \int \frac{T_c T_o}{GJ}\,ds
\end{aligned}
\tag{298}
$$

where M_a, etc., are the bending moments at any section of ACB due to a unit load in direction X_a, etc., M_o and T_o are the bending and torsional moments at any section due to the load on the beam.

Only the results of the solution will be given here.[4] Refer to Figure 229 and let the reactions at B be the redundants, also $X_a = M_{ba}$, $X_b = T_{ba}$, $X_c = F_{ba}$, where F_{ba} is the supporting force at B. If support A is fixed, $W_a = W_b = W_c = 0$.

When $\delta_a = \alpha_b$ in plane AB, $\delta_b = \delta_c = 0$,

$$
M_{ba} = \frac{\delta_{bb}\,\delta_{cc} - \delta_{bc}^2}{D} \cdot \alpha_b = A_1\alpha_b \tag{299a}
$$

$$
T_{ba} = \frac{\delta_{bc}\,\delta_{ca} - \delta_{ba}\,\delta_{cc}}{D} \cdot \alpha_b = A_2\alpha_b \tag{299b}
$$

$$
F_{ba} = \frac{\delta_{ba}\,\delta_{cb} - \delta_{bb}\,\delta_{ca}}{D} \cdot \alpha_b = A_3\alpha_b \tag{299c}
$$

[4] See "Moment Distribution in Bow Girders" by E. Curiel-Benfield, a thesis submitted in partial fulfillment of requirements for the degree of Master of Science at M.I.T.

where $D = \delta_{aa}(\delta_{bb}\,\delta_{cc} - \delta_{bc}\,\delta_{cb}) + \delta_{ba}(\delta_{cb}\,\delta_{ac} - \delta_{ab}\,\delta_{cc}) +$
$\delta_{ca}(\delta_{bc}\,\delta_{ab} - \delta_{ac}\,\delta_{bb})$.

When $\delta_a = \delta_c = 0$ and $\delta_b = \beta_b$ in plane perpendicular to AB,

$$M_{ba} = \frac{\delta_{cb}\,\delta_{ac} - \delta_{ab}\,\delta_{cc}}{D} \cdot \beta_b = B_1\beta_b \tag{300a}$$

$$T_{ba} = \frac{\delta_{aa}\,\delta_{cc} - \delta_{ac}^{\;2}}{D} \cdot \beta_b = B_2\beta_b \tag{300b}$$

$$F_{ba} = \frac{\delta_{ca}\,\delta_{ab} - \delta_{cb}\,\delta_{aa}}{D} \cdot \beta_b = B_3\beta_b \tag{300c}$$

or

$$\begin{aligned} M_{ba} &= A_1\alpha_b & \qquad M_{ba} &= T_{ba}\frac{B_1}{B_2} \\ T_{ba} &= M_{ba}\frac{A_2}{A_1} & \qquad T_{ba} &= B_2\beta_b \\ F_{ba} &= M_{ba}\frac{A_3}{A_1} & \qquad F_{ba} &= T_{ba}\frac{B_3}{B_2} \end{aligned} \tag{301}$$

Refer to Figure 229; the bow girder ACB is symmetrical about OC and is in equilibrium; therefore, when $\delta_a = \alpha_b$, $\delta_b = \delta_c = 0$,

$$M_{ab} = -M_{ba} - F_{ba} \times AB = -M_{ba}\left(1 + \frac{A_3}{A_1} \times AB\right) \tag{302}$$

$$T_{ab} = -T_{ba} = -M_{ba} \times \frac{A_2}{A_1} \tag{303}$$

When $\delta_a = \delta_c = 0$, $\delta_b = \beta_b$,

$$M_{ab} = -M_{ba} - F_{ba} \times AB = -T_{ba}\left(\frac{B_1}{B_2} + \frac{B_3}{B_2} \cdot AB\right) \tag{304}$$

$$T_{ab} = -T_{ba} \tag{305}$$

Similarly, when support A rotates through an angle α_a in plane AB

$$\begin{aligned} M_{ab} &= A_1\alpha_a & \qquad M_{ba} &= -M_{ab}\left(1 + \frac{A_3}{A_1} \cdot AB\right) \\ T_{ab} &= -M_{ab} \cdot \frac{A_2}{A_1} & \qquad T_{ba} &= M_{ab} \cdot \frac{A_2}{A_1} \\ F_{ab} &= M_{ab} \cdot \frac{A_3}{A_1} \end{aligned} \tag{306}$$

When support A rotates through an angle β_a in a plane perpendicular to AB,

$$M_{ab} = -T_{ab} \cdot \frac{B_1}{B_2} \qquad M_{ba} = -T_{ab}\left(\frac{B_1}{B_2} + \frac{B_3}{B_2} \cdot AB\right)$$

$$T_{ab} = B_2\beta_a \qquad T_{ba} = -T_{ab} \tag{307}$$

$$F_{ab} = -T_{ab} \cdot \frac{B_3}{B_2}$$

BOW GIRDER

376. Bow Girder. The term *bow girder* will be used in the application of the equations above as limited to a beam whose horizontal axis is an arc of a circle (Fig. 229).

$$\text{If } M_{ba} = 1,\ M_a = \cos(\phi - \theta) \text{ and } T_a = -\sin(\phi - \theta)$$

$$\text{If } T_{ba} = 1,\ M_b = \sin(\phi - \theta) \text{ and } T_b = \cos(\phi - \theta) \tag{308}$$

$$\text{If } F_{ba} = 1,\ M_c = R \sin\theta \text{ and } T_c = R(1 - \cos\theta)$$

Substituting equations 308 into equations 298 and integrating between 0 and 2ϕ, where $ds = R\,d\theta$, one obtains

$$\delta_{aa} = \frac{R}{2EI}(2\phi + \sin 2\phi) + \frac{R}{2GJ}(2\phi - \sin 2\phi)$$

$$\delta_{ab} = 0$$

$$\delta_{ac} = \frac{R^2}{2EI}\sin\phi(2\phi + \sin 2\phi) + \frac{R^2}{2GJ}\sin\phi(2\phi - \sin 2\phi)$$

$$\delta_{bb} = \frac{R}{2EI}(2\phi - \sin 2\phi) + \frac{R}{2GJ}(2\phi + \sin 2\phi) \tag{309}$$

$$\delta_{bc} = \frac{R^2}{2EI}(\sin\phi + \sin\phi\cos 2\phi - 2\phi\cos\phi) + \frac{R^2}{2GJ}(3\sin\phi - \sin\phi\cos 2\phi - 2\phi\cos\phi)$$

$$\delta_{cc} = \frac{R^3}{EI}\left(\phi - \frac{1}{4}\sin 4\phi\right) + \frac{R^3}{GJ}\left(3\phi + \frac{1}{4}\sin 4\phi - 2\sin 2\phi\right)$$

For a uniformly distributed load w, consider end A to be fixed and end B to be free. The bending moment at D due to the load w over the length BD is obtained by integration of a load wds acting at E.

$$M_o = -R^2 w \int_o^\theta \sin(\theta - \psi)\, d\psi = -R^2 w(1 - \cos\theta) \qquad (310a)$$

$$T_o = -R^2 w \int_o^\theta (1 - \cos\psi)\, d\psi = -R^2 w(\theta - \sin\theta) \qquad (310b)$$

Substituting in equations 298,

$$\delta_{ao} = -\frac{R^3 w}{EI}(\sin\phi + \sin^3\phi - \phi\cos\phi) - \frac{R^3 w}{GJ}(3\sin\phi - \sin^3\phi - 3\phi\cos\phi)$$

$$\delta_{bo} = \frac{R^3 w}{2}\left(\frac{1}{EI} - \frac{1}{GJ}\right)(2\phi - \sin 2\phi)\sin\phi \qquad (311)$$

$$\delta_{co} = \frac{R^4 w}{2EI}(\sin^2 2\phi + 2\cos 2\phi - 2) - \frac{R^4 w}{2GJ}[\sin^2 2\phi - 4\phi\sin 2\phi + (2\phi)^2]$$

Equations 309 and 311 are substituted into equations 297 to solve for the fixed-end moments, where $\delta_a = \delta_b = \delta_c = 0$ and $W_a = W_b = W_c = 0$. The solutions are much easier if numerical values are used.

ILLUSTRATIVE PROBLEM 75

377. Bow Girder. *Fixed-End Moments.* Assume a bow girder with $AB = 20$ ft., $CE = 5$ ft., and $w = 2000$ lb. per ft. Then $R = 12.5$ ft. and $\phi = 53.13°$. Assume the girder section to be 20 in. square. In that case, $I = \frac{b^4}{12} = 13{,}330$ (in.)^4 and $J = 0.1406b^4 = 22{,}500$ (in)^4. If Poisson's ratio equals $\frac{1}{6}$, $G = \frac{3}{7}E$, and $GJ = 0.7235EI$.

Substitution of $\phi = 0.928$ radians and the functions of ϕ and 2ϕ in equations 309 and 311 give the results tabulated below.

$$\delta_{aa} = +\frac{25.34}{EI} \qquad \delta_{bb} = +\frac{29.91}{EI} \qquad \delta_{ao} = -\frac{2062}{EI}w$$

$$\delta_{ab} = 0 \qquad \delta_{bc} = +\frac{121.13}{EI} \qquad \delta_{bo} = -\frac{267.6}{EI}w$$

$$\delta_{ac} = +\frac{253.4}{EI} \qquad \delta_{cc} = +\frac{4045}{EI} \qquad \delta_{co} = -\frac{33{,}550}{EI}w$$

The fixed-end moments can be found by substituting these values into equations 297 and noting that $\delta_a = \delta_b = \delta_c = 0$ and $W_a = W_b = W_c = 0$. The three equations are

$$+25.34M_{ba} \qquad\qquad + 253.4F_{ba} = +2062w$$

$$+29.91T_{ba} + 121.13F_{ba} = +267.6w$$

$$+253.4M_{ba} + 121.13T_{ba} + 4045F_{ba} = +33{,}550w$$

The solution of these three simultaneous equations gives

$$M_{ba} = -34.73w \qquad T_{ba} = -38.06w \qquad F_{ba} = 11.61w$$

Since the girder and loading are symmetrical about OY, the force and couples at support A equal

$$M_{ab} = +34.73w \qquad T_{ab} = -38.06w \qquad F_{ab} = 11.61w$$

Owing to the symmetry, $F_{ab} = F_{ba} = wR\phi$. If one takes moments about the line AB, $T_{ab} = T_{ba}$ can be found by statics. Therefore, it is only necessary to use the first equation above to solve for M_{ba}. It should be noted that the so-called bending moments M_{ab} and M_{ba} do not act perpendicular to the cross sections at A and B, nor do the torsional moments T_{ab} and T_{ba} act in the cross-sectional planes.

378. Moment Distribution. The directions OC and AB of Figure 229 have been chosen to act parallel to the other beams framing into the joints A and B. If the complete continuous frame is analyzed by the method of moment distribution, the moment M_{ba} distributed to the bow girder ACB will have carry-over moments of M_{ab}, T_{ab}, and T_{ba}. These moments can be found by use of equations 302 and 303. Similarly the carry-over moments due to a distributed moment of M_{ab}, or T_{ab}, or T_{ba} can be computed by equations 302 to 307.

BALCONY BEAM

379. Balcony Beam. A beam composed of straight members, similar to that of Figure 230, can be analyzed by the same method, that of

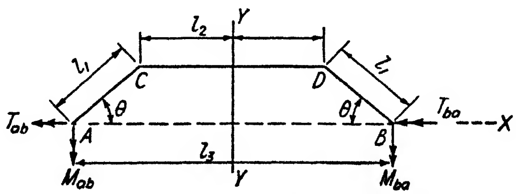


FIG. 230

virtual work, equations 297 to 307 inclusive being used. For the symmetrical case of Figure 230:

$$\delta_{aa} = \frac{1}{EI}[2l_1 \cos^2 \theta + l_2] + \frac{1}{GJ}[2l_1 \sin^2 \theta]$$

$$\delta_{ab} = 0$$

$$\delta_{ac} = \frac{1}{EI}\left[l_1^2 \cos \theta(1 + \cos 2\theta) + l_1 l_2(1 + \cos \theta)\frac{l_2^2}{2}\right] + \frac{1}{GJ}[l_1 l_3 \sin^2 \theta]$$

$$\delta_{bb} = \frac{1}{EI}[2l_1 \sin^2 \theta] + \frac{1}{GJ}[2l_1 \cos^2 \theta + l_2] \qquad (312)$$

$$\delta_{bc} = \frac{1}{EI}[(-l_1 \sin \theta)(l_2 \cos \theta + l_1 \cos 2\theta)] + \frac{1}{GJ}[(l_1 \cos \theta)(l_2 + l_3 \sin \theta)]$$

$$\delta_{cc} = \frac{1}{EI}\left[l_1^3\left(\frac{2}{3} + \cos 2\theta(1 + \cos 2\theta)\right) + \right.$$
$$\left. l_1^2 l_2 \cos \theta(1 + \cos \theta + \cos 2\theta) + l_1 l_2^2 \cos \theta(1 + \cos \theta) + \frac{l_2^3}{3}\right] +$$
$$\frac{1}{GJ}[l_1^2 l_2 \cos^2 \theta + l_1 l_3^2 \sin^2 \theta]$$

For a uniformly distributed load w, with end A fixed, take moments about a section distant s from right end of each member:

Member DB: $M_o = -\dfrac{ws^2}{2}$ and $T_o = 0$

Member CD: $M_o = -\dfrac{w}{2}[s^2 + 2sl_1 + l_1^2 \cos \theta]$

$$T_o = -\frac{wl_1^2}{2}\sin \theta \qquad (313)$$

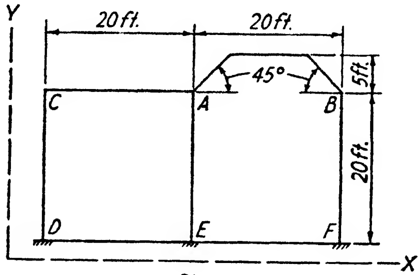
Member AC: $M_o = -\dfrac{w}{2}[s^2 + 2s(l_1 + l_2) + l_1^2 \cos 2\theta + l_2^2 \cos \theta + 2l_1 l_2 \cos \theta]$

$$T_o = -\frac{w}{2}[l_2^2 \sin \theta + 2l_1 l_2 \sin \theta + l_1^2 \sin 2\theta]$$

Numerical values of l_1, l_2, and the functions of θ should be placed in the above equations for M_o and T_o before substituting in equations 298 for δ_{ao}, δ_{bo}, and δ_{co}.

ILLUSTRATIVE PROBLEM 76

380. A Balcony Beam as Part of a Continuous Frame. Given the floor plan of Figure 231, compute the fixed-end moments for the balcony beam AB and determine by moment distribution the moments in the other members of the frame resulting from a uniformly distributed load of 2000 lb. per ft. acting on AB. Assume the joints at D, E, and F to be fixed. The interior beam AE has the dimensions of Figure 225, all exterior beams have sections shown by Figure 227, the balcony beam



Plan

FIG. 231

has a section 20 in. square. All upper columns are 20 in. square and all lower columns are 23 in. square. The far ends of all columns are fixed. All columns are 15 ft. long.

381. Balcony Beam. *Fixed-End Moments.* As in Problem 75, let $GJ = 0.7235EI$. Substituting in equation 312,

$$\delta_{aa} = +\frac{26.84}{EI} \qquad \delta_{bb} = +\frac{30.67}{EI}$$

$$\delta_{ab} = 0 \qquad \delta_{bc} = +\frac{131.5}{EI}$$

$$\delta_{ac} = +\frac{268.9}{EI} \qquad \delta_{cc} = +\frac{4326}{EI}$$

As explained in Article 375, M_a, etc., are the bending moments at any section due to a unit load in direction X_a, etc. From the geometry of Figure 230,

Member DB: $M_a = \cos\theta = 0.707$ $\qquad M_b = \sin\theta = 0.707$

$M_c = s$ $\qquad T_a = -\sin\theta = -0.707$

$T_b = \cos\theta = 0.707$ $\qquad T_c = 0$

Member CD: $M_a = \cos 0° = 1$ $\qquad M_b = 0$

$M_c = s + l_1 \cos\theta = s + 5$ $\qquad T_a = 0$

$T_b = \cos 0° = 1$ $\qquad T_c = l_1 \sin\theta = 5$

Member AC: $M_a = \cos\theta = 0.707$ $\qquad M_b = -\sin\theta = -0.707$

$M_c = s + l_2 \cos\theta + l_1 \cos 2\theta = s + 7.07$ $\qquad T_a = \sin\theta = 0.707$

$T_b = \cos\theta = 0.707$ $\qquad T_c = l_2 \sin\theta = 14.14$

By equations 313,

$$\text{Member } DB: \quad M_o = -\frac{ws^2}{2} \quad \text{and} \quad T_o = 0$$

$$\text{Member } CD: \quad M_o = -\frac{w}{2}(s^2 + 14.14s + 35.35)$$

$$T_o = -17.68w$$

$$\text{Member } AC: \quad M_o = -\frac{w}{2}(s^2 + 34.14s + 170.7)$$

$$T_o = -110.4w$$

Substituting these values in equation 298,

$$\delta_{ao} = -\frac{w}{EI}\left[\int_o^{l_1} (0.707)\left(\frac{s^2}{2}\right) ds + \int_o^{l_2} (1)\left(\frac{1}{2}\right)(s^2 + 14.14s + 35.35)\, ds + \int_o^{l_1} (0.707)\left(\frac{1}{2}\right)(s^2 + 34.14s + 170.7)\, ds\right] - \frac{w}{GJ}\left[0 + 0 + \int_o^{l_1} (0.707)(110.4)\, ds\right]$$

$$= -\frac{2272}{EI}w$$

Similarly,

$$\delta_{bo} = -\frac{280}{EI}w \quad \text{and} \quad \delta_{co} = -\frac{36{,}946}{EI}w$$

The supporting forces $F_{ab} = F_{ba} = \frac{w}{2}(2l_1 + l_2) = 12.07w = 24{,}140$ lb.

If the supports at A and B are fixed, take moments about the line AB, noting that $T_{ab} = T_{ba}$:

$$w\left[\frac{2l_1^2 \sin\theta}{2} + l_2(l_1 \sin\theta)\right] + T_{ab} + T_{ba} = 0$$

$$T_{ab} = T_{ba} = -42.68w = -85{,}400 \text{ ft.-lb.}$$

The moments M_{ab} and M_{ba} can be found by use of the equation 297*a*.

$$0 = -2272w + 26.84M_{ba} + 268.9 \times 12.07w$$

$$M_{ba} = -35.9w = -71{,}800 \text{ ft.-lb.} \quad \text{and} \quad M_{ab} = +71{,}800 \text{ ft.-lb.}$$

382. Balcony Beam. *Moment Distribution Coefficients.* A moment M_{ba} distributed to the balcony beam at support B will cause carry-over moments of T'_{ba}, M'_{ab}, and T'_{ab} at both supports. This couple M_{ba} acts in the plane AB and equals $A_1\alpha_b$ (equation 299*a*), where A_1 is the stiffness factor K_B to be used for rotation of joint B through an angle α_b in the plane AB. To obtain the numerical value of A_1 it is necessary to use equation 299*a* and the value of D given immediately below equations 299.

$$D = \frac{1}{(EI)^3}\{26.84[30.67 \times 4326 - (131.5)^2] + 0 + 268.9[0 - 268.9 \times 30.67]\}$$

$$= \frac{876{,}780}{(EI)^3}$$

By equations 299 and 300,

$$A_1 = \frac{30.67 \times 4326 - (131.5)^2}{876{,}780} = +0.1315EI \qquad \text{and} \qquad B_1 = +0.04033EI$$

$$A_2 = +0.04033EI \qquad\qquad B_2 = +0.04987EI$$

$$A_3 = -0.009406EI \qquad\qquad B_3 = -0.004026EI$$

At support B, because of a rotation α_b in the plane AB, by equations 299 to 303,

$$M_{ba} = 0.1315EI\alpha_b = K_B\alpha_b \qquad M'_{ab} = 0.430M_{ba}$$

$$T'_{ba} = 0.3065M_{ba} \qquad T'_{ab} = -0.3065M_{ba}$$

At support B, because of a rotation β_b in a plane perpendicular to AB,

$$T_{ba} = 0.04987EI\beta_b = K_T\beta_b \qquad M'_{ab} = 0.8075T_{ba}$$

$$M'_{ba} = 0.8075T_{ba} \qquad T'_{ab} = -T_{ba}$$

At support A, because of a rotation α_a in plane AB,

$$M_{ab} = 0.1315EI\alpha_a = K_B\alpha_a \qquad M'_{ba} = 0.430M_{ab}$$

$$T'_{ab} = -0.3065M_{ab} \qquad T'_{ba} = 0.3065M_{ab}$$

At support A, because of a rotation β_a in a plane perpendicular to AB,

$$T_{ab} = 0.04987EI\beta_a = K_T\beta_b \qquad M'_{ba} = -0.8075T_{ab}$$

$$M'_{ab} = -0.8075T_{ab} \qquad T'_{ba} = -T_{ab}$$

383. Solution of Continuous Frame. The necessary cross-sectional properties of the members shown in Figure 231 are given in Table Q.

Table Q

	Cross Section	I (in.)^4	J (in.)^4	Stiffness	
				K_B	K_T
Balcony beam AB	20 in. square	13,330	22,550	1,753E	665E
Interior beam AE	Fig. 225	50,900	17,220	10,250E	369E
Exterior beams	Fig. 227	32,190	13,620	6,438E	292E
Upper columns	20 in. square	13,330	.....	3,556E	.....
Lower columns	23 in. square	23,320	.....	6,219E	.....

TABLE R

	Joint B							
Member	BB'		BB"		BF		BA	
Moment	M_x	M_y	M_x	M_y	T_x	M_y	M_{ba}	T_{ba}
$\frac{K}{\Sigma K}$	.300	.211	.527	.369	.025	.381	.148	.039
Moment = ft. Kips								
Fixed End Moment							-71.8	-85.4
Distribution	+21.6	+18.0	+37.8	+31.5	+1.8	+32.6	+10.6	+3.3
Carry-over					due to	M_{ab}	-3.0	-2.1
					due to	T_{ab}	-2.2	-2.7
					due to	M_{ba}		+3.3
					due to	T_{ba}	+2.7	
Distribution	+0.7	+0.3	+1.3	+0.5	+0.1	+0.6	+0.4	+0.1
Carry-over					due to	M_{ab}	-0.2	-0.2
					due to	T_{ab}	-0.1	-0.1
					due to	M_{ba}		+0.1
					due to	T_{ba}	+0.1	
Distribution	+0.1	+0	+0.1	+0.1	+0	+0.1	+0	+0
Final Moment	+22.4	+18.3	+39.2	+32.1	+1.9	+33.3	-63.5	-83.7

	Joint A									
Member	AB		AA'		AA"		AE		AC	
Moment	M_{ab}	T_{ab}	M_x	M_y	M_x	M_y	T_x	M_y	M_x	T_y
$\frac{K}{\Sigma K}$	.096	.032	.194	.169	.339	.296	.020	.489	.351	.014
Moment = ft. Kips										
Fixed End Moment	+71.8	-85.4								
Distribution	-6.9	+2.7	-13.9	+14.4	-24.4	+25.3	-1.4	+41.8	-25.2	+1.2
Carry-over		+2.1								
	-2.2									
	+4.6	-3.3								
	+2.7	-3.3								
Distribution	-0.5	+0.1	-1.0	+0.8	-1.7	+1.3	-0.1	+2.2	-1.8	+0.1
Carry-over		+0.2							+2.5	0
	-0.1									
	+0.2	-0.1								
	+0.1	-0.1								
Distribution	-0.3	0	-0.5	0	-0.9	0	-0.1	0	-0.9	0
Final Moment	+69.4	-87.1	-15.4	+15.2	-27.0	+26.6	-1.6	+44.0	-25.4	+1.3

	Joint C							
Member	CA		CD		CC'		CC"	
Moment	M_x	T_y	T_x	M_y	M_x	M_y	M_x	M_y
$\frac{K}{\Sigma K}$	.390	.018	.018	.390	.215	.215	.377	.377
Moment = ft. Kips								
Fixed End Moment								
Distribution								
Carry-over	-12.6	-1.2						
Distribution	+4.9	+0	+0.2	+0.5	+2.7	+0.3	+4.8	+0.4
Carry-over	-0.9	-0.1						
Distribution	+0.4	0	+0	+0.1	+0.2	0	+0.3	0
Final Moment	-8.2	-1.3	+0.2	+0.6	+2.9	+0.3	+5.1	+0.4

Table R gives the moment distribution over the frame of Figure 231 of a load of $w = 2000$ lb. per ft. on the balcony beam AB. Columns labeled BB', etc., are upper columns, those labeled BB'', etc., are lower columns. The carry-over factors for all columns and beams are $(+\frac{1}{2})$ for bending and (-1) for torsion, except for the balcony beam.

The results show little change from the fixed-end values of M_{ab}, T_{ab}, and T_{ba}, but a 10 per cent decrease of M_{ba}. Before the bending and torsional moment diagrams are plotted for the balcony girder, the moments M_{ab} and T_{ab} should be resolved into true bending and torsional couples perpendicular or coinciding with the cross section at A; also M_{ba} and T_{ba} at section B should be resolved in the same manner.

CHAPTER 15

ARCHES AND RIGID FRAME BRIDGES

The arch and rigid frame bridge are two examples of the structure whose center line is curved, or changes direction, in the vertical plane. There is the added complication that the cross sections vary in depth, this variation being especially marked in the rigid frame bridge. Most designers agree that curved members with variable section are best analyzed by a virtual work or least work solution. Virtual work was used in the previous chapter to determine the fixed-end moments for the bow girder and will now be employed to establish a method of arch design.

384. Arches. From the earliest examples to recent ones arches have been compression members. They are, even now, usually so loaded that all the particles in a cross section have compressive stresses. Therefore, concrete is a legitimate material with which to construct arches. Plain concrete arches, like masonry ones, should be of such a shape that the line of the resultant pressure at each section lies within the middle third. In such a case all particles in the section have compressive stresses. The reinforced concrete arch can have tensile stresses, but it is good practice to keep the resultant pressure close to the center of gravity of the section because variations in the position of live loads or moving loads will cause variations in the stress distribution at a section.

385. Arch Nomenclature. The arch *springs* from the foundation or abutment. The thickest section of the arch proper is known as the *springing* (Fig. 232). It is easily located in the masonry arches, but in

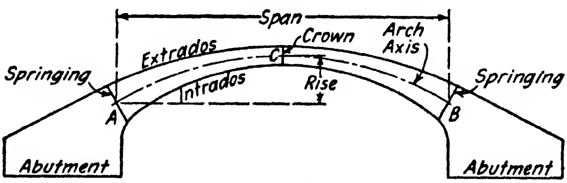


FIG. 232

concrete construction, with abutment and arch poured continuously, it will be defined as the last section in the abutment which remains fixed in position. The upper surface of the arch is the *extrados*; the inner surface is the *intrados*. The *arch axis* is the line passing through the center of gravity of each section. The *crown* is the highest section. For design, the *span* is the distance between the centers of gravity of the two springing sections, and the *rise* is the vertical distance from the center of gravity of the crown to that of the springing. Out in the field the term span is sometimes used as the clear distance between the vertical faces of the abutment, and rise as the vertical distance from the top of the vertical abutment face to the highest point of the intrados.

386. Design Theory. Hingeless Arch. The reinforced concrete arch is continuous from abutment to abutment and is statically indeterminate. The arch is statically indeterminate because at each springing section there may act:

1. A shear force V_S.
2. A normal force N_S at the center of gravity.
3. A bending couple M_S.

There will be six unknowns. For a curved member, such as an arch, the solution by least work is recommended.

The derivation will be made for an arch which is symmetrical about the crown section (Fig. 233). The half arches will be considered separately, and the shear force V_C, normal force H_C, and couple M_C acting

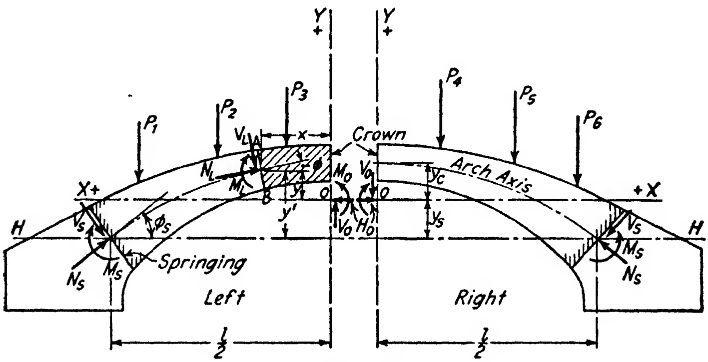


FIG. 233

at the crown will be obtained. Since the two sides of the crown section do not displace relative to each other, the theory of least work may be used.

387. Assumptions. The following assumptions are made.

I. The normal stresses at any section are distributed as in a section of a straight beam subjected to the same forces. This is sufficiently accurate for curved beams with a large radius of curvature.

II. The normal force N at any section will be taken equal to H_C, the normal force at the crown section, in computing the work done by the normal forces. This approximation makes for simplicity and is justified since the work due to the normal force is a small part of the total work.

III. The work done by the shear force V will be neglected. This is the customary procedure in work solutions, since its effect on the computed forces is negligible.

388. Derivation Nomenclature. In addition to the terms previously defined, let

s = length of arch measured on the arch axis
W = work done
H_O = normal force at origin O (Fig. 233)
V_O = shear force at origin O
M_O = bending moment at origin O
M_L = bending moment at any section of left half-arch
M_R = bending moment at any section of right half-arch
m_L = sum of moments about the center of gravity of a section in the left half-arch of the loads between the section and the crown
m_R = sum of moments about the center of gravity of a section in the right half-arch of the loads between the section and the crown
X = one of the unknowns.

389. Determination of Crown Forces by Least Work. In Figure 233 let the OY axis be taken through the crown section for both half-arches. It would seem natural to take the OX axis through the center of gravity of the crown, but there is a gain in simplicity if the OX axis be taken below the crown at such a distance y_c that $\int \frac{y\,ds}{EI} = 0$. The y dimensions are *plus*, measured upwards. The x dimensions are *plus*, measured to the left for the left half-arch, and *plus*, measured to the right for the right half-arch.

If an unknown force X acts at a section of a member, the displacement δ of this section in the direction of X is given by work equations from textbooks on indeterminate structures. The displacement due to bending in the member is

$$\delta + L = \frac{\partial W}{\partial X} = \int_0^{s/2} \frac{M}{EI} \cdot \frac{\partial M}{\partial X}\,ds$$

where L = the displacement of the supports.

The displacement of the same section in the direction of a normal force X due to the normal forces N acting throughout the member equals

$$\delta + L = \frac{\partial W}{\partial X} = \int_0^{s/2} \frac{N}{EA} \cdot \frac{\partial N}{\partial X} ds$$

Taking the portion of the arch between the crown and a given section AB as a rigid body, the external forces acting are

(a) Forces V_O and H_O and couple M_O acting at the crown section but transferred to O.
(b) Loads P_2, P_3, etc.
(c) Forces N_L and V_L, and couple M_L, acting at section AB.

Taking moments about the center of the section AB,

$$\begin{aligned} &\text{Left half-arch:} \qquad && M_L = M_O - m_L - H_O y + V_O x \\ &\text{Right half-arch:} \qquad && M_R = M_O - m_R - H_O y - V_O x \end{aligned}$$

also

$$N = H_O \text{ for the least-work derivation}$$

If the two sides of the crown sections do not move apart in a horizontal direction and the abutments do not move horizontally,

$$\delta + L = 0 = \frac{\delta W}{\delta H_O} = \int_0^{s/2} \frac{M_L}{EI} \cdot \frac{\partial M_L}{\partial H_O} ds + \int_0^{s/2} \frac{M_R}{EI} \cdot \frac{\partial M_R}{\partial H_O} ds + \cdot$$
$$2 \int_0^{s/2} \frac{N}{EA} \cdot \frac{\partial N}{\partial H_O} ds$$

$$0 = \int_0^{s/2} (M_O - m_L - H_O y + V_O x)(-y) \frac{ds}{EI} +$$
$$\int_0^{s/2} (M_O - m_R - H_O y - V_O x)(-y) \frac{ds}{EI} + 2H_O \int_0^{s/2} \frac{ds}{EA}$$

But

$$\int \frac{y\,ds}{EI} = 0 \text{ by assumption}$$

M_O is a constant multiplied by this term, the terms $+V_O x$ and $-V_O x$ cancel, and the moments m_L and m_R vary from section to section. Therefore,

$$0 = \int_0^{s/2} \frac{(m_L + m_R)}{EI} y\,ds + 2H_O \int_0^{s/2} \frac{y^2\,ds}{EI} + 2H_O \int_0^{s/2} \frac{ds}{EA} \tag{314}$$

If the abutments do not move vertically, and the two sides of the crown sections do not move relatively in a vertical direction,

$$\delta + L = 0 = \frac{\delta W}{\delta V_O} = \int_0^{s/2} \frac{M_L}{EI} \cdot \frac{\partial M_L}{\partial V_O} ds + \int_0^{s/2} \frac{M_R}{EI} \cdot \frac{\partial M_R}{\partial V_O} ds + 2\int_0^{s/2} \frac{N}{EA} \cdot \frac{\partial N}{\partial V_O} ds$$

$$0 = \int_0^{s/2} (M_O - m_L - H_O y + V_O x)(x) \frac{ds}{EI} + \int_0^{s/2} (M_O - m_R - H_O y - V_O x)(-x) \frac{ds}{EI} + 0$$

$$0 = \int_0^{s/2} (m_R - m_L) \frac{x\, ds}{EI} + 2V_O \int_0^{s/2} \frac{x^2\, ds}{EI} \tag{315}$$

If the abutments do not rotate and the two sides of the crown section do not rotate relative to each other,

$$\frac{\partial W}{\partial M_O} = 0 = \int_0^{s/2} \frac{M_L}{EI} \cdot \frac{\partial M_L}{\partial M_O} ds + \int_0^{s/2} \frac{M_R}{EI} \cdot \frac{\partial M_R}{\partial M_O} ds + 2\int_0^{s/2} \frac{N}{EA} \cdot \frac{\partial N}{\partial M_O} ds$$

$$0 = \int_0^{s/2} (M_O - m_L - H_O y + V_O x)(1)\, ds + \int_0^{s/2} (M_O - m_R - H_O y - V_O x)(1)\, ds$$

$$0 = 2M_O \int_0^{s/2} \frac{ds}{EI} - \int_0^{s/2} \frac{(m_L + m_R)\, ds}{EI} \tag{316}$$

Solving equations 314, 315, and 316,

$$H_O = -\frac{\displaystyle\int_0^{s/2} \frac{(m_L + m_R) y\, ds}{I}}{\displaystyle 2\int_0^{s/2} \frac{y^2\, ds}{I} + 2\int_0^{s/2} \frac{ds}{A}}$$

$$V_O = -\frac{\displaystyle\int_0^{s/2} \frac{(m_R - m_L) x\, ds}{I}}{\displaystyle 2\int_0^{s/2} \frac{x^2\, ds}{I}}$$

$$M_O = \frac{\displaystyle\int_0^{s/2} \frac{(m_L + m_R)\, ds}{I}}{\displaystyle 2\int_0^{s/2} \frac{ds}{I}}$$

If the force H_o is transferred to the center of the crown section and denoted H_c, then H_c, V_c, and M_c are the forces and couple at the crown section:

$$H_c = H_O \qquad V_c = V_O \qquad \text{and} \qquad M_c = M_O - H_O y_c$$

where y_c = distance of center of gravity of crown section above the OX axis.

The OX axis can be located by taking moments of the term $\dfrac{ds}{EI}$ about the horizontal axis HH through the center of gravity of the springing sections. This axis is a distance y_s below the OX axis.

Let y' = distance from HH to any section. Moments about HH give

$$\int \frac{y'\, ds}{EI} = \int (y + y_s) \frac{ds}{EI} = \int \frac{y\, ds}{EI} + y_s \int \frac{ds}{EI} = 0 + y_s \int \frac{ds}{EI}$$

$$y_s = \frac{\displaystyle\int \frac{y'\, ds}{I}}{\displaystyle\int \frac{ds}{I}} \tag{317}$$

The integrals are difficult to evaluate, and designs are made by using a number of short lengths Δs instead of the infinitesimal length ds. In each length Δs the moment of inertia I and the area A are assumed to be constant. The design equations become

$$H_O = H_c = \frac{\displaystyle -\Sigma \frac{(m_L + m_R)y}{I}}{\displaystyle 2\Sigma \frac{y^2}{I} + 2\Sigma \frac{1}{A}} \tag{318}$$

$$V_O = V_c = \frac{\displaystyle \Sigma \frac{(m_L - m_R)x}{I}}{\displaystyle 2\Sigma \frac{x^2}{I}} \tag{319}$$

$$M_c = \frac{\Sigma \dfrac{(m_L + m_R)}{I}}{2\Sigma \dfrac{1}{I}} - H_O y_c = M_O - H_O y_c \tag{320}$$

These summations are for the half-arch.

390. Temperature Stresses. The reinforced concrete arch, fixed at its ends by the abutments, will have additional stresses due to temperature changes. These might have been included in the previous derivation, but it is convenient to keep the temperature analysis separate. The temperature change affects the normal stresses of the cross section. The resultant of these stresses may be taken as a normal force N_T at the center of gravity and a couple M_T. For either half-arch,

$$M_T = M'_O - H'_O y$$

$$N_T = H'_O \cos \phi$$

where ϕ = the angle between the section in question and the crown section.

Once again, the two sides of the crown section do not separate and the abutments are assumed to be unyielding. For the symmetrical arch,

$$\frac{\partial W}{\partial H'_O} = 0 = 2\int_0^{s/2} (M'_O - H'_O y)(-y)\frac{ds}{EI} + 2\int_0^{s/2} (H'_O \cos \phi)(\cos \phi)\frac{ds}{EA} - 2\epsilon t \int_0^{s/2} \cos \phi \, ds$$

where ϵ = coefficient of expansion

t = average temperature rise in the length ds.

The first term is the horizontal displacement of the crown section due to the temperature couples in the half-arch. The second term gives the displacement due to the thrust. The third cares for the lengthening of the arch due to a temperature rise and is tension (negative in the equation).

$$0 = -2M'_O \int_0^{s/2} \frac{y\,ds}{EI} + 2H'_O \int_0^{s/2} \frac{y^2\,ds}{EI} + 2H'_O \int_0^{s/2} \frac{\cos^2 \phi\,ds}{EA} - 2\epsilon t \int_0^{s/2} \cos \phi\,ds$$

$$0 = 0 + 2H'_O \int_0^{s/2} \frac{y^2\,ds}{EI} + 2H'_O \int_0^{s/2} \frac{\cos^2\phi\,ds}{EA} - \epsilon tl$$

where $l = \text{span} = 2\int_0^{s/2} \cos\phi\,ds.$

Also,

$$\frac{\partial W}{\partial M'_O} = 0 = 2\int_0^{s/2} (M'_O - H'_O y)(1)\frac{ds}{EI}$$

$$0 = 2M'_O \int_0^{s/2} \frac{ds}{EI} - 2H'_O \int_0^{s/2} \frac{y\,ds}{EI}$$

$$0 = 2M'_O \int_0^{s/2} \frac{ds}{EI} - 0$$

therefore

$$M'_O = 0$$

and

$$H'_O = \frac{\epsilon tlE}{2\int_0^{s/2} \frac{y^2\,ds}{I} + 2\int_0^{s/2} \frac{\cos^2\phi\,ds}{A}}$$

If a finite number of divisions Δs are used, $l = 2n\Delta s$, where n is the number of Δs divisions in the half-arch. The term $\cos^2\phi$ can be approximated as unity.

Using a finite number of short divisions the value of H'_c becomes

$$H'_c = \frac{\epsilon tnE}{\Sigma \frac{y^2}{I} + \Sigma \frac{1}{A}} \qquad (321)$$

These summations are for the half-arch.

391. Shrinkage. The shrinkage stresses are considerable in a long curved member, such as an arch, held firmly at the ends, and these stresses should be considered in the design. It is customary to reinforce the concrete arch with equal areas of steel on the extrados and intrados faces. The shrinkage of an arch will, therefore, be analogous to that of a column, and the symmetrical section permits the use of average stresses. Consider the arch as a series of n straight members of length Δs with a total length equal to $l = 2n\Delta s$. Equations 117 and 118 (Art. 180) give the tensile stress f_t in the concrete and the compressive stress f'_s in the steel due to a shrinkage strain s.

The change in length will be sl. The allowance for additional stresses due to the fact that the arch is actually curved can be handled, as were changes in length due to temperature, by substituting the value sl for ϵtl in equation 321 to obtain H'_c.

The shrinkage coefficient was given an average value of $s = 0.0004$ in. per in. in Article 181, Chapter 9. Arches usually consist of large masses of concrete exposed to the weather and not completely dried out; so a value of $s = 0.0002$ is probably safe. Shrinkage is often cared for in computations by specifying that it be regarded as a temperature change of 10° to 30°.

392. Plastic Flow. Compressive distortions of the arch due to the dead loads will continue for some years after the arch is constructed. It is stated in Article 189, Chapter 9, that the effect of plastic flow can be included in computations by using a corrected modulus of elasticity ratio, perhaps $n' = 50$ to 90. This can be used in the shrinkage procedure outlined above, and the ultimate effect of shrinkage is much reduced.

In the equations for M_c, H_c, and V_c derived above, the modulus of elasticity does not appear, except for the temperature effect in equation 321. The moment of inertia I of a reinforced section is only slightly affected by the value of n. Therefore, the magnitude of the moments and thrusts at a section due to dead or live loads need not be corrected for plastic flow. However, the computation of the *stresses* at a section due to these forces and couples should be made using the modified value n' when dead loads are considered. Live loads usually vary so rapidly that plastic flow need not be considered.

The result of the shrinkage, plastic flow, and temperature changes is to produce resultant stresses in the steel considerably greater than those due to the loads. These may approach the yield point of the steel. Plastic flow will prevent the total stress from greatly exceeding the elastic limit stresses, but there results a shifting of stress conditions in the section, and final values are problematical. It is advisable to estimate the effect of shrinkage and flow in the design.[1]

393. Proportions of Arch. The equations for H_c, V_c, and M_c, derived above, can best be used if the shape of the arch axis and the dimensions of the cross sections are known. There are empirical equations for the thickness of the crown and springing, and approximate solutions for the curve of the arch axis that enable the designer to choose arch proportions that will give the minimum bending moments. The bending mo-

[1] For a comprehensive discussion of the effect of shrinkage, temperature changes, and plastic flow see "Plain and Reinforced Concrete Arches," by Charles S. Whitney, *Jour. A.C.I.*, March 1932, p. 479.

ment will be zero when the normal thrust N passes through the center of gravity of the section. The dead weight is always present and, with earth-filled arches, is a large portion of the total load. It is customary to adopt an arch axis coinciding closely with the line of thrust due to the dead loads.

394. Crown and Springing Thickness. The crown thickness can be tentatively assumed by an empirical equation based on existing arches. That of Mr. F. M. Weld [2] gives

$$t_c = \sqrt{l} + \frac{l}{10} + \frac{w_l}{200} + \frac{w_c}{400} \tag{322}$$

where t_c = crown thickness in inches
l = clear span in feet
w_l = live load in pounds per square foot
w_c = dead load at crown in pounds per square foot.

The thickness of the section at the springing is about 1.5 to 3 times the crown thickness, a value of 2 being commonly used.

Mr. C. S. Whitney [3] has derived an equation for the thickness t_x at any section in terms of the crown thickness t_c for arches with a distributed loading.

$$t_x = \frac{t_c}{\sqrt[3]{\left[1 - \left(1 - \frac{t_c^3}{t_s^3 \cos \phi_s}\right) k\right] \cos \phi}} \tag{323}$$

where ϕ = angle between tangent to the arch axis and the horizontal (Fig. 234)

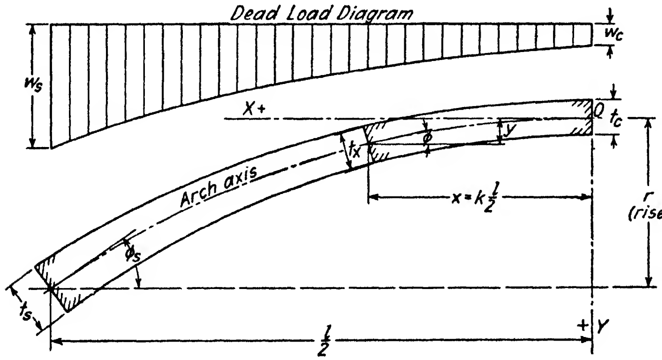


FIG. 234

[2] *Engineering Record*, November 4, 1905.
[3] *Trans. A.S.C.E.*, 1925.

ϕ_s = angle at springing
t_c = crown section thickness
t_s = springing thickness
k = ratio of horizontal distance x to the length $\frac{l}{2}$.

395. Arch Axis. Mr. Whitney also proposes an equation for the curve of the arch axis for distributed loads. For any section whose center is a horizontal distance x from the crown the vertical distance y of the axis below the crown center is (Fig. 234):

$$y = \frac{r}{\left(\frac{w_s}{w_c} - 1\right)}(\cosh kC - 1) \tag{324}$$

where $C = \cosh^{-1}\frac{w_s}{w_c}$
w_c = dead load intensity at crown
w_s = dead load intensity at springing
r = rise of arch axis.

This is not in a convenient form for the designer. For design Mr. Whitney has prepared a table which enables the designer to select the arch axis if the ratios $\frac{w_s}{w_c}$, $\frac{y}{r}$ and $\frac{y_1}{r}$ are known. The dimension y_1 is the y coordinate of the section at $x = \frac{l}{4}$ from the crown.

Mr. Victor M. Cochrane[4] proposes an equation for the axis of arches with distributed loads.

$$y = k^2 r\left(\frac{l + 3k^3 r}{l + 3r}\right) \tag{325}$$

ILLUSTRATIVE PROBLEM 77

396. Design of an Earth-Filled Arch. Design a symmetrical arch to carry a roadway. The concrete pavement is 6 in. thick. The fill between the pavement and the arch weighs 120 lb. per cu. ft. and is 1 ft. thick at the crown. The span between the vertical faces of the abutments is 145 ft., and the rise from the top of this vertical face to the highest point of the intrados is 30 ft. The live load is 100 lb. per sq. ft.

The allowable foundation pressure is 5 tons per square foot. It has been decided to try a 3000-lb. concrete. The steel ratio will be assumed as $p = 0.015$ at the crown and will be kept a constant area throughout the arch. An arch 1 ft. wide will be used in the computations.

[4] *Proceedings, Engineering Society of Western Pennsylvania*, November 1916.

397. Selection of Trial Arch. *Crown Thickness.* Substituting in Mr. Weld's equation 322,

$$t_c = \sqrt{150} + \frac{150}{10} + \frac{100}{200} + \frac{120 + 150(0.5 + 2.5)}{400} = 29.1 \text{ in.}$$

Adopt tentatively a crown thickness of 30 in. The springing will be assumed as 75 in. thick.

Arch Axis. Using equations 324 and 325 and dividing the horizontal distance between crown and springing into tenths, Table S has been computed for the trial arch axis. It has been assumed that the span of the axis will be 5 ft. more than the clear span (or l = 150 ft.), and that the rise of the axis r = 30 ft.

By Cochrane:

$$y = \frac{30k^2}{(150 + 3 \times 30)}(150 + 3 \times 30k^3) = 0.125k^2(150 + 90k^3)$$

By Whitney:

$$\frac{w_s}{w_c} = \frac{28 \times 120 + 150(0.5 + 10.0)}{120 + 150(0.5 + 2.5)} = \frac{4935}{570} = 8.67$$

This assumes 28 ft. of fill on the arch at the springing and a vertical depth of concrete of 10 ft. in the arch.

$$C = \cosh^{-1} 8.67 = 2.85$$

$$y = \frac{30}{7.67}(\cosh 2.85k - 1)$$

The two results are shown in Table S and differ only slightly. In Figure 235 the trial axis is plotted, for the most part between the Cochrane and Whitney values.

Table S

Arch Axis Vertical Dimensions y			Section Thickness t_x	
k	Cochrane	Whitney	ϕ	Whitney
Crown	0	0	0	30.0 in.
0.1	0.19	0.16	3.0°	31.0
0.2	0.75	0.65	5.5°	32.1
0.3	1.72	1.52	9.5°	33.5
0.4	3.12	2.83	12.5°	35.1
0.5	5.04	4.67	16.5°	37.1
0.6	7.62	7.27	21.3°	39.8
0.7	11.07	10.73	28.5°	43.7
0.8	15.70	15.4	37.5°	49.5
0.9	21.8	21.6	43.5°	58.0
Spring	30.0	30.0	50.0°	75

Thickness of Sections. Equation 323 will be used to obtain the trial section depths. By the previous assumptions the crown thickness $t_c = 30$ in. and springing $t_s = 75$ in. By Figure 235 the angle ϕ_s at the springing is $\tan^{-1}\dfrac{30}{25.2}$, or $\phi_s = 50°$. The cos ϕ_s equals 0.643.

$$t_x = \frac{30}{\sqrt[3]{\left[1-\left(1-\dfrac{(30)^3}{(75)^3 \times 0.643}\right)k\right]\cos\phi}} = \frac{30}{\sqrt[3]{[1-0.90k]\cos\phi}}$$

Dead Loads

Horizontal Division	Average Ordinate	Load – lb.
0 – 1	3.9	P_1 = 4,400
1 – 2	4.1	P_2 = 4,600
2 – 3	4.6	P_3 = 5,200
3 – 4	5.55	P_4 = 6,200
4 – 5	7.05	P_5 = 7,900
5 – 6	9.05	P_6 = 10,200
6 – 7	11.65	P_7 = 13,000
7 – 8	15.25	P_8 = 17,100
8 – 9	20.55	P_9 = 23,100
9 – 10	28.15	P_{10} = 31,600
Total		123,300

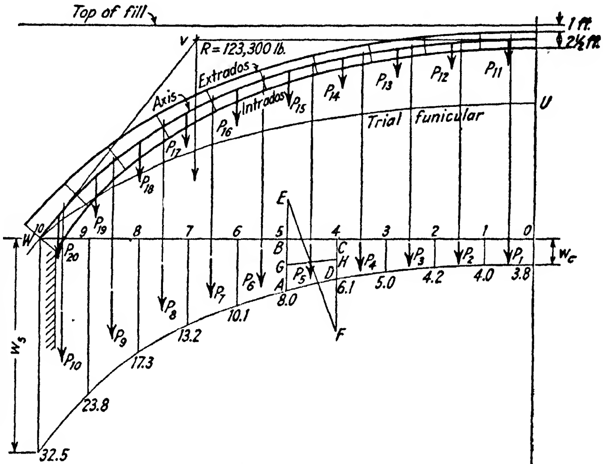


FIG. 235

Dead Loads by Δs divisions

Arch Division	Average Ordinate	Division Width	Load – lb.
1	3.95	8.4	P_{11} = 4,980
2	4.35	8.4	P_{12} = 5,480
3	5.0	8.3	P_{13} = 6,230
4	6.3	8.2	P_{14} = 7,750
5	8.25	8.2	P_{15} = 10,150
6	10.6	7.5	P_{16} = 11,910
7	12.55	7.5	P_{17} = 14,110
8	16.75	6.8	P_{18} = 17,080
9	23.25	6.3	P_{19} = 21,880
10	29.35	5.4	P_{20} = 23,750
Total			123,320

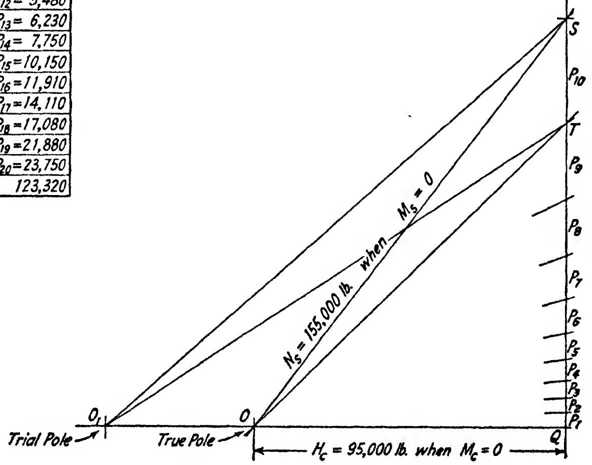


FIG. 236

The values of $\tan \phi$ are scaled on Figure 235 for each of the division points. The angles ϕ and thicknesses t_x are listed in Table S, and the thicknesses are plotted in Figure 235 to give the trial extrados and intrados.

398. Dead Loads. The dead load is plotted in Figure 235 in terms of concrete weights. Each foot of fill will be equivalent to $\frac{120}{150} = 0.8$ ft. of concrete. Thus, at the crown there are $2\frac{1}{2}$ ft. of arch, 1 ft. of fill, and 0.5 ft. of pavement. At the springing there are 10 ft. of arch (vertical), 27.5 ft. of fill, and 0.5 ft. of pavement. For convenience these ordinates have been plotted to a horizontal base line below the arch. The dead loads are divided into ten divisions by the original horizontal dimensions of one tenth the half-span. These loads P_1, P_2, etc., for a foot width of arch are equal to the average height times the width of 7.5 ft. multiplied by the unit weight of concrete. Thus,

$$P_1 = \frac{3.8 + 4.0}{2} \times 7.5 \times 150 = 4390 \text{ lb.}$$

and

$$P_{10} = \frac{23.8 + 32.5}{2} \times 7.5 \times 150 = 31{,}600 \text{ lb.}$$

Their lines of action are found graphically. For example, the load P_5 is equal to the area $ABCD$ multiplied by 150 lb. per cu. ft. It acts at the center of gravity of the area. This can be found graphically by laying off on the line AB a distance BE equal to CD, and laying off in the opposite direction on CD a distance DF equal to AB. Locate also the middle point G of AB and the middle point H of CD. The intersection of the lines EF and GH is the center of gravity of the trapezoid.

These forces $P_1, P_2 \ldots P_{10}$ are plotted in order as a force diagram in Figure 236. Other forces acting on the half-arch are the horizontal thrust H_c at the crown and a thrust at the springing, both being assumed to act at the arch axis. Hence, there are no bending moments at crown or springing. The shear V_c at the crown equals zero, if both halves of a symmetrical arch are equally loaded. At the end Q of the P_1 vector draw a horizontal line. Choose any pole O_1 on this line and draw the rays. This assumes $H_c = O_1Q$ and the thrust at the springing equals O_1S. Starting at the center line of the springing draw the string parallel to O_1S until it intersects the line of action of the load P_{10}. The next string, parallel to O_1T, is drawn between the loads P_{10} and P_9. The final string parallel to O_1Q intersects the crown section at U. The pole O_1 is not correctly located as the force H_c was assumed to act at the center of the crown section. To correct, locate the line of action of the resultant R of all the loads by prolonging the end strings parallel to O_1S and O_1Q. From the center of the crown draw a new final string parallel to O_1Q; at the intersection with the line of action of R draw VW to the center of the springing. The ray parallel to VW through S locates the true pole O. The second funicular polygon passes through the centers of the springing and crown and coincides throughout with our trial axis. Therefore, the present arch axis will be adopted for complete analysis, since the bending moments at all sections due to the dead loads are eliminated. This is the end of the preliminary analysis for the selection of arch dimensions.

399. Live Loads. A variation in the position of the live loads will affect the magnitude of the forces and couples at the cross sections. The position of the live load to give maximum forces or couples at any given section can best be found by the use of influence lines. The sections chosen for analysis will be the crown, springing, and the section halfway between on the arch axis which will be called the quarter-section.

400. Influence Lines for Live Load. In Figure 237 let a load of unity be applied to the left of the crown. In the analysis of Article 389 the moment m_L due to the loads becomes at any section B between the load and springing

$$m_L = -1(x - a)$$

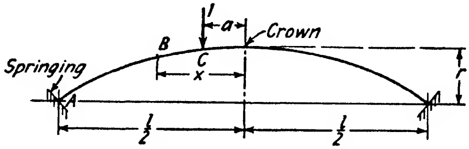


FIG. 237

The negative sign agrees with the original derivation. For sections between the load and the crown, or for any section in the right half-arch

$$m_L = m_R = 0$$

Substitution in equations 318, 319, and 320 gives

$$H_c = \frac{-\Sigma_A^C \dfrac{(x-a)y}{I}}{2\Sigma \dfrac{y^2}{I} + 2\Sigma \dfrac{1}{A}} = \frac{-\Sigma_A^C \dfrac{xy}{I} + a\Sigma_A^C \dfrac{y}{I}}{2\Sigma \dfrac{y^2}{I} + 2\Sigma \dfrac{1}{A}} \tag{326}$$

$$V_c = \frac{\Sigma_A^C \dfrac{(x-a)x}{I}}{2\Sigma \dfrac{x^2}{I}} = \frac{\Sigma_A^C \dfrac{x^2}{I} - a\Sigma_A^C \dfrac{x}{I}}{2\Sigma \dfrac{x^2}{I}} \tag{327}$$

$$M_c = \frac{\Sigma_A^C \dfrac{(x-a)}{I}}{2\Sigma \dfrac{1}{I}} - H_c y_c = \frac{\Sigma_A^C \dfrac{x}{I} - a\Sigma_A^C \dfrac{1}{I}}{2\Sigma \dfrac{1}{I}} - H_c y_c \tag{328}$$

401. Stress Computation. The arch adopted will now be analyzed for the stresses due to all dead and live loads and temperature changes. Since the summations take much time, especially for the live-load investigation, the stresses will be obtained at the crown, springing, and quarter-sections only.

Ten divisions Δs of the half-arch axis will be used in the summation. The half-arch axis is scaled as 84.0 ft. long. Each division will be taken as 8.4 ft. long and is spaced out on the arch axis (Fig. 235) and the cross-section drawn. The center of each division is also marked. The cross-section constants are listed in Table T; the dimensions used are at the center of each Δs division. The total cross section will be used to compute area and moment of inertia. This is equivalent to assuming that there is no tension in the section. Since these values are average for the length Δs it is not necessary to be more accurate. The area of any section equals

$$A = bt_x + (n-1)pbt_c = t_x + 9 \times 0.015 \times 1 \times 2.5$$

$$= t_x + 0.338 \text{ sq. ft.} \tag{329}$$

TABLE T. CROSS-SECTION DATA (FOOT UNITS)

Δs	y'	t_x	$\frac{t^3_x}{12}$	$0.0844(t_x - 0.5)^2$	I	$\frac{y'}{I}$	$\frac{1}{I}$
1	29.8	2.54	1.366	0.351	1.717	17.36	0.582
2	29.4	2.65	1.551	0.390	1.941	15.13	0.516
3	28.6	2.75	1.733	0.428	2.161	13.22	0.463
4	27.2	2.92	2.074	0.495	2.569	10.58	0.389
5	25.0	3.09	2.456	0.567	3.023	8.26	0.331
6	22.5	3.32	3.046	0.671	3.717	6.06	0.269
7	18.9	3.65	4.06	0.84	4.90	3.86	0.204
8	14.4	4.12	5.82	1.11	6.93	2.08	0.144
9	9.1	4.80	9.21	1.56	10.77	0.85	0.093
10	3.1	5.70	15.43	2.28	17.71	0.17	0.056
Total	....	....	.....	....	.....	77.57	3.047

The moment of inertia of any section equals

$$I = \frac{b(t_x)^3}{12} + 0.338\left(\frac{t_x}{2} - \frac{3}{12}\right)^2 = \frac{t_x^3}{12} + 0.0844(t_x - 0.5)^2 \text{ (ft.)}^4 \tag{330}$$

By equation 317,

$$y_s = \frac{\Sigma \frac{y'}{I}}{\Sigma \frac{1}{I}} = \frac{77.57}{3.047} = 25.47 \text{ ft.}$$

$$y_c = 30 - 25.47 = 4.53 \text{ ft.}$$

Table U gives data determined from the arch axis dimensions. These distances are again taken to the center of the Δs divisions and referred to the X axis of Figure 233 which is 4.53 ft. below the center of the crown.

402. Analysis for Dead Loads. In order to figure the forces and couple at the crown section it will be necessary to sum up the moments of the dead loads about the center of each Δs division. The dead-load diagram will be divided anew into trapezoids. In Table V the moment at the center of the fourth Δs will be equal to

TABLE U. ARCH AXIS DIMENSIONS

Δs	x	y or $(y' - y_s)$	$\frac{x^2}{I}$	$\frac{y^2}{I}$	$\frac{1}{A}$
1	4.2	4.3	10.28	10.87	0.347
2	12.6	3.9	81.8	7.92	0.334
3	20.9	3.1	202	4.49	0.324
4	29.2	1.7	332	1.15	0.307
5	37.4	−0.5	462	0.08	0.292
6	45.3	−3.0	552	2.39	0.273
7	52.9	−6.6	571	8.83	0.251
8	60.0	−11.1	520	17.70	0.224
9	66.5	−16.4	411	24.90	0.195
10	72.5	−22.4	297	28.24	0.166
Total	401.5	....	3439	106.57	2.713

the moment at the third, plus the product of the total dead weight of the first three divisions times the horizontal distance between the third and the fourth.

By equations 318, 319, 320,

$$H_c = \frac{+2 \times 10{,}540{,}000}{2(106.57 + 2.71)} = 96{,}450 \text{ lb.}$$

$$V_c = 0$$

$$M_c = \frac{2 \times 1{,}320{,}000}{2 \times 3.047} - 96{,}450 \times 4.53 = -3700 \text{ ft.-lb}$$

The value H_c should not check the value $H_c = 95{,}000$ lb. obtained in the tentative analysis assuming H_c to act through the center of the crown, since the resultant of H_c and M_c does not act at the center. From the equilibrium polygon the resultant dead load equals 123,300 lb. and scales 23.3 ft. from the center of the springing. At the springing the normal force N_s makes an angle $\phi_s = 50°$ with the horizontal.

$$N_s = 96{,}450 \times 0.643 + 123{,}300 \times 0.766 = 156{,}500 \text{ lb.}$$

$$M_s = -123{,}300 \times 23.3 + 96{,}450 \times 30 - 3700 = +16{,}500 \text{ ft.-lb.}$$

At the springing the normal force also acts close to the center line, as was indicated by the trial funicular polygon.

It is customary to check also a section about half way between crown and springing. In this problem the quarter section is at the junction of the fifth and sixth Δs. Taking as rigid body the portion of the arch between this section and the crown, the dead load totals 34,600 lb. (Table V) and the arch slopes at an angle $\phi = 18.5°$.

The center of this section is 6 ft. below and 41.5 ft. to the left of the crown center.

$$N_Q = 96{,}450 \cos \phi + 34{,}600 \sin \phi = 91{,}500 + 11{,}000 = 102{,}500 \text{ lb.}$$

$$M_Q = 96{,}450 \times 6 - (467{,}500 + 34{,}600 \times 4.1) - 3700 = -34{,}360 \text{ ft.-lb.}$$

The moment of the dead load is handled as was that at the center of the sixth Δs in Table V.

TABLE V. MOMENTS OF DEAD LOAD

$m_L = m_R$

Δs	Total Dead Load	Moment Arm $x_2 - x_1$ (Table U)	Moment	Total Moment m_L	$\frac{m_L}{I}$	$\frac{m_L y}{I}$
1	4,980	0	0	0	0	0
2		8.4	41,800	41,800	21,500	84,000
	5,480					
3	10,460	8.3	86,800	128,600	59,500	186,000
	6,230					
4	16,690	8.3	138,500	267,100	103,900	179,000
	7,750					+ 449,000
5	24,440	8.2	200,400	467,500	154,400	− 74,000
	10,150					
6	34,590	7.9	273,500	741,000	200,000	− 595,000
	11,910					
7	46,500	7.6	353,500	1,094,500	223,500	− 1,470,000
	14,110					
8	60,610	7.1	430,000	1,524,500	220,200	− 2,440,000
	17,080					
9	77,690	6.5	505,000	2,029,500	188,600	− 3,090,000
	21,880					
10	99,570	6.0	597,500	2,627,000	148,300	− 3,320,000
	23,750					−10,989,000
						+449,000
Total	123,320				1,319,900	−10,540,000

403. Stresses Due to Dead Loads. *Crown.* By equation 329:

$$A = (2.5 + 0.338)144 = 410 \text{ sq. in.}$$

By equation 330:

$$I = \left[\frac{(2.5)^3}{12} + 0.0844(2)^2\right](12)^4 = 34{,}100 \text{ (in.)}^4$$

$$\text{Maximum } f_c = \frac{96{,}450}{410} + \frac{3700 \times 12 \times 15}{34{,}100} = 255 \text{ lb. per sq. in. (bottom)}$$

$$\text{Minimum } f_c = 235 - 20 = 215 \text{ lb. per sq. in. (top)}$$

Springing.

$$A = (6.25 + 0.338)144 = 950 \text{ sq. in.}$$

$$I = \left[\frac{(6.25)^3}{12} + 0.0844(5.75)^2\right](12)^4 = 479{,}000 \text{ (in.)}^4$$

$$\text{Maximum } f_c = \frac{156{,}500}{950} + \frac{16{,}500 \times 12 \times 37.5}{479{,}000} = 181 \text{ lb. per sq. in. (top)}$$

$$\text{Minimum } f_c = 165 - 16 = 149 \text{ lb. per sq. in. (bottom)}$$

Quarter-Section: Depth of section = 3.25 ft.

$$A = (3.25 + 0.338)144 = 517 \text{ sq. in.}$$

$$I = \left[\frac{(3.25)^3}{12} + 0.0844(2.75)^2\right](12)^4 = 72{,}700 \text{ (in.)}^4$$

$$\text{Maximum } f_c = \frac{102{,}500}{517} + \frac{34{,}360 \times 12 \times 19.5}{72{,}700} = 309 \text{ lb. per sq. in. (bottom)}$$

$$\text{Minimum } f_c = 198 - 111 = 87 \text{ lb. per sq. in. (top)}$$

All the sections examined have compression over the whole area.

404. Analysis for Live Loads. The live load is assumed to be 100 lb. per sq. ft. It will be necessary to solve for the live-load placement that gives the maximum normal stress, or shear, or bending moment. This analysis will be made for the crown, springing, and quarter-section by using influence lines.

405. Influence Lines for Crown Analysis. The live load will be divided into increments which are spaced equally on the horizontal span. The unit load will be applied at the center of each increment. The summations should be made from the springing to the load in each case; but it will give substantially the same results if the Δs divisions of the arch are used, including each time those whose centers are between the load and the springing (Fig. 235). Table W gives the computations; and in Figure 238 are plotted the results of the computations for the crown sections. For instance, in the H_c diagram, the value of $H_c = 0.22$ at 20 ft. from the springing means that $H_c = 0.22$ lb. if the unit load is located 20 ft. horizontally from the springing. The maximum H_c occurs with the live load spread over the whole span and will be equal to the area under the H_c curve multiplied by the intensity $w = 100$ lb. per ft. The shear V_c is a maximum if the live load is spread over a half-span.

The maximum positive moment M_c at the crown occurs when the live load extends $75 - 51 = 24$ ft. either side of the crown (24 ft. $= 0.16l$). The maximum

TABLE W. INFLUENCE LINE VALUES FOR CROWN (Use Tables T and U)

Load at $a/75$ (1)	Load at a (2)	As Divisions Included (3)	$\sum_A^C \frac{xy}{I}$ (4)	$a\sum_A^C \frac{y}{I}$ (5)	$H_c = \frac{-(4)+(5)}{218.8}$ (6)	$\sum_A^C \frac{x^2}{I}$ (7)	$a\sum_A^C \frac{x}{I}$ (8)	$V_c = \frac{(7)-(8)}{6878}$ (9)	$\sum_A^C \frac{x}{I}$ (10)	$a\sum_A^C \frac{1}{I}$ (11)	$\frac{(10)-(11)}{6.094}$ (12)	4.53H_c (13)	$M_c =$ (12) − (13) (14)
0.95	71.25	10	− 91.5 −101.1	−1.26×a = − 90.0 −1.52	0.0069	297 411	4.09×a = 291 6.18	0.0009	4.09	0.056×a = 4.02 0.093	0.0116	0.0310	−0.0195
0.85	63.75	9 to 10	−192.6 − 96.0	−2.78×a = −176.3 −1.60	0.072	708 520	10.27×a = 656 8.66	0.0076	10.27	0.149×a = 9.51 0.144	0.125	0.326	−0.201
0.75	56.25	8 to 10	−288.6 − 70.8	−4.38×a = −246.4 −1.34	0.193	1228 571	18.93×a = 1065 10.80	0.0237	18.93	0.293×a = 16.50 0.204	0.399	0.874	−0.475
0.65	48.75	7 to 10	−359.4 − 36.2	−5.72×a = −279.0 −0.80	0.367	1799 552	29.73×a = 1448 12.19	0.0510	29.73	0.497×a = 24.22 0.269	0.905	1.660	−0.755
0.55	41.25	6 to 10	−395.6 − 5.6	−6.52×a = −268.0 −0.15	0.583	2351 462	41.92×a = 1730 12.36	0.0903	41.92	0.766×a = 31.60 0.331	1.695	2.637	−0.942
0.45	33.75	5 to 10	−401.2 + 19.9	−6.67×a = −224.7 +0.68	0.807	2813 332	54.28×a = 1830 11.37	0.143	54.28	1.097×a = 37.00 0.389	2.835	3.654	−0.819
0.35	26.25	4 to 10	−381.3 + 30.4	−5.99×a = −157.3 +1.45	1.023	3145 202	65.65×a = 1722 9.67	0.207	65.65	1.486×a = 39.00 0.463	4.38	4.64	−0.260
0.25	18.75	3 to 10	−350.9 + 25.4	−4.54×a = −85.2 +2.02	1.213	3347 82	75.32×a = 1413 6.49	0.281	75.32	1.949×a = 36.55 0.516	6.36	5.50	+0.860
0.15	11.25	2 to 10	−325.5 + 10.6	−2.52×a = −28.4 +2.52	1.357	3429 10	81.81×a = 921 2.45	0.365	81.81	2.465×a = 27.74 0.582	8.88	6.14	+2.74
0.05	3.75	1 to 10	−314.9	0 = 0	1.44	3439	84.26×a = 316 0	0.455	84.26	3.047×a = 11.42	11.94	6.52	+5.42
Crown	0	1 to 10	−314.9	0 0	1.44	3439	84.26×0 = 0	0.500	84.26	3.047×0 = 0	13.83	6.52	+7.31

negative moment occurs when the live load extends from each springing to the 51-ft. horizontal distance. The positive moment is much the greater. The loadings for maximum H_c and M_c are not the same. The maximum combined effect will be obtained by some intermediate loading. This maximum combined effect will give, however, stresses only slightly more than those obtained by dealing with the maximum moment M_c and the accompanying H_c. The latter procedure will be adopted when the stresses are computed.

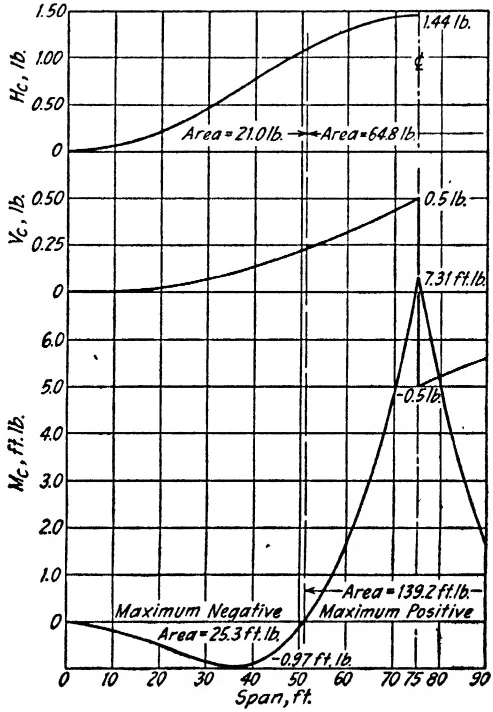


FIG. 238

406. Influence Lines for Springing. By Figures 233 and 237 it is evident that at the left springing

$$N_s = H_c \cos \phi_s + (1 - V_c) \sin \phi_s$$

$$M_s = M_c + 30H_c + 75V_c - 1(75 - a)$$

If the unit load is applied to the right half-arch, the normal force and couple at the left springing equal

$$N_s = H_c \cos \phi_s - V_c \sin \phi_s$$

$$M_s = M_c + 30H_c + 75V_c$$

The computations for the normal force and the couple are given in Table X, and the results are plotted in Figure 239. The maximum positive moment M_s at the

TABLE X. INFLUENCE LINE VALUES FOR LEFT SPRINGING

Load Point Left	$0.643H_c$	$0.766(1 - V_c)$	$0.766V_c$	N_s	M_c	$30H_c$	$75V_c$	$-1(75 - a)$	M_s
0.95	0.0044	0.765	....	0.769	−0.020	0.206	0.065	− 4.75	− 4.50
0.85	0.047	0.760	....	0.808	−0.201	2.216	0.568	−11.25	− 8.72
0.75	0.124	0.749	....	0.873	−0.475	5.79	1.78	−18.75	−11.66
0.65	0.236	0.726	....	0.962	−0.755	11.01	3.83	−26.25	−12.17
0.55	0.374	0.698	....	1.072	−0.942	17.49	6.78	−33.75	−10.42
0.45	0.519	0.652	....	1.175	−0.819	24.21	10.72	−41.25	− 7.24
0.35	0.658	0.608	....	1.266	−0.260	30.69	15.52	−48.75	− 2.80
0.25	0.780	0.550	....	1.330	+0.860	36.39	21.06	−56.25	+ 2.06
0.15	0.872	0.486	....	1.358	+2.74	40.71	27.37	−63.75	+ 7.07
0.05	0.926	0.418	....	1.344	+5.42	43.20	34.15	−71.25	+11.52
Crown Right	0.926	0.383	0.383	1.309	+7.31	43.20	37.5	−75.00	+13.01
0.05	0.926	....	0.349	1.275	+5.42	43.20	−34.15	....	+14.47
0.15	0.872	....	0.280	1.152	+2.74	40.71	−27.37	....	+16.08
0.25	0.780	....	0.215	0.995	+0.860	36.39	−21.06	....	+16.19
0.35	0.658	....	0.159	0.817	−0.260	30.69	−15.52	....	+14.91
0.45	0.519	....	0.114	0.629	−0.819	24.21	−10.72	....	+12.67
0.55	0.374	....	0.069	0.443	−0.942	17.49	− 6.78	....	+ 9.77
0.65	0.236	....	0.039	0.275	−0.755	11.01	− 3.83	....	+ 6.42
0.75	0.124	....	0.018	0.142	−0.475	5.79	− 1.78	....	+ 3.53
0.85	0.047	....	0.0058	0.053	−0.201	2.216	− 0.568	....	+ 1.55
0.95	0.0044	....	0.0007	0.005	−0.020	0.206	− 0.065	....	+ 0.121

left springing occurs when the live load covers the right half-arch and extends to 53 ft. (0.355l) from the left springing. The maximum negative moment is smaller and occurs with the live load on the first 53 ft. from the left springing.

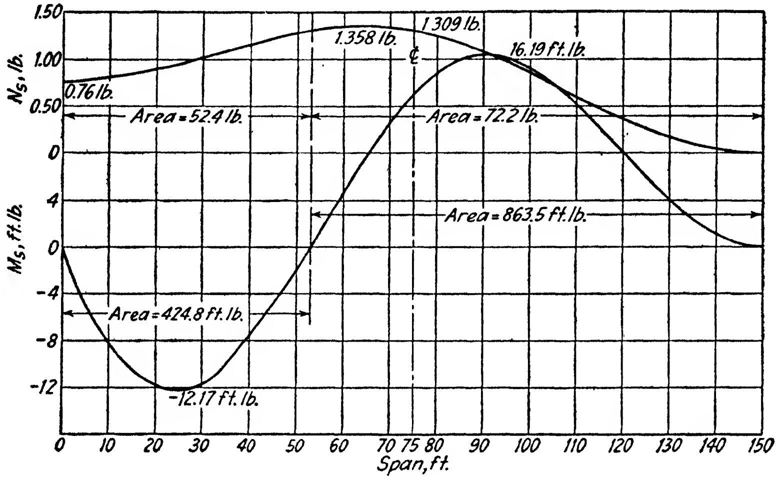


FIG. 239

407. Influence Lines for the Quarter-Section. The maximum values of the normal force and couple at the quarter-section can be found. Take the portion of the arch between the left quarter-section and the crown as a rigid body (Figs. 233 and 237). If the load of unity acts on this body,

$$N_Q = H_c \cos 18.5° + (1 - V_c) \sin 18.5°$$

$$M_Q = M_c + 6H_c + 41.5V_c - 1(41.5 - a)$$

If the unit load does not act on the rigid body, these equations become

$$N_Q = 0.948H_c - 0.317V_c$$

$$M_Q = M_c + 6H_c + 41.5V_c$$

The computations for the normal force and the couple are given in Table Y. The results are plotted in Figure 240. The quarter-section occurs within 3 in. of the load point $a = 0.55$. The values of this load point will be used for the quarter-section.

The maximum negative moment at the left quarter-section occurs with the live load acting on the whole right half-arch and ending at 47 ft. (0.31l) from the left springing. The maximum positive moment is less and occurs if the live load acts over the first 47 ft. from the left springing.

TABLE Y. INFLUENCE LINE VALUES FOR LEFT QUARTER-SECTION

Load Point	$0.948H_c$	$0.317V_c$	$0.317(1 - V_c)$	N_Q	M_c	$6H_c$	$41.5V_c$	$-1(41.5-a)$	M_Q
Left									
0.95	0.0065	0.0003	....	0.0062	−0.0195	0.0411	0.0362	....	0.0578
0.85	0.068	0.0024	....	0.066	−0.201	0.432	0.314	....	0.545
0.75	0.183	0.0075	....	0.175	−0.475	1.158	0.983	....	1.666
0.65	0.348	0.0162	....	0.332	−0.755	2.202	2.117	....	3.564
Quarter	......	....	....	0.523					
0.55	0.552	0.0286	0.288	0.840	−0.942	3.498	3.744	0	6.300
0.45	0.765	....	0.272	1.037	−0.819	4.842	6.19	− 7.75	2.46
0.35	0.970	....	0.251	1.221	−0.260	6.138	8.59	−15.25	−0.78
0.25	1.150	....	0.228	1.378	+0.860	7.278	11.66	−22.75	−2.95
0.15	1.285	....	0.201	1.486	+2.74	8.14	15.13	−30.25	−4.24
0.05	1.365	....	0.173	1.538	+5.42	8.64	18.89	−37.75	−4.80
Crown									
Right	1.365	....	0.159	1.524	+7.31	8.64	20.75	−41.50	−4.80
0.05	1.365	−0.144	....	1.509	+5.42	8.64	−18.89	....	−4.82
0.15	1.285	−0.116	....	1.401	+2.74	8.14	−15.13	....	−4.25
0.25	1.150	−0.089	....	1.239	+0.860	7.278	−11.66	....	−3.52
0.35	0.970	−0.066	....	1.036	−0.260	6.138	− 8.59	....	−2.71
0.45	0.765	−0.047	....	0.812	−0.819	4.842	− 6.19	....	−2.17
0.55	0.552	−0.029	....	0.581	−0.942	3.498	− 3.744	....	−1.188
0.65	0.348	−0.016	....	0.364	−0.755	2.202	− 2.117	....	−0.670
0.75	0.183	−0.0075	....	0.191	−0.475	1.158	− 0.983	....	−0.300
0.85	0.068	−0.0024	....	0.070	−0.201	0.432	− 0.314	....	−0.083
0.95	0.0065	−0.0003	....	0.0068	−0.0195	0.0411	− 0.0362	....	−0.0146

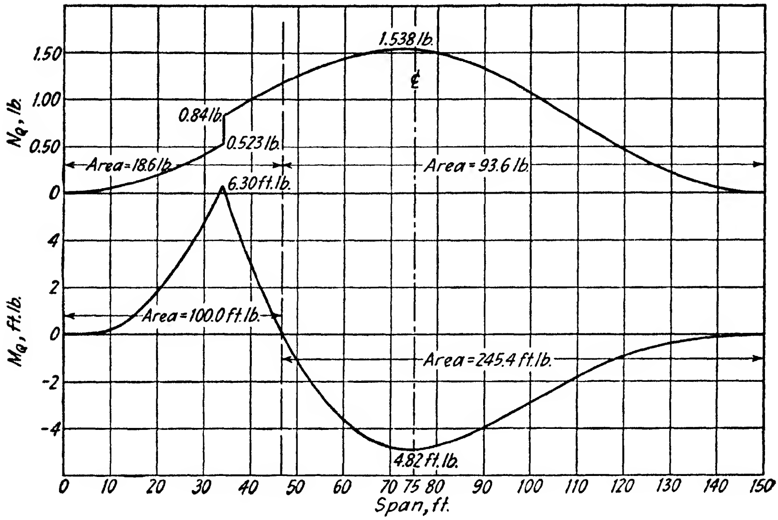


FIG. 240

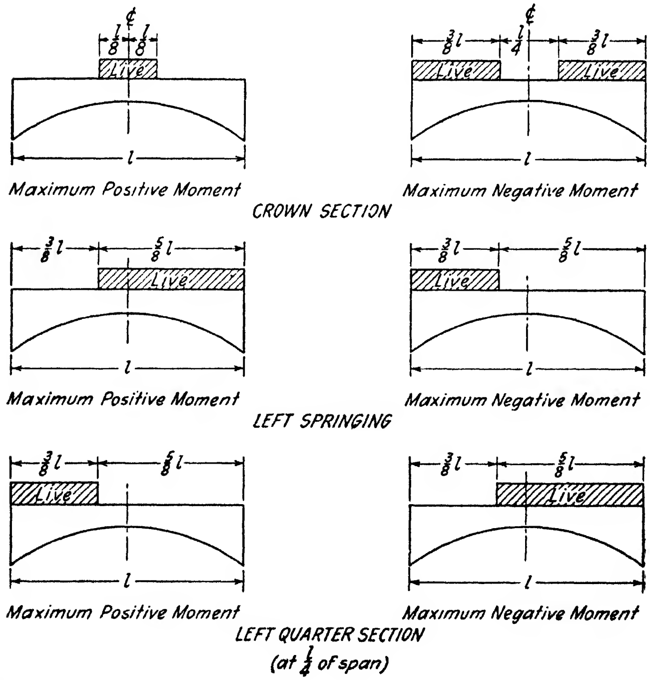


FIG. 241

408. Maximum Moments by Approximation. The problem illustrated above has a fairly long span for earth-fill arches. The analysis by influence lines is justified. Analysis of many arch designs leads to the loadings of Figure 241 to give approximate maximum moments.

If these approximate loadings were used in this problem the crown moment would be figured for a live load extending $0.125l$ each side of the crown, whereas the influence line calls for a distance $0.16l$. The positive moment is the greater, and the approximation would give a moment which is too small.

At the springing the approximation ends the loading at $0.375l$ from the springing. The influence lines of Figure 239 end the loading at $0.355l$. The positive moment is again the greater, and the approximation gives slightly smaller values. It is a reasonable approximation, however.

The comparison cannot be made at the quarter-section as the approximation deals with a section $\frac{l}{4} = 37.5$ ft. from the crown, whereas the section used here at one quarter of the arch axis is 41.5 ft. from the crown.

409. Stresses Due to Live Loads. The areas of the moment and thrust influence lines are tabulated in Table Z. After being multiplied by $w = 100$ lb. per ft., the

TABLE Z. LIVE-LOAD AND TEMPERATURE STRESSES (POUNDS PER SQUARE INCH)

Section	Maximum Moment			Corresponding Thrust		Section Data		Live-Load Stresses		Temperature Stresses	
		Area	Ft.-lb.	Area	Lb.	A	$\frac{I}{y}$	Top	Bottom	Top	Bottom
Crown	$+M$	139.2	13,920	64.8	6480	410	$\frac{34,100}{15}$	89	−58	203	−249
	$-M$	25.3	− 2,530	21.0	2100			− 8	19	−203	249
Springing	$+M$	863.5	86,350	72.2	7220	950	$\frac{479,000}{37.5}$	89	−73	233	−220
	$-M$	424.8	−42,480	52.4	5240			−34	45	−233	220
Quarter	$+M$	100.0	10,000	18.6	1860	517	$\frac{72,700}{19.5}$	36	−29	62	− 28
	$-M$	245.4	−24,540	93.6	9360			−61	97	− 62	28

maximum moments and the corresponding thrusts are found. The stresses are also computed. Thus, for the crown section with positive bending (Art. 403):

$$\text{Maximum } f_c = \frac{6480}{410} + \frac{13,920 \times 12 \times 15}{34,100} = 89 \text{ lb. per sq. in.}$$

$$\text{Minimum } f_c = 16 - 74 = -58 \text{ lb. per sq. in.}$$

410. Temperature Stresses. An earth-filled arch will have temperature changes somewhat less than those of the atmosphere, since only the intrados and the sides are exposed. Assuming the arch to be poured about 50° to 60° F., we shall assume a temperature variation in the arch of 40° rise or drop. The coefficient of expansion will be taken as $\epsilon = 0.000006$. By equation 321,

$$H'_c = \frac{0.000006 \times 40 \times 10 \times 3{,}000{,}000 \times 144}{109.3} = 9480 \text{ lb.}$$

$$M'_c = M'_o - H_c y = 0 - 9480 \times 4.53 = -42{,}900 \text{ ft.-lb.}$$

At the springing for 40° rise,

$$N'_s = H'_c \cos \phi_s = 9480 \times 0.643 = 6100 \text{ lb.}$$

$$M'_s = 30H'_c + M'_c = 9480 \times 30 - 42{,}900 = 241{,}500 \text{ ft.-lb.}$$

At the quarter-section for 40° rise,

$$N' = 9480 \times 0.948 = 9000 \text{ lb.}$$

$$M' = 9480 \times 6 - 42{,}900 = 14{,}000 \text{ ft.-lb.}$$

The stresses due to these forces and couples are also listed in Table Z. For example, the maximum stress at the crown due to a 40° rise equals

$$f_c = \frac{9480}{410} + \frac{42{,}900 \times 12 \times 15}{34{,}100} = 249 \text{ lb. per sq. in.} \quad \text{(bottom)}$$

The stresses for a 40° drop are equal in magnitude but opposite in sign.

411. Resultant Stresses. Table AA lists the values of the stresses previously computed. Compressive stresses are positive. It will be noticed that the resultant stresses due to the dead loads produce compression over the whole of the trial sections. This is also true for the live and dead loads acting together.

For the unusual case that the most severe live-load placement will coincide with a maximum temperature change, there will be tensile stresses at the crown, quarter-section, and springing. Those at the springing have the greatest magnitude, and that case will now be checked, allowing for the loss of area due to resultant tension in the section.

Springing. The resultant normal force and couple due to dead and live loads plus a 40° temperature rise are

$$N_s = 156{,}500 + 7200 + 6100 = 169{,}800 \text{ lb.}$$

$$M_s = 16{,}500 + 86{,}350 + 241{,}500 = 344{,}350 \text{ ft.-lb.}$$

The resultant force and couple for dead and live loads plus a 40° temperature drop are

$$N_s = 156{,}500 + 5240 - 6100 = 155{,}640 \text{ lb.}$$

$$M_s = 16{,}500 - 42{,}480 - 241{,}500 = -267{,}480 \text{ ft.-lb.}$$

Resultant Stresses at Springing. Temperature Rise.

$$\text{Eccentricity ratio } \frac{e}{t} = \frac{344{,}350 \times 12}{169{,}800 \times 75} = 0.325 \quad \text{(Case II)}$$

$$p = \frac{A_s}{bt} = \frac{0.015 \times 12 \times 30}{12 \times 75} = 0.006 \quad \text{and} \quad np = 0.06$$

$$\frac{d'}{t} = \frac{3}{75} = 0.04 \quad \text{and} \quad \left(\frac{a}{t}\right)^2 = \left(\frac{34.5}{75}\right)^2 = 0.21$$

$k = 0.66$ $\qquad C = 9.0$

$f_c = 550$ lb. per sq. in. $\qquad f_s = 2500$ lb. per sq. in. (tension)

The value $f_c = 550$ lb. per sq. in. supersedes $f_c = 503$ in Table AA.

TABLE AA. MAXIMUM FIBER STRESSES (POUNDS PER SQUARE INCH)

Section		Dead	Live		Max. D. & L.	Min. D. & L.	Temp.		Max. D., L., and Temp.	Min. D., L., and Temp.
Crown	Top	215	89	−8	304	207	203	−203	507	4
	Bottom	255	19	−58	274	197	249	−249	523	−52
Springing	Top	181	89	−34	270	147	233	−233	503	−86
	Bottom	149	45	−73	194	76	220	−220	414	−144
Quarter	Top	87	36	−61	123	26	62	−62	185	−36
	Bottom	309	97	−29	406	280	28	−28	434	252

Temperature Drop.

$$\frac{e}{t} = \frac{267{,}480 \times 12}{155{,}640 \times 75} = 0.275 \quad \text{(Case II)}$$

$k = 0.76$ $\qquad C = 9.07$

$f_c = 430$ lb. per sq. in. $\qquad f_s = 1130$ lb. per sq. in. (tension)

The stress $f_c = 430$ lb. per sq. in. supersedes the value $f_c = 414$ in Table AA.

By similar computations it is possible to check the crown, springing, and quarter-sections for the loadings that give maximum tensions. Though there is tension on these sections, the probability that maximum live-load conditions and maximum temperature changes will occur more than momentarily is too slight to justify a revision of the moments of inertia which have been figured on the assumption of compression over the full area.

412. Allowable Stresses. The allowable stress for dead and live loads will vary with the eccentricity. Assuming the neutral axis to be at the edge of the section at the springing, the allowable stress f_c is about 1100 lb. per sq. in.

When the effect of temperature changes is included, the allowable stress can be increased, perhaps 20 to 25 per cent.

This design uses a mix somewhat richer than those from which Mr. Cochrane's or Mr. Whitney's approximate designs are based. The steel ratio at the crown of 0.015 is also somewhat higher than the more usual $p = 0.01$. It is possible to reduce the steel ratio somewhat without starting the design anew. A change in mix should require a fresh start. Since the approximate equations are based on concretes about 2000 lb. per sq. in. in strength with a steel ratio about $p = 0.01$, the sections can be reduced in depth if richer mixes or more steel are adopted.

413. Shrinkage and Flow.[5] The shrinkage stress at the crown can be computed by the procedure outlined in Articles 391 and 392. Assume that

$$
\begin{aligned}
s &= 0.0002 \\
n &= 10 \text{ for elastic deformations} \\
n' &= 40 \text{ for combined plastic and elastic deformations}
\end{aligned}
$$

The stresses due to direct shrinkage are computed by equations 117 and 118 (Art. 180) at each Δs division. If t_x is expressed in *inches*, the stress equations become

$$\text{Concrete } f_t = \left(\frac{np}{1 + (n-1)p}\right) sE_c = \frac{32{,}400}{12t_x + 210.6}$$

$$\text{Steel} \quad f'_s = \left(\frac{1-p}{1 + (n-1)p}\right) sE_s = \left(\frac{12t_x - 5.4}{12t_x + 210.6}\right) 6000$$

The concrete stresses vary from 56.3 to 31.4 lb. per sq. in. from crown to springing; the steel stresses vary from 3750 to 4750 lb. per sq. in. The summation of the strains $\left(\frac{f'_s}{E_s}\right)$ for the ten Δs divisions equals 1381×10^{-6} in. per in. Substituting this strain for the uniform temperature strain ϵt in equation 321,

$$H'_c = \frac{1381 \times 10^{-6} \times 750{,}000 \times 144}{109.3} = -1370 \text{ lb.} \quad \text{(tension)}$$

$$M'_c = 0 - (-1370)4.53 = +6200 \text{ ft.-lb.}$$

This thrust and moment due to shrinkage are always present and should be included with the dead load thrust and moment. Combining,

$$H_c = 96{,}450 - 1370 = 95{,}080 \text{ lb.} \quad \text{(compression)}$$

$$M_c = -3700 + 6200 = 2500 \text{ ft.-lb.}$$

The stresses due to the elastic loads can then be revised:

$$\text{Maximum } f_c = \frac{95{,}080}{410} + \frac{2500 \times 12 \times 15}{34{,}100} = 245 \text{ lb. per sq. in.} \quad \text{(bottom)}$$

$$\text{Minimum } f_c = 232 - 13 = 219 \text{ lb. per sq. in.} \quad \text{(top)}$$

[5] Space does not permit a more extended discussion of the effects of shrinkage and plastic flow. The student is referred to "Plain and Reinforced Concrete Arches," by Charles S. Whitney, *Jour. A.C.I.*, March 1932, p. 479.

If the effect of plastic flow is included, the stresses are computed by using $n' = 40$. At the crown section,

$$\text{Area} = bt[1 + (n - 1)p] = 12 \times 30[1 + 39(0.015)] = 570 \text{ sq. in.}$$

$$I = \frac{bt^3}{12} + (n-1)pbt\left(\frac{t}{2} - 3\right)^2 = \frac{12(30)^3}{12} + 39(0.015)12 \times 30(12)^2 = 57{,}300 \text{ (in.)}^4$$

$$\text{Maximum } f_c = \frac{95{,}080}{570} + \frac{2500 \times 12 \times 15}{57{,}300} = 175 \text{ lb. per sq. in.} \quad \text{(bottom)}$$

$$\text{Minimum } f_c = 167 - 8 = 159 \text{ lb. per sq. in.} \quad \text{(top)}$$

The effect of shrinkage and plastic flow may be estimated as

$$\text{(Bottom) } f_c = -56 + 175 - 245 = -126 \text{ lb. per sq. in.} \quad \text{(tension)}$$

$$\text{(Top) } f_c = -56 + 159 - 219 = -116 \text{ lb. per sq. in.} \quad \text{(tension)}$$

In this particular problem the correction is a tensile stress of considerable magnitude. The stress in the steel can be found by plotting the concrete stresses to get the concrete stress at the same level as the steel and multiplying by n'. Thus, the bottom steel has a stress correction:

$$f'_s = 3740 + 174 \times 40 - 244 \times 10$$

$$= 3740 + 6960 - 2440 = 8260 \text{ lb. per sq. in.} \quad \text{(compression)}$$

This correction will be added to the computed stresses in the compression steel which do not exceed a possible maximum of $f'_s = 1.25nf_c$ and the combined stress does not exceed $f_s = 20{,}000$ lb. per sq. in. This correction will reduce the stress in steel whose computed stresses are tensile.

414. Abutments. The abutments are loaded with earth pressures from above and on the base, also by lateral earth pressure on the back face. There may be water pressure on the front face in certain cases. In addition, at the springing section acts the normal force N_s and couple M_s, which vary as the live load and temperature vary. The abutment should be of such size and shape that it is in equilibrium for all values of N_s and M_s without exceeding the allowable soil pressures. If the assumption that the abutments do not yield is justified, the abutment should be massive enough to fix the springing section. The adoption of pressures for design which are well below the allowable will help to reduce settlement. For this reason the location of the abutment must be very carefully selected. On poor soils the arch should be designed by equations which allow for movement of the abutments.

415. Open Spandrel Arches. Many arches are built without earth fills. The superstructure loads are brought to the arch by columns or walls. The analysis and design follow the methods illustrated above, but some of the live and dead loads will be concentrated forces.

THE RIGID FRAME BRIDGE

416. The Rigid Frame Bridge. The type of bridge known as the rigid frame bridge has been increasingly used in modern construction. The usual cross section has the shape of Figure 242. Such a section is superior to the arch in that the clearance under the deck can be kept close to the maximum rise throughout the clear span. The abrupt change of direction between the center lines of the deck and walls renders it impossible to keep the line of thrust nearly coincident with these center lines, and large bending moments occur at the junction of deck and wall. These large restraining moments at the ends of the deck give a low value to the positive moment at the center of the deck span, and the center section can be made relatively thin.

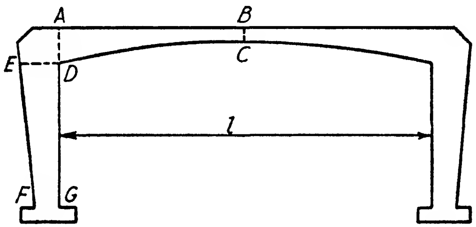


FIG. 242

If the footings are relatively narrow as in Figure 242, the support at the base of the wall is assumed to be hinged. Sometimes a deliberate attempt is made to construct a hinged connection between wall and footing. A massive footing justifies the assumption of a fixed base.

The Portland Cement Association [6] suggests trial dimensions for the mathematical analysis. They are

1. Determine clear span l.

2. Assume section depth at BC equal to $\frac{l}{35}$ for usual soils. This may be reduced to $\frac{l}{40}$, if the footing does not settle.

3. Assume AD and DE to be about $\frac{l}{15}$.

4. Assume FG from 18 in. for 30-ft. span, to 30 in. for 60-ft. spans, up to 40 in. for 90-ft. spans.

417. Design of Rigid Frame Bridge. The design of the rigid frame bridge is usually made by virtual work, the equations being the same used previously for arch design.[7] It is usual to take an even number of Δs blocks between sections FG and DE and again between sections

[6] *Analysis of Rigid Frame Concrete Bridges*, Fourth Edition, 1936.
[7] *The Rigid Frame Bridge*, by Arthur G. Hayden.

AD and *BC*. The mass of concrete between sections *DE* and *AD* is treated as a single Δs block. The analysis should include influence lines for moving loads, and the effect of dead loads and earth pressure against the walls. Temperature stresses and the readjustments due to shrinkage and plastic flow should be included in the final summations.

Space does not permit a complete solution of a rigid frame bridge design in this text and the reader is referred to Mr. Hayden's book for illustrative designs using virtual work, and to the Portland Cement Association publications, for an illustrative design using the column analogy and moment-distribution methods for the solution. Comparisons of the time spent in design cannot be made directly between these two methods as the labor of computation is considerably reduced in the moment-distribution illustrative problem by the use of charts giving coefficients obtained as the average of many analyses.

APPENDIX

1941 Building Regulations for Reinforced Concrete (A.C.I. Code)

The following articles of the 1941 *Building Regulations for Reinforced Concrete*[1] have been reprinted by kind permission of the American Concrete Institute. They were selected from the regulations for use as design standards in the text. The A.C.I. article numbers have been used, and the references in the text distinguish them by the prefix A.C.I.

305. Allowable Unit Stresses in Concrete. (*a*) The unit stresses in pounds per square inch on concrete to be used in the design shall not exceed the values of Table 305(*a*) where f'_c equals the minimum specified ultimate compressive strength at 28 days, or at the earlier age at which the concrete may be expected to receive its full load.

306. Allowable Unit Stresses in Reinforcement. Unless otherwise provided in these Regulations, steel for concrete reinforcement shall not be stressed in excess of the following limits:

(*a*) *Tension*
(f_s = Tensile unit stress in longitudinal reinforcement) and
(f_v = Tensile unit stress in web reinforcement)

20,000 p.s.i. for Rail-Steel Concrete Reinforcement Bars, Billet-Steel Concrete Reinforcement Bars (of intermediate and hard grades), Axle-Steel Concrete Reinforcement Bars (of intermediate and hard grades), and Cold-Drawn Steel Wire for Concrete Reinforcement.

18,000 p.s.i. for Billet-Steel Concrete Reinforcement Bars (of structural grade), and Axle-Steel Concrete Reinforcement Bars (of structural grade).

(*b*) *Tension in One-Way Slabs of Not More Than 12 Feet Span*
(f_s = Tensile unit stress in main reinforcement)

For the main reinforcement, $\frac{3}{8}$ inch or less in diameter, in one-way slabs, 50 per cent of the minimum yield point specified in the Standard Specifications of the American Society for Testing Materials for the particular kind and grade of reinforcement used, but in no case to exceed 30,000 p.s.i.

(*c*) *Compression, Vertical Column Reinforcement*
(f_s = Nominal working stress in vertical column reinforcement)

[1] The complete regulations can be obtained from The American Concrete Institute, 7400 Second Boulevard, Detroit 2, Michigan.

TABLE 305(*a*). ALLOWABLE UNIT STRESSES IN CONCRETE

Description		Allowable Unit Stresses: For Any Strength of Concrete as Fixed by Test in Accordance with Section 302, $n = \frac{30,000}{f'_c}$	When Strength of Concrete is Fixed by the Water-Content in Accordance with Section 302: f'_c = 2000 p.s.i., n = 15	f'_c = 2500 p.s.i., n = 12	f'_c = 3000 p.s.i., n = 10	f'_c = 3750 p.s.i., n = 8
Flexure: f_c						
Extreme fiber stress in compression	f_c	$0.45f'_c$	900	1125	1350	1688
Shear: v						
Beams with no web reinforcement and without special anchorage of longitudinal steel	v_c	$0.02f'_c$	40	50	60	75
Beams with no web reinforcement but with special anchorage of longitudinal steel	v_c	$0.03f'_c$	60	75	90	113
Beams with properly designed web reinforcement but without special anchorage of longitudinal steel	v	$0.06f'_c$	120	150	180	225
Beams with properly designed web reinforcement and with special anchorage of longitudinal steel	v	$0.12f'_c$	240	300	360	450
Flat slabs at distance d from edge of column capital or drop panel.*	v_c	$0.03f'_c$	60	75	90	113
Footings.†	v_c	$0.03f'_c$ but not to exceed 75 p.s.i.	60	75	75	75
Bond‡: u						
In beams and slabs and one-way footings: Plain bars	u	$0.04f'_c$ but not to exceed 160 p.s.i.	80	100	120	150
Deformed bars	u	$0.05f'_c$ but not to exceed 200 p.s.i.	100	125	150	188
In two-way footings: Plain bars (hooked)	u	$0.045f'_c$ but not to exceed 160 p.s.i.	90	113	135	160
Deformed bars (hooked)	u	$0.056f'_c$ but not to exceed 200 p.s.i.	112	140	168	200
Bearing: f_c						
On full area	f_c	$0.25f'_c$	500	625	750	938
On one-third area or less §	f_c	$0.375f'_c$	750	938	1125	1405

* See Section 807. † See Section 905(*a*) and 808(*a*).

‡ Where special anchorage is provided [see Section 903(*a*)], one and one-half times these values in bond may be used in beams, slabs and one-way footings, but in no case to exceed 200 p.s.i. for plain bars and 250 p.s.i. for deformed bars. The values given for two-way footings include an allowance for special anchorage.

§ The allowable bearing stress on an area greater than one-third but less than the full area shall be interpolated between the values given.

The author adds the following to clarify the allowable shear stresses given in Table 305(*a*). These shear stresses perform two functions: (*a*) as allowable shear stresses, (*b*) as measures of allowable tension (assuming t = constant × v).

Shear.

Plain concrete	$v_c = 0.02f'_c$
Reinforced concrete—ordinary anchorage	maximum $v = 0.06f'_c$
Reinforced concrete—special anchorage	maximum $v = 0.12f'_c$

Diagonal Tension. Shear stress as measure of allowable tension.

Plain concrete—allowable tensile stress	$t = 0.03f'_c$
Reinforced concrete—shear equivalent of tension.	
Concrete—ordinary anchorage	$v_c = 0.02f'_c$
special anchorage	$v_c = 0.03f'_c$
Total on particle—ordinary anchorage	$v = 0.06f'_c$
special anchorage	$v = 0.12f'_c$

Forty per cent of the minimum yield point specified in the Standard Specifications of the American Society for Testing Materials for the particular kind and grade of reinforcement used, but in no case to exceed 30,000 p.s.i.

(f_r = Allowable unit stress in the metal core of composite and combination columns):

Structural steel sections	16,000 p.s.i.
Cast iron sections	10,000 p.s.i.
Steel pipe	See limitations of Section 1106(*b*)

(*d*) *Compression, Flexural Members.* For compression reinforcement in flexural members see Section 706(*b*).

FORMS AND DETAILS OF CONSTRUCTION

501. Design of Forms. (*a*) Forms shall conform to the shape, lines, and dimensions of the members as called for on the plans, and shall be substantial and sufficiently tight to prevent leakage of mortar. They shall be properly braced or tied together so as to maintain position and shape.

502. Removal of Forms. (*a*) Forms shall be removed in such manner as to insure the complete safety of the structure. Where the structure as a whole is supported on shores, the removable floor forms, beam and girder sides, column and similar vertical forms may be removed after twenty-four hours, providing the concrete is sufficiently hard not to be injured thereby. In no case shall the supporting forms or shoring be removed until the members have acquired sufficient strength to support safely their weight and the load thereon. The results of suitable control tests may be used as evidence that the concrete has attained such sufficient strength.

503. Pipes, Conduits, etc., Embedded in Concrete. (*a*) Pipes which will contain liquid, gas or vapor at other than room temperature shall not be embedded in concrete necessary for structural stability or fire protection. Drain pipes and pipes whose contents will be under pressure greater than atmospheric pressure by more than one pound per square inch shall not be embedded in structural concrete except in passing through from one side to the other of a floor, wall or beam. Electric conduits and other pipes whose embedment is allowed shall not, with their fittings, displace that concrete of a column on which stress is calculated or which is required for fire protection, to greater extent than four per cent of the area of the cross section. Sleeves or other pipes passing through floors, walls or beams shall not be of such size or in such location as unduly to impair the strength of the construction; such sleeves or pipes may be considered as replacing structurally the displaced concrete, provided they are not exposed to rusting or other deterioration, are of uncoated iron or steel not thinner than standard wrought-iron pipe, have a nominal inside diameter not over two inches, and are spaced not less than three diameters on centers. Embedded pipes or conduits other than those merely passing through, shall not be larger in outside diameter than one-third the thickness of the slab, wall or beam in which they are embedded; shall not be spaced closer than three diameters on centers, nor so located as unduly to impair the strength of the construction. Circular uncoated or galvanized electric conduit of iron or steel may be considered as replacing the displaced concrete.

504. Cleaning and Bending Reinforcement. (*a*) Metal reinforcement, at the time concrete is placed, shall be free from rust scale or other coatings that will destroy or reduce the bond. Bends for stirrups and ties shall be made around a pin having a diameter not less than two times the minimum thickness of the bar. Bends for other bars shall be made around a pin having a diameter not less than six times the minimum thickness of the bar, except that for bars larger than one inch, the pin shall be not less than eight times the minimum thickness of the bar. All bars shall be bent cold.

505. Placing Reinforcement. (*a*) Metal reinforcement shall be accurately placed and adequately secured in position by concrete or metal chairs and spacers. The minimum clear distance between parallel bars shall be one and one-half times the diameter for round bars and twice the side dimension for square bars. If special anchorage as required in Section 903 is provided, the minimum clear distance between parallel bars shall be equal to the diameter for round bars and one and one-half times the side dimension for square bars. In no case shall the clear distance between bars be less than one inch, nor less than one and one-third times the maximum size of the coarse aggregate.

(*b*) When wire or other reinforcement, not exceeding one-fourth inch in diameter is used as reinforcement for slabs not exceeding ten feet in span, the reinforcement may be curved from a point near the top of the slab over the support to a point near the bottom of the slab at mid-span; provided such reinforcement is either continuous over, or securely anchored to, the support.

506. Splices and Offsets in Reinforcement. (*a*) In slabs, beams and girders, splices of reinforcement at points of maximum stress shall generally be avoided. Splices shall provide sufficient lap to transfer the stress between bars by bond and shear. In such splices the minimum spacing of bars shall be as specified in Section 505.

(*b*) Where changes in the cross section of a column occur, the longitudinal bars shall be offset in a region where lateral support is afforded. Where offset, the slope of the inclined portion shall not be more than 1 in 6, and in the case of tied columns the ties shall be spaced not over three inches on centers for a distance of one foot below the actual point of offset.

507. Concrete Protection for Reinforcement. (*a*) The reinforcement of footings and other principal structural members in which the concrete is deposited against the ground shall have not less than three inches of concrete between it and the ground contact surface. If concrete surfaces after removal of the forms are to be exposed to the weather or be in contact with the ground, the reinforcement shall be protected with not less than two inches of concrete for bars more than five-eighths inch in diameter and one and one-half inches for bars five-eighths inch or less in diameter.

(*b*) The concrete protective covering for reinforcement at surfaces not exposed directly to the ground or weather shall be not less than three-fourths inch for slabs and walls; and not less than one and one-half inches for beams, girders and columns. In concrete joist floors in which the clear distance between joists is not more than thirty inches, the protection of metal reinforcement shall be at least three-fourths inch.

(*c*) If the code of which these regulations form a part specifies, as fire-protective covering of the reinforcement, thicknesses of concrete greater than those given in this section, then such greater thicknesses shall be used.

(*d*) Concrete protection for reinforcement shall in all cases be at least equal to the diameter of round bars, and one and one-half times the side dimension of square bars.

(*e*) Exposed reinforcement bars intended for bonding with future extensions shall be protected from corrosion by concrete or other adequate covering.

508. Construction Joints. (*a*) Joints not indicated on the plans shall be so made and located as to least impair the strength of the structure. Where a joint is to be made, the surface of the concrete shall be thoroughly cleaned and all laitance removed. In addition to the foregoing, vertical joints shall be thoroughly wetted but not saturated, and slushed with a coat of neat cement grout immediately before placing of new concrete.

(*b*) At least two hours must elapse after depositing concrete in the columns or walls before depositing in beams, girders, or slabs supported thereon. Beams, girders, brackets, column capitals, and haunches shall be considered as part of the floor system and shall be placed monolithically therewith.

(*c*) Construction joints in floors shall be located near the middle of the spans of slabs, beams, or girders, unless a beam intersects a girder at this point, in which case the joints in the girders shall be offset a distance equal to twice the width of the beam. In this last case provision shall be made for shear by use of inclined reinforcement.

DESIGN—GENERAL CONSIDERATIONS

600. Notation.

f'_c = Ultimate compressive strength of concrete at age of 28 days, unless otherwise specified.

n = Ratio of modulus of elasticity of steel to that of concrete $= \dfrac{E_s}{E_c}$; assumed as equal to $\dfrac{30{,}000}{f'_c}$.

601. Assumptions. (*a*) The design of reinforced concrete members shall be made with reference to working stresses and safe loads. The accepted theory of flexure as applied to reinforced concrete shall be applied to all members resisting bending. The following assumptions shall be made:

1. The steel takes all the tensile stress.
2. In determining the ratio n for design purposes, the modulus of elasticity for the concrete shall be assumed as $1000 f'_c$, and that for steel as 30,000,000 p.s.i.

602. Design Loads. (*a*) The provisions for design herein specified are based on the assumption that all structures shall be designed for all dead- and live-loads coming upon them, the live-loads to be in accordance with the general requirements of the building code of which this forms a part, with such reductions for girders and lower story columns as are permitted therein.

603. Resistance to Wind Forces. (*a*) The resisting elements in structures required to resist wind forces shall be limited to the integral structural parts.

(*b*) The moments, shears, and direct stresses resulting from wind forces determined in accordance with recognized methods shall be added to the maximum stresses which obtain at any section for dead- and live-loads.

(*c*) In proportioning the component parts of the structure for the maximum combined stresses, including wind stresses, the unit stresses shall not exceed the allowable stresses for combined live- and dead-loads provided in Sections 305, 306 and 1110 by more than one-third. The structural members and their connections shall be so proportioned as to provide suitable rigidity of structure.

FLEXURAL COMPUTATIONS

701. General Requirements. (*a*) All members of frames or continuous construction shall be designed to resist at all sections the maximum moments and shears produced by dead load, live load and wind load, as determined by the theory of elastic frames in which the simplified assumptions of Section 702 may be used.

(*b*) Approximate methods of frame analysis are satisfactory for buildings of usual types of construction, spans and story heights.

(*c*) In the case of two or more approximately equal spans (the larger of two adjacent spans not exceeding the shorter by more than 20 per cent) with loads uniformly distributed, where the unit live load does not exceed three times the unit dead load, design for the following moments and shears is satisfactory: [2]

Positive moment at center of span

End spans	$\frac{1}{14} wl'^2$
Interior spans	$\frac{1}{16} wl'^2$

Negative moment at exterior face of first interior support

Two spans	$\frac{1}{9} wl'^2$
More than two spans	$\frac{1}{10} wl'^2$
Negative moment at other faces of interior supports	$\frac{1}{11} wl'^2$
Negative moment at face of all supports for, (*a*) slabs with spans not exceeding ten feet, and (*b*) beams and girders where ratio of sum of column stiffnesses to beam stiffness exceeds eight	$\frac{1}{12} wl'^2$
Shear in end members at first interior support	$1.15 \dfrac{wl'}{2}$
Shear at other supports	$\dfrac{wl'}{2}$

702. Conditions of Design.[3] (*a*) *Arrangement of Live Load.* 1. The live load may be considered to be applied only to the floor under consideration, and the far ends of the columns may be assumed as fixed.

2. Consideration may be limited to combinations of dead load on all spans with full live load on two adjacent spans and with full live load on alternate spans.

(*b*) *Span length.* 1. The span length, l, of members that are not built integrally with their supports shall be the clear span plus the depth of the slab or beam but shall not exceed the distance between centers of supports.

2. In analysis of continuous frames, center to center distances, l and h, may be used in the determination of moments. Moments at faces of supports may be used for design of beams and girders.

3. Solid or ribbed slabs with clear spans of not more than ten feet that are built integrally with their supports may be designed as continuous slabs on knife edge supports with spans equal to the clear spans of the slab and the width of beams otherwise neglected.

(*c*) *Stiffness.* 1. The stiffness, K, of a member is defined as EI divided by l or h.

2. In computing the value of I of slabs, beams, girders, and columns, the reinforcement may be neglected. In T-shaped sections allowance shall be made for the effect of flange.

[2] l' = clear span for positive moment and the average of the two adjacent clear spans for negative moment.

[3] Chapter 7 deals with floor members only. For moments in columns see Section 1108.

3. Any reasonable assumption may be adopted as to relative stiffness of columns and of floor system. The assumption made shall be consistent throughout the analysis.

(*d*) *Haunched Floor Members.* 1. When members are widened near the supports, the additional width may be neglected in computing moments, but may be considered as resisting the resulting moments and shears.

2. When members are deepened near the supports, they may be analyzed as members of constant depth provided the minimum depth only is considered as resisting the resulting moments; otherwise an analysis taking into account the variation in depth is required. In any case, the actual depth may be considered as resisting shear.

(*e*) *Limitations.* 1. Wherever at any section positive reinforcement is indicated by analysis, the amount provided shall be not less than $.005b'd$ except in slabs of uniform thickness.

2. Not less than $0.005b'd$ of negative reinforcement shall be provided at the outer end of all members built integrally with their supports.

3. Where analysis indicates negative reinforcement along the full length of a span, the reinforcement need not be extended beyond the point where the required amount is $0.0025b'd$ or less.

4. In slabs of uniform thickness the minimum amount of reinforcement in the direction of the span shall be:

For structural, intermediate and hard grades and rail steel	$0.0025bd$
For steel having a minimum yield point of 56,000 p.s.i.	$0.002bd$

703. Depth of Beam or Slab. (*a*) The depth of the beam or slab shall be taken as the distance from the centroid of the tensile reinforcement to the compression face of the structural members. Any floor finish not placed monolithically with the floor slab shall not be included as a part of the structural member. When the finish is placed monolithically with the structural slab in buildings of the warehouse or industrial class, there shall be placed an additional depth of one-half inch over that required by the design of the member.

704. Distance between Lateral Supports. (*a*) The clear distance between lateral supports of a beam shall not exceed thirty-two times the least width of compression flange.

705. Requirements for T-Beams. (*a*) In T-beam construction the slab and beam shall be built integrally or otherwise effectively bonded together. The effective flange width to be used in the design of symmetrical T-beams shall not exceed one-fourth of the span length of the beam, and its overhanging width on either side of the web shall not exceed eight times the thickness of the slab nor one-half the clear distance to the next beam.

(*b*) For beams having a flange on one side only, the effective overhanging flange width shall not exceed one-twelfth of the span length of the beam, nor six times the thickness of the slab, nor one-half the clear distance to the next beam.

(*c*) Where the principal reinforcement in a slab which is considered as the flange of a T-beam (not a joist in concrete joist floors) is parallel to the beam, transverse reinforcement shall be provided in the top of the slab. This reinforcement shall be designed to carry the load on the portion of the slab assumed as the flange of the T-beam. The spacing of the bars shall not exceed five times the thickness of the flange, nor in any case eighteen inches.

(*d*) Provision shall be made for the compressive stress at the support in continuous T-beam construction, care being taken that the provisions of Section 505 relating to the spacing of bars, and 404(*d*), relating to the placing of concrete shall be fully met.

(*e*) The overhanging portion of the flange of the beam shall not be considered as effective in computing the shear and diagonal tension resistance of T-beams.

(*f*) Isolated beams in which the T-form is used only for the purpose of providing additional compression area, shall have a flange thickness not less than one-half the width of the web and a total flange width not more than four times the web thickness.

706. Compression Steel in Flexural Members. (*a*) Compression steel in beams, girders, or slabs shall be anchored by ties or stirrups not less than one-fourth inch in diameter spaced not farther apart than 16 bar diameters, or 48 tie diameters. Such stirrups or ties shall be used throughout the distance where the compression steel is required.

(*b*) The effectiveness of compression reinforcement in resisting bending may be taken at twice the value indicated from the calculations assuming a straight-line relation between stress and strain and the modular ratio given in Section 601, but not of greater value than the allowable stress in tension.

707. Shrinkage and Temperature Reinforcement. (*a*) Reinforcement for shrinkage and temperature stresses normal to the principal reinforcement shall be provided in floor and roof slabs where the principal reinforcement extends in one direction only. Such reinforcement shall provide for the following minimum ratios of reinforcement area to concrete area *bd*, but in no case shall such reinforcing bars be placed farther apart than five times the slab thickness nor more than eighteen inches:

Floor slabs where plain bars are used	0.0025
Floor slabs where deformed bars are used	0.002
Floor slabs where wire fabric is used, having welded intersections not farther apart in the direction of stress than twelve inches	0.0018
Roof slabs where plain bars are used	0.003
Roof slabs where deformed bars are used	0.0025
Roof slabs where wire fabric is used, having welded intersections not farther apart in the direction of stress than twelve inches	0.0022

708. Concrete Joist Floor Construction. (*a*) Concrete joist floor construction consists of concrete joists and slabs placed monolithically with or without burned clay or concrete tile fillers. The joists shall not be farther apart than thirty inches face to face. The ribs shall be straight, not less than four inches wide, nor of a depth more than three times the width.

(*b*) When burned clay or concrete tile fillers, of material having a unit compressive strength at least equal to that of the designed strength of the concrete in the joists are used, and the fillers are so placed that the joints in alternate rows are staggered, the vertical shells of the fillers in contact with the joists may be included in the calculations involving shear or negative bending moment. No other portion of the fillers may be included in the design calculations.

(*c*) The concrete slab over the fillers shall be not less than one and one-half inches in thickness, nor less in thickness than one-twelfth of the clear distance between joists. Shrinkage reinforcement in the slab shall be provided as required in Section 707.

(*d*) Where removable forms or fillers not complying with (*b*) are used, the thickness of the concrete slab shall not be less than one-twelfth of the clear distance between joists and in no case less than two inches. Such slab shall be reinforced at right angles to the joists with a minimum of .049 square inch of reinforcing steel per foot of width, and in slabs on which the prescribed live load does not exceed fifty pounds per square foot, no additional reinforcement shall be required.

(*e*) When the finish used as a wearing surface is placed monolithically with the structural slab in buildings of the warehouse or industrial class, the thickness of the concrete over the fillers shall be one-half inch greater than the thickness used for design purposes.

(*f*) Where the slab contains conduits or pipes, the thickness shall not be less than one inch plus the total over-all depth of such conduits or pipes at any point. Such conduits or pipes shall be so located as not to impair the strength of the construction.

710. Maximum Spacing of Principal Slab Reinforcement. (*a*) In slabs other than concrete joist floor construction or flat slabs, the principal reinforcement shall not be spaced farther apart than three times the slab thickness, nor shall the ratio of reinforcement be less than specified in Section 707(*a*).

SHEAR AND DIAGONAL TENSION

801. Shearing Unit Stress. (*a*) The shearing unit stress v, as a measure of diagonal tension, in reinforced concrete flexural members shall be computed by formula (12):

$$v = \frac{V}{bjd} \qquad (12)$$

(*b*) For beams of I or T section, b' shall be substituted for b in formula (12).

(*c*) In concrete joist floor construction, where burned clay or concrete tile are used, b' may be taken as a width equal to the thickness of the concrete web plus the thicknesses of the vertical shells of the concrete or burned clay tile in contact with the joist as in Section 708(*b*).

(*d*) When the value of the shearing unit stress computed by formula (12) exceeds the shearing unit stress v_c permitted on the concrete of an unreinforced web (see Section 305), web reinforcement shall be provided to carry the excess.

802. Types of Web Reinforcement. (*a*) Web reinforcement may consist of:

1. Stirrups or web reinforcement bars perpendicular to the longitudinal steel.
2. Stirrups or web reinforcement bars welded or otherwise rigidly attached to the longitudinal steel and making an angle of 30 degrees or more thereto.
3. Longitudinal bars bent so that the axis of the inclined portion of the bar makes an angle of 15 degrees or more with the axis of the longitudinal portion of the bar.
4. Special arrangements of bars with adequate provisions to prevent slip of bars or splitting of the concrete by the reinforcement. See Section 804(*f*).

(*b*) Stirrups or other bars to be considered effective as web reinforcement shall be anchored at both ends, according to the provisions of Section 904.

803. Stirrups. (*a*) The area of steel required in stirrups placed perpendicular to the longitudinal reinforcement shall be computed by formula (13).[4]

$$A_v = \frac{V's}{f_v jd} \qquad (13)$$

(*b*) Inclined stirrups shall be proportioned by formula (15), Section 804(*d*).

(*c*) Stirrups placed perpendicular to the longitudinal reinforcement shall not be used alone as web reinforcement when the shearing unit stress (v) exceeds $0.08f'_c$.

[4] V' = excess of the total shear over that permitted on the concrete.

804. Bent Bars. (*a*) When the web reinforcement consists of a single bent bar or of a single group of bent bars the required area of such bars shall be computed by formula (14).

$$A_v = \frac{V'}{f_v \sin \alpha} \tag{14}$$

(*b*) In formula (14) V' shall not exceed $0.040 f'_c bjd$.

(*c*) Only the center three-fourths of the inclined portion of such bar, or group of bars, shall be considered effective as web reinforcement.

(*d*) Where there is a series of parallel bent bars, the required area shall be determined by formula (15).

$$A_v = \frac{V's}{f_v jd (\sin \alpha + \cos \alpha)} \tag{15}$$

(*e*) When bent bars, having a radius of bend of not more than two times the diameter of the bar, are used alone as web reinforcement, the allowable shearing unit stress shall not exceed $0.060f'_c$. This shearing unit stress may be increased at the rate of $0.01f'_c$ for each increase of four bar diameters in the radius of bend until the maximum allowable shearing unit stress is reached. See Section 305(*a*).

(*f*) The shearing unit stress permitted when special arrangements of bars are employed shall be that determined by making comparative tests, to destruction, of specimens of the proposed system and of similar specimens reinforced in conformity with the provisions of this code, the same factor of safety being applied in both cases.

805. Combined Web Reinforcement. (*a*) Where more than one type of reinforcement is used to reinforce the same portion of the web, the total shearing resistance of this portion of the web shall be assumed as the sum of the shearing resistances computed for the various types separately. In such computations the shearing resistance of the concrete shall be included only once, and no one type of reinforcement shall be assumed to resist more than $\frac{2V'}{3}$.

806. Spacing of Web Reinforcement. (*a*) Where web reinforcement is required it shall be so spaced that every 45 degree line (representing a potential crack) extending from the mid-depth of the beam to the longitudinal tension bars shall be crossed by at least one line of web reinforcement. If a shearing unit stress in excess of $0.06f'_c$ is used, every such line shall be crossed by at least two such lines of web reinforcement.

807. Shearing Stress in Flat Slabs. (*a*) In flat slabs, the shearing unit stress on a vertical section which lies at a distance $t_2 - 1\frac{1}{2}$ inches beyond the edge of the column capital and parallel or concentric with it, shall not exceed the following values when computed by formula (12) (in which d shall be taken as $t_2 - 1\frac{1}{2}$ inches):

1. $0.03f'_c$, when at least 50 per cent of the total negative reinforcement in the column strip passes directly over the column capital.

2. $0.025f'_c$, when 25 per cent or less of the total negative reinforcement in the column strip passes directly over the column capital.

3. For intermediate percentages, intermediate values of the shearing unit stress shall be used.

(*b*) In flat slabs, the shearing unit stress on a vertical section which lies at a distance of $t_3 - 1\frac{1}{2}$ inches beyond the edge of the drop panel and parallel with it shall not exceed $0.03f'_c$ when computed by formula (12) (in which d shall be taken as $t_3 - 1\frac{1}{2}$ inches). At least 50 per cent of the cross-sectional area of the negative reinforcement in the column strip must be within the width of strip directly above the drop panel.

808. Shear and Diagonal Tension in Footings. (*a*) In isolated footings the shearing unit stress computed by formula (12) on the critical section [see 1205(*a*)], shall not exceed $0.03f'_c$, nor in any case shall it exceed 75 p.s.i.

BOND AND ANCHORAGE

901. Computation of Bond Stress in Beams. (*a*) In flexural members in which the tensile reinforcement is parallel to the compression face, the bond stress at any cross section shall be computed by formula (16).

$$u = \frac{V}{\Sigma o\, jd} \tag{16}$$

in which V is the shear at that section.

(*b*) Adequate end anchorage shall be provided for the tensile reinforcement in all flexural members to which formula (16) does not apply, such as footings, brackets and other tapered or stepped beams in which the tensile reinforcement is not parallel to the compression face.

902. Ordinary Anchorage Requirements. (*a*) Tensile negative reinforcement in any span of a continuous, restrained, or cantilever beam, or in any member of a rigid frame shall be adequately anchored by bond, hooks or mechanical anchors in or through the supporting member. Within any such span every reinforcing bar shall be extended at least twelve diameters beyond the point at which it is no longer needed to resist stress. In cases where the length from the point of maximum tensile stress in the bar to the end of the bar is not sufficient to develop this maximum stress by bond, the bar shall extend into a region of compression and be anchored by means of a standard hook or it shall be bent across the web at an angle of not less than 15 degrees with the longitudinal portion of the bar and either made continuous with the positive reinforcement or anchored in a region of compression.

(*b*) Of the positive reinforcement in continuous beams not less than one-fourth the area shall extend along the same face of the beam into the support a distance of ten or more bar diameters, or shall be extended as far as possible into the support and terminated in standard hooks, or other adequate anchorage.

(*c*) In simple beams, or at the outer or freely supported ends of end spans of continuous beams, at least one-half the positive reinforcement shall extend along the same face of the beam into the support a distance of twelve or more bar diameters, or shall be extended as far as possible into the support and terminated in standard hooks.

903. Special Anchorage Requirements. (*a*) Where increased shearing or bond stresses are permitted because of the use of special anchorage (see Section 305), every bar shall be terminated in a standard hook in a region of compression, or it shall be bent across the web at an angle of not less than 15 degrees with the longitudinal portion of the bar and made continuous with the negative or positive reinforcement.

904. Anchorage of Web Reinforcement. (*a*) Single separate bars used as web reinforcement shall be anchored at each end by one of the following methods:

1. Welding to longitudinal reinforcement.
2. Hooking tightly around the longitudinal reinforcement through 180 degrees.

3. Embedment above or below the mid-depth of the beam on the compression side, a distance sufficient to develop the stress to which the bar will be subjected at a bond stress of not to exceed $.04f'_c$ on plain bars nor $.05f'_c$ on deformed bars.

4. Standard hook [see Section 906(*a*)], considered as developing 10,000 p.s.i., plus embedment sufficient to develop by bond the remainder of the stress to which the bar is subjected. The unit bond stress shall not exceed that specified in Table 305(*a*). The effective embedded length shall not be assumed to exceed the distance between the mid-depth of the beam and the tangent of the hook.

(*b*) The extreme ends of bars forming simple U or multiple stirrups shall be anchored by one of the methods of Section 904(*a*) or shall be bent through an angle of at least 90 degrees tightly around a longitudinal reinforcing bar not less in diameter than the stirrup bar, and shall project beyond the bend at least twelve diameters of the stirrup bar.

(*c*) The loops or closed ends of such stirrups shall be anchored by bending around the longitudinal reinforcement through an angle of at least 90 degrees, or by being welded or otherwise rigidly attached thereto.

(*d*) Hooking or bending stirrups or separate web reinforcement bars around the longitudinal reinforcement shall be considered effective only when these bars are perpendicular to the longitudinal reinforcement.

(*e*) Longitudinal bars bent to act as web reinforcement shall, in a region of tension, be continuous with the longitudinal reinforcement. The tensile stress in each bar shall be fully developed in both the upper and the lower half of the beam by one of the following methods:

1. As specified in Section 904(*a*), (3).
2. As specified in Section 904(*a*), (4).
3. By bond, at a unit bond stress not exceeding $.04f'_c$ on plain bars nor $.05f'_c$ on deformed bars, plus a bend of radius not less than two times the diameter of the bar, parallel to the upper or lower surface of the beam, plus an extension of the bar of not less than twelve diameters of the bar terminating in a standard hook. This short radius bend extension and hook shall together not be counted upon to develop a tensile unit stress in the bar of more than 10,000 p.s.i.
4. By bond, at a unit bond stress not exceeding $.04f'_c$ on plain bars nor $.05f'_c$ on deformed bars, plus a bend of radius not less than two times the diameter of the bar, parallel to the upper or lower surface of the beam and continuous with the longitudinal reinforcement. The short radius bend and continuity shall together not be counted upon to develop a tensile unit stress in the bar of more than 10,000 p.s.i.
5. The tensile unit stress at the beginning of a bend may be increased from 10,000 p.s.i. when the radius of bend is two bar diameters, at the rate of 1000 p.s.i. tension for each increase of one and one-half bar diameters in the radius of bend, provided that the length of the bar in the bend and extension is sufficient to develop this increased tensile stress by bond at the unit stresses given in Section 904(*e*), (3).

(*f*) In all cases web reinforcement shall be carried as close to the compression surface of the beam as fireproofing regulations and the proximity of other steel will permit.

905. Anchorage of Bars in Footing Slabs. (*a*) All bars in footing slabs shall be anchored by means of standard hooks. The outer faces of these hooks shall be not less than three inches nor more than six inches from the face of the footing.

906. Hooks. (*a*) The terms "hook" or "standard hook" as used herein shall mean either

1. A complete semicircular turn with a radius of bend on the axis of the bar of not less than three and not more than six bar diameters, plus an extension of at least four bar diameters at the free end of the bar, or
2. A 90-degree bend having a radius of not less than four bar diameters plus an extension of twelve bar diameters.

Hooks having a radius of bend of more than six bar diameters shall be considered merely as extensions to the bars, and shall be treated as in Section 904(*e*), (5).

(*b*) In general, hooks shall not be permitted in the tension portion of any beam except at the ends of simple or cantilever beams or at the freely supported ends of continuous or restrained beams.

(*c*) No hook shall be assumed to carry a load which would produce a tensile stress in the bar greater than 10,000 p.s.i.

(*d*) Hooks shall not be considered effective in adding to the compressive resistance of bars.

(*e*) Any mechanical device capable of developing the strength of the bar without damage to the concrete may be used in lieu of a hook. Tests must be presented to show the adequacy of such devices.

FLAT SLABS

1000. Notation.

A = The distance from the center line of the column, in the direction of any span, to the intersection of a 45-degree diagonal line from the center of the column to the bottom of the flat slab or drop panel, where such line lies wholly within the column, capital, or bracket, provided such capital or bracket is structurally capable of resisting shears and moments without excessive unit stress. In no case shall A be greater than one-eighth the span in the direction considered.

A_{av} = Average of the two values of A for the two columns at the ends of a column strip, in the direction of the spans considered.

c = Diameter or width of column capital at the under side of the slab or drop panel. No portion of the column capital shall be considered for structural purposes which lies outside the largest right circular cone, with 90 degrees vertex angle, that can be included within the outlines of the column capital.

L = Span length of slab center to center of columns in the direction of which bending is considered.

M_o = Sum of the positive and the average negative bending moments at the critical design sections of a flat slab panel. See Section 1003(*b*).

W = Total dead and live load uniformly distributed over a single panel area.

W_{av} = The average of the total load on two adjacent panels.

x = Coefficient of span L which gives the distance from the center of column to the critical section for negative bending in design according to Section 1002(*a*).

1001. Scope. (*a*) The term flat slab shall mean a reinforced concrete slab supported by columns with or without flaring heads or column capitals, with or without depressed or drop panels and generally without beams or girders.

(*b*) Recesses or pockets in flat slab ceilings, located between reinforcing bars and forming cellular or two-way ribbed ceilings, whether left open or filled with permanent fillers, shall not prevent a slab from being considered a flat slab; but allowable unit stresses shall not be exceeded.

(*c*) This chapter provides for two methods of design of flat slab structures.

1. Any type of flat slab construction may be designed by application of the principles of continuity, using the method outlined in Section 1002, or using other recognized methods of elastic analysis. In either case, the design must be subject to the provisions of Sections 1005(*a*) and (*c*), 1006, 1008 and 1009.
2. The common cases of flat slab construction described in Section 1003 may be designed by the use of moment coefficients, given in Sections 1003 and 1004, and subject to the provisions of Sections 1005, 1006, 1007, 1008 and 1009.

1002. Design of Flat Slabs as Continuous Frames. (*a*) Except in the cases of flat slab construction where specified coefficients for bending may be used, as provided in Section 1003, bending and shear in flat slabs and their supports shall be determined by an analysis of the structure as a continuous frame, and all sections shall be proportioned to resist the moments and shears thus obtained. In the analysis, the following assumptions may be made:

1. The structure may be considered divided into a number of bents, each consisting of a row of columns and strips of supported slabs, each strip bounded laterally by the center line of the panel on either side of the row of columns. The bents shall be taken longitudinally and transversely of the building.
2. Each such bent may be analyzed in its entirety; or each floor thereof and the roof may be analyzed separately with its adjacent columns above and below, the columns being assumed fixed at their remote ends. Where slabs are thus analyzed separately, in bents more than four panels long, it may be assumed in determining the bending at a given support that the slab is fixed at any support two panels distant therefrom beyond which the slab continues.
3. The joints between columns and slabs may be considered rigid and this rigidity may be assumed to extend in the slabs a distance A from the center of the columns, and in the column to the intersection of the sides of the column and the 45-degree line defining A. The change in length of columns and slabs due to direct stress, and deflections due to shear, may be neglected. Where metal column capitals are used, account may be taken of their contributions to stiffness and resistance to bending and shear.
4. The supporting columns may be assumed free from settlement or lateral movement unless the amount thereof can be reasonably determined.
5. The moment of inertia of slab or column at any cross section may be assumed to be that of the gross section of the concrete. Variation in the moments of inertia of the slabs and columns along their axes shall be taken into account.
6. Where the load to be supported is definitely known, the structure shall be analyzed for that load. Where the live load is variable but does not exceed three-quarters of the dead load, or the nature of the live load is such that all panels will be loaded simultaneously, the maximum bending may be assumed to obtain at all sections under full live load. Elsewhere, maximum positive bending near mid-span of a panel may be assumed to obtain under full live load in the panel and in alternate panels; and maximum negative bending at a support may be assumed to obtain under full live load in the adjacent panels only.

7. Where neither beams nor girders help to transfer the slab load to the supporting column, the critical section for negative bending may be assumed as not more than the distance xL from the column center, where

$$x = 0.073 + 0.57\frac{A}{L} \tag{17}$$

In slabs supported by beams, girders, or walls, the critical section for negative bending shall be assumed at the face of such support.

8. The numerical sum of the maximum positive and the average maximum negative bending moments for which provision is made in the design in the direction of either side of a rectangular panel shall be assumed as not less than

$$\frac{1}{10}W_{av}L\left(1 - \frac{4A_{av}}{3L}\right)^2 \tag{18}$$

9. The bending at critical sections across the slabs of each bent may be apportioned between the column strip and middle strip, as defined in Section 1005, in the ratio of the specified coefficients which affect such apportionment in the special cases of flat slabs provided for in Section 1003.

10. The maximum bending in columns may be assumed to obtain under full live load in alternate panels. Columns shall be proportioned to resist the maximum bending combined with the maximum direct load consistent therewith; and for maximum direct load combined with the bending under full load, the direct load subject to allowable reductions, in the manner provided in Chapter 11. In computing moments in columns at any floor, the far ends of the columns may be considered fixed.

(*b*) The foregoing provisions outline the method to be followed in analyzing and designing flat slabs in the general case. In all instances the design must conform to the requirements for panel strips and critical design sections, slab thickness and drop panels, capitals and brackets, arrangement of reinforcement and openings in flat slabs, as provided in Sections 1005(*a*) and (*c*), 1006, 1008 and 1009.

1003. Design of Flat Slabs by Moment Coefficients. (*a*) In those cases of flat slab construction which fall within the following limitations as to continuity and dimensions, the bending moments at critical sections may be determined by the use of specified coefficients as provided in Section 1004.

1. The ratio of length to width of panel does not exceed 1.33.
2. The slab is continuous for at least three panels in each direction.
3. The successive span lengths in each direction differ by not more than twenty per cent of the shorter span.

(*b*) In such slabs, the numerical sum of the positive and negative bending moments in the direction of either side of an interior rectangular panel shall be assumed as not less than

$$M_o = 0.09\,WL\left(1 - \frac{2c}{3L}\right)^2 \tag{19}$$

(*c*) Three-fourths of the width of the strip shall be taken as the width of the section in computing compression due to bending, except that, on a section through a drop panel, three-fourths of the width of the drop panel shall be taken. Account shall be taken of any recesses which reduce the compressive area. Tension reinforcement distributed over the entire strip shall be included in the computations.

(*d*) The design of slabs under the procedure given in this section is subject to the provisions of all subsequent sections of this chapter (Sections 1004 to 1009).

1004. Bending Moment Coefficients. (*a*) The bending moments at the critical sections of the middle and column strips of an interior panel shall be assumed as given in Table 1004(*a*).

(*b*) The bending moments at critical sections of strips, in an exterior panel, at right angles to the discontinuous edge, where the exterior supports consist of reinforced concrete columns or reinforced concrete bearing walls integral with the slab, the ratio of stiffness of the support to that of the slab being at least as great as the ratio of the live load to the dead load and not less than one, shall be assumed as given in Table 1004(*b*). Where a flat slab is so supported by a wall providing restraint at the discontinuous edge, the coefficient for negative bending at the edge shall be assumed more nearly equal in the column and middle strips, the sum remaining as given in Table 1004(*b*), but that for the column strip shall not be less than $0.30M_o$. Bending in middle strips parallel to a discontinuous edge, except in a corner panel, shall be assumed the same as in an interior panel. M_o shall be determined as provided in Section 1003(*b*) for an interior panel.

(*c*) The bending moments at critical sections of strips, in an exterior panel, at right angles to the discontinuous edge, where the exterior supports are masonry bearing walls or other construction which provide only negligible restraint to the slab, shall be assumed as given in Table 1004(*b*) with the following modifications.

1. On critical sections at the face of the exterior support, negative bending in each strip shall be assumed as $0.05M_o$.
2. The coefficients for positive bending shall be increased by forty per cent.
3. The coefficients for negative bending at the first interior columns shall be increased thirty per cent.

(*d*) The bending moments in panels with marginal beams or walls, in the strips parallel and close thereto, and in the beams, shall be determined upon the basis of assumptions presented in Table 1004(*c*).

(*e*) For design purposes any of the moment coefficients of Tables 1004(*a*), 1004(*b*), and 1004(*c*) may be varied by not more than six per cent, but the numerical sum of the positive and negative moments in a panel shall not be taken as less than the amount specified.

(*f*) Panels supported by marginal beams on opposite edges shall be designed as solid one or two-way slabs to carry the entire panel load.

(*g*) The ratio of reinforcement in any strip shall not be less than 0.0025.

General Requirements

1005. Panel Strips and Critical Design Sections. (*a*) A flat slab panel shall be considered as consisting of strips in each direction as follows:

A middle strip one half panel in width, symmetrical about panel center line and extending through the panel in the direction of the span for bending.

A column strip consisting of the two adjacent quarter-panels either side of the column center lines.

(*b*) The critical sections for bending are located as follows:

Sections for negative bending shall be taken along the edges of the panel, on column center lines between capitals and around the perimeters of column capitals.

Sections for positive bending shall be taken at mid-span of the strips.

TABLE 1004(a). BENDING MOMENTS IN INTERIOR FLAT SLAB PANEL

With drop panel		
Column strip	Negative moment	$0.50M_o$
	Positive moment	$0.20M_o$
Middle strip	Negative moment	$0.15M_o$
	Positive moment	$0.15M_o$
Without drop panel		
Column strip	Negative moment	$0.46M_o$
	Positive moment	$0.22M_o$
Middle strip	Negative moment	$0.16M_o$
	Positive moment	$0.16M_o$

TABLE 1004(b). BENDING MOMENTS IN EXTERIOR FLAT SLAB PANEL

With drop panel		
Column strip	Exterior negative	$0.45M_o$
	Positive moment	$0.25M_o$
	Interior negative	$0.55M_o$
Middle strip	Exterior negative	$0.10M_o$
	Positive moment	$0.19M_o$
	Interior negative	$0.165M_o$
Without drop panel		
Column strip	Exterior negative	$0.41M_o$
	Positive moment	$0.28M_o$
	Interior negative	$0.50M_o$
Middle strip	Exterior negative	$0.10M_o$
	Positive moment	$0.20M_o$
	Interior negative	$0.176M_o$

TABLE 1004(c). BENDING MOMENTS IN PANELS WITH MARGINAL BEAMS OR WALLS

		Marginal Beams with Depth Greater than $1\frac{1}{2}$ Times the Slab Thickness; or Bearing Wall		Marginal Beams with Depth $1\frac{1}{2}$ Times the Slab Thickness or Less	
(a) Load to be carried by marginal beam or wall		Loads directly superimposed upon it plus a uniform load equal to one-quarter of the total live and dead panel load		Loads directly superimposed upon it exclusive of any panel load	
		With Drop	Without Drop	With Drop	Without Drop
(b) Moment to be used in the design of half column strip adjacent and parallel to marginal beam or wall	Neg.	$0.125M_o$	$0.115M_o$	$0.25M_o$	$0.23M_o$
	Pos.	$0.05M_o$	$0.055M_o$	$0.10M_o$	$0.11M_o$
(c) Negative moment to be used in design of middle strip continuous across a beam or wall	Neg.	$0.195M_o$	$0.208M_o$	$0.15M_o$	$0.16M_o$

(c) Only the reinforcement which crosses a critical section within a strip may be considered effective to resist bending in the strip at that section. Reinforcement which crosses such section at an angle with the center-line of the strip shall be assumed to contribute to the resistance of bending only its effective area in the direction of the strip, as defined in Chapter 1.

1006. Slab Thickness and Drop Panels. (a) The thickness of a flat slab and the size and thickness of the drop panel, where used, shall be such that the compressive stress due to bending at the critical sections of any strip and the shear about the column capital and the drop panel shall not exceed the unit stresses allowed in concrete of the quality used.

(b) The shearing stresses in the slab outside the capital or drop panel shall be computed as provided in Section 807.

(c) Slab thickness shall not, however, be less than

$$\frac{L}{40} \text{ with drop panels}$$

or

$$\frac{L}{36} \text{ without drop panels}$$

(d) The thickness of the drop panel below the slab shall not be more than one-fourth the distance from the edge of the column capital to the edge of the drop panel.

1007. Capitals and Brackets. (a) Where a column is without a flaring concrete capital the distance c shall be taken as the diameter of the column. Structural metal embedded in the slab or drop panel may be regarded as contributing to resistance in bending and shear.

(b) Where a reinforced concrete beam frames into a column without capital or bracket on the same side with the beam, the value of c may be taken as the width of the column plus twice the projection of the beam above or below the slab or drop panel for computing bending in strips parallel to the beam.

(c) Brackets capable of transmitting the negative bending and the shear in the column strips to the columns without excessive unit stress may be substituted for column capitals at exterior columns. The value of c where brackets are used shall be taken as twice the distance from the center of the column to a point where the bracket is $1\frac{1}{2}$ inches thick, but not more than the thickness of the column plus twice the depth of the bracket.

(d) The average of the diameters c of the column capitals at the four corners of a panel shall be used in determining the bending in the middle strips of the panel. The average of the diameters c of the two column capitals at the ends of a column strip shall be used in determining bending in the strip.

1008. Arrangement of Reinforcement. (a) Slab reinforcement shall be provided to resist the bending and bond stresses not only at critical sections, but also at intermediate sections.

(b) Bars shall be spaced evenly across strips or bands and the spacing shall not exceed three times the slab thickness.

(c) In exterior panels the reinforcement perpendicular to the discontinuous edge for positive bending, shall extend to the edge and have embedment of at least six inches in spandrel beams, walls or columns. All such reinforcement for negative bending shall be bent, hooked or otherwise anchored in spandrel beams, walls or columns.

1009. Openings in Flat Slabs. Openings of any size may be cut through a flat slab if provision is made for the total positive and negative resisting moments, as required in Section 1002 or 1003, without exceeding the allowable stresses as given in Sections 305 and 306.

REINFORCED CONCRETE COLUMNS AND WALLS

1100. Notation.

A_c = Area of core of a spirally reinforced column measured to the outside diameter of the spiral; net area of concrete section of a composite column.
A_g = The overall or gross area of spirally reinforced or tied columns; the total area of the concrete encasement of combination columns.
A_r = Area of the steel or cast-iron core of a composite column; the area of the steel core in a combination column.
A_s = Effective cross-sectional area of reinforcement in compression in columns.
C = Ratio of allowable concrete stress, f_a, in axially loaded column to allowable fiber stress for concrete in flexure.
$D = \frac{t^2}{2R^2}$ = a factor, usually varying from 3 to 9. (The term R as used here is the radius of gyration of the entire column section.)
d = The least lateral dimension of a concrete column.
e = Eccentricity of the resultant load on a column, measured from the gravity axis.
$F = \frac{\textit{Yield point of pipe}}{45{,}000}$ [See Section 1106(b).]
f_a = Average allowable stress in the concrete of an axially loaded reinforced concrete column.
f_c = Computed concrete fiber stress in an eccentrically loaded column.
f'_c = Ultimate compressive strength of concrete at age of 28 days, unless otherwise specified.
f_p = Maximum allowable concrete fiber stress in an eccentrically loaded column.
f_r = Allowable unit stress in the metal core of a composite column.
f'_r = Allowable unit stress on unencased steel columns and pipe columns.
f_s = Nominal working stress in vertical column reinforcement.
f'_s = Useful limit stress of spiral reinforcement.
h = Unsupported length of column.
K = Least radius of gyration of a metal pipe section (in pipe columns).
$n = \frac{30{,}000}{f'_c}$
N = Axial load applied to reinforced concrete column.
p' = Ratio of volume of spiral reinforcement to the volume of the concrete core (out to out of spirals) of a spirally reinforced concrete column.
p_g = Ratio of the effective cross-sectional area of vertical reinforcement to the gross area A_g.
P = Total allowable axial load on a column whose length does not exceed ten times its least cross-sectional dimension.
P' = Total allowable axial load on a long column.
R = Least radius of gyration of a section.
t = Overall depth of column section.

1101. Limiting Dimensions. (*a*) The following sections on reinforced concrete and composite columns, except Section 1107(*a*), apply to a short column for which the unsupported length is not greater than ten times the least dimension. When the unsupported length exceeds this value, the design shall be modified as shown in Section 1107(*a*). Principal columns in buildings shall have a minimum diameter of twelve inches, or in the case of rectangular columns, a minimum thickness of ten inches, and a minimum gross area of 120 square inches. Posts that are not continuous from story to story shall have a minimum diameter or thickness of six inches.

1102. Unsupported Length of Columns. (*a*) For purposes of determining the limiting dimensions of columns, the unsupported length of reinforced concrete columns shall be taken as the clear distance between floor slabs, except that

1. In flat slab construction, it shall be the clear distance between the floor and the lower extremity of the capital.
2. In beam and slab construction, it shall be the clear distance between the floor and the under side of the deeper beam framing into the column in each direction at the next higher floor level.
3. In columns restrained laterally by struts, it shall be the clear distance between consecutive struts in each vertical plane; provided that to be an adequate support, two such struts shall meet the column at approximately the same level, and the angle between vertical planes through the struts shall not vary more than 15 degrees from a right angle. Such struts shall be of adequate dimensions and anchorage to restrain the column against lateral deflection.
4. In columns restrained laterally by struts or beams, with brackets used at the junction, it shall be the clear distance between the floor and the lower edge of the bracket, provided that the bracket width equals that of the beam or strut and is at least half that of the column.

(*b*) For rectangular columns, that length shall be considered which produces the greatest ratio of length to depth of section.

1103. Spirally Reinforced Columns. (*a*) *Allowable Load.* The maximum allowable axial load, P, on columns with closely spaced spirals enclosing a circular concrete core reinforced with longitudinal bars shall be that given by formula (20).

$$P = A_g\ (0.225f'_c + f_s p_g) \tag{20}$$

Wherein A_g = the gross area of the column

f'_c = compressive strength of the concrete

f_s = nominal working stress in vertical column reinforcement, to be taken at forty per cent of the minimum specification value of the yield point; viz., 16,000 p.s.i. for intermediate grade steel and 20,000 p.s.i. for rail or hard grade steel.[5]

p_g = ratio of the effective cross-sectional area of vertical reinforcement to the gross area, A_g.

(*b*) *Vertical Reinforcement.* The ratio p_g shall not be less than 0.01 nor more than 0.08. The minimum number of bars shall be six, and the minimum diameter shall

[5] Nominal working stresses for reinforcement of higher yield point may be established at forty per cent of the yield point stress, but not more than 30,000 p.s.i., when the properties of such reinforcing steels have been definitely specified by standards of A.S.T.M. designation. If this is done, the lengths of splice required by Section 1103(*c*) shall be increased accordingly.

be $\frac{5}{8}$ inch. The center to center spacing of bars within the periphery of the column core shall not be less than $2\frac{1}{2}$ times the diameter for round bars or three times the side dimension for square bars. The clear spacing between bars shall not be less than $1\frac{1}{2}$ inches or $1\frac{1}{2}$ times the maximum size of the coarse aggregate used. These spacing rules also apply to adjacent pairs of bars at a lapped splice; each pair of lapped bars forming a splice may be in contact, but the minimum clear spacing between one splice and the adjacent splice should be that specified for adjacent single bars.

(*c*) *Splices in Vertical Reinforcement.* Where lapped splices in the column verticals are used, the minimum amount of lap shall be as follows:

1. For deformed bars—with concrete having a strength of 3000 p.s.i. or above, twenty-four diameters of bar of intermediate grade steel and thirty diameters of bar of hard grade steel. For bars of higher yield point, the amount of lap shall be increased in proportion to the nominal working stress. When the concrete strengths are less than 3000 p.s.i., the amount of lap shall be one-third greater than the values given above.

2. For plain bars—the minimum amount of lap shall be twenty-five per cent greater than that specified for deformed bars.

3. Welded splices or other positive connections may be used instead of lapped splices. Welded splices shall preferably be used in cases where the bar diameter exceeds $1\frac{1}{4}$ inch. An approved welded splice shall be defined as one in which the bars are butted and welded and that will develop in tension at least the yield point stress of the reinforcing steel used.

4. Where changes in the cross section of a column occur, the longitudinal bars shall be offset in a region where lateral support is afforded by a concrete capital, floor slab or by metal ties or reinforcing spirals. Where bars are offset, the slope of the inclined portion from the axis of the column shall not exceed 1 in 6 and the bars above and below the offset shall be parallel to the axis of the column.

(*d*) *Spiral Reinforcement.* The ratio of spiral reinforcement, p', shall not be less than the value given by formula (21).

$$p' = 0.45\left(\frac{A_g}{A_c} - 1\right)\frac{f'_c}{f'_s} \tag{21}$$

Wherein p' = ratio of volume of spiral reinforcement to the volume of the concrete core (out to out of spirals).

f'_s = useful limit stress of spiral reinforcement, to be taken as 40,000 p.s.i. for hot rolled rods of intermediate grade, 50,000 p.s.i. for rods of hard grade, and 60,000 p.s.i. for cold drawn wire.

The spiral reinforcement shall consist of evenly spaced continuous spirals held firmly in place and true to line by at least three vertical spacer bars. The spirals shall be of such size and so assembled as to permit handling and placing without being distorted from the designed dimensions. The material used in spirals shall have a minimum diameter of $\frac{1}{4}$ inch for rolled bars or No. 4 W. & M. gage for drawn wire. Anchorage of spiral reinforcement shall be provided by $1\frac{1}{2}$ extra turns of spiral rod or wire at each end of the spiral unit. Splices, when necessary shall be made in spiral rod or wire by welding or by a lap of $1\frac{1}{2}$ turns. The center to center spacing of the spirals shall not exceed one-sixth of the core diameter. The clear spacing between spirals shall not exceed 3 inches nor be less than $1\frac{3}{8}$ inches or $1\frac{1}{2}$ times the maximum size of coarse aggregate used. The reinforcing spiral shall extend from the floor level in any story or from the top of the footing in the basement, to the level of the lowest horizontal reinforcement in the slab, drop panel or beam above. In a column with a capital, it shall extend to a plane at which the diameter or width of the capital is twice that of the column.

(*e*) *Protection of Reinforcement.* The column reinforcement shall be protected everywhere by a covering of concrete cast monolithically with the core, for which the thickness shall not be less than $1\frac{1}{2}$ inches nor less than $1\frac{1}{2}$ times the maximum size of the coarse aggregate, nor shall it be less than required by the fire protection and weathering provisions of Section 507.

(*f*) *Isolated Column with Multiple Spirals.* In case two or more interlocking spirals are used in a column, the outer boundary of the column shall be taken as a rectangle of which the sides are outside the extreme limits of the spiral at a distance equal to the requirements of Section 1103(*e*).

(*g*) *Limits of Section of Column Built Monolithically with Wall.* For a spiral column built monolithically with a concrete wall or pier, the outer boundary of the column section shall be taken either as a circle at least $1\frac{1}{2}$ inches outside the column spiral or as a square or rectangle of which the sides are at least $1\frac{1}{2}$ inches outside the spiral or spirals.

(*h*) *Equivalent Circular Columns.* As an exception to the general procedure of utilizing the full gross area of the column section, it shall be permissible to design a circular column and to build it with a square, octagonal, or other shaped section of the same least lateral dimension. In such case, the allowable load, the gross area considered, and the required percentages of reinforcement shall be taken as those of the circular column.

1104. Tied Columns. (*a*) *Allowable Load.* The maximum allowable axial load on columns reinforced with longitudinal bars and separate lateral ties shall be 80 per cent of that given by formula (20). The ratio, p_g, to be considered in tied columns shall not be less than 0.01 nor more than 0.04. The longitudinal reinforcement shall consist of at least four bars, of minimum diameter of $\frac{5}{8}$ inch. Splices in reinforcing bars shall be made as described in Section 1103(*c*).

(*b*) *Lateral Ties.* Lateral ties shall be at least $\frac{1}{4}$ inch in diameter and shall be spaced apart not over 16 bar diameters, 48 tie diameters or the least dimension of the column. When there are more than four vertical bars, additional ties shall be provided so that every longitudinal bar is held firmly in its designed position and has lateral support equivalent to that provided by a 90-degree corner of a tie.

(*c*) *Limits of Column Section.* In a tied column which for architectural reasons has a larger cross section than required by considerations of loading, a reduced effective area, A_g, not less than one-half of the total area may be used in applying the provisions of Section 1104(*a*).

1105. Composite Columns. (*a*) *Allowable Load.* The allowable load on a composite column, consisting of a structural steel or cast-iron column thoroughly encased in concrete reinforced with both longitudinal and spiral reinforcement, shall not exceed that given by formula (22).

$$P = 0.225A_cf'_c + f_sA_s + f_rA_r \tag{22}$$

Wherein A_c = net area of concrete section
$= A_g - A_s - A_r$
A_s = cross-sectional area of longitudinal bar reinforcement.
A_r = cross-sectional area of the steel or cast-iron core.
f_r = allowable unit stress in metal core, not to exceed 16,000 p.s.i. for a steel core; or 10,000 p.s.i. for a cast-iron core.

The remaining notation is that of Section 1103.

(*b*) *Details of Metal Core and Reinforcement.* The cross-sectional area of the metal core shall not exceed 20 per cent of the gross area of the column. If a hollow metal core is used it shall be filled with concrete. The amounts of longitudinal and spiral reinforcement and the requirements as to spacing of bars, details of splices and thickness of protective shell outside the spiral shall conform to the limiting values specified in Section 1103(*b*), (*c*) and (*d*). A clearance of at least three inches shall be maintained between the spiral and the metal core at all points except that when the core consists of a structural steel H-column, the minimum clearance may be reduced to two inches.

(*c*) *Splices and Connections of Metal Cores.* Metal cores in composite columns shall be accurately milled at splices and positive provision shall be made for alignment of one core above another. At the column base, provision shall be made to transfer the load to the footing at safe unit stresses in accordance with Section 305(*a*). The base of the metal section shall be designed to transfer the load from the entire composite column to the footing, or it may be designed to transfer the load from the metal section only, provided it is so placed in the pier or pedestal as to leave ample section of concrete above the base for the transfer of load from the reinforced concrete section of the column by means of bond on the vertical reinforcement and by direct compression on the concrete. Transfer of loads to the metal core shall be provided for by the use of bearing members such as billets, brackets or other positive connections; these shall be provided at the top of the metal core and at intermediate floor levels where required. The column as a whole shall satisfy the requirements of formula (22) at any point; in addition to this, the reinforced concrete portion shall be designed to carry, in accordance with formula (20), all floor loads brought onto the column at levels between the metal brackets or connections. In applying formula (20), the value of A_g shall be interpreted as the area of the concrete section outside the metal core, and the allowable load on the reinforced concrete section shall be further limited to $0.35f'_cA_g$. Ample section of concrete and continuity of reinforcement shall be provided at the junction with beams or girders.

(*d*) *Allowable Load on Metal Core Only.* The metal cores of composite columns shall be designed to carry safely any construction or other loads to be placed upon them prior to their encasement in concrete.

1106. Combination Columns. (*a*) *Steel Columns Encased in Concrete.* The allowable load on a structural steel column which is encased in concrete at least $2\frac{1}{2}$ inches thick over all metal (except rivet heads) reinforced as hereinafter specified, shall be computed by formula (23).

$$P = A_rf'_r\left[1 + \frac{A_g}{100A_r}\right] \tag{23}$$

Wherein A_r = cross-sectional area of steel column.
f'_r = allowable stress for unencased steel column.
A_g = total area of concrete section.

The concrete used shall develop a compressive strength, f'_c, of at least 2000 p.s.i. at 28 days. The concrete shall be reinforced by the equivalent of welded wire mesh having wires of No. 10 W. and M. gage, the wires encircling the column being spaced not more than four inches apart and those parallel to the column axis not more than eight inches apart. This mesh shall extend entirely around the column at a distance of one inch inside the outer concrete surface and shall be lap-spliced at least forty wire diameters and wired at the splice. Special brackets shall be used to receive the entire floor load at each floor level. The steel column shall be designed to carry safely any construction or other loads to be placed upon it prior to its encasement in concrete.

(*b*) *Pipe Columns.* The allowable load on columns consisting of steel pipe filled with concrete shall be determined by formula (24).

$$P = 0.225f'_cA_c + f'_rA_r \tag{24}$$

The value of f'_r shall be given by formula (25).

$$f'_r = \left[18{,}000 - 70\frac{h}{K}\right]F \tag{25}$$

Wherein f'_r = allowable unit stress in metal pipe.
h = unsupported length of column.
K = least radius of gyration of metal pipe section.
$F = \dfrac{\textit{yield point of pipe}}{45{,}000}$.

If the yield point of the pipe is not known, the factor F shall be taken as 0.5.

1107. Long Columns. (*a*) The maximum allowable load, P', on axially loaded reinforced concrete or composite columns having a length, h, greater than ten times the least lateral dimension, d, shall be given by formula (26).

$$P' = P\left[1.3 - .03\frac{h}{d}\right] \tag{26}$$

where P is the allowable axial load on a short column as given by formulas (20) and (22).

The maximum allowable load, P', on eccentrically loaded columns in which $\frac{h}{d}$ exceeds ten shall also be given by formula (26), in which P is the allowable eccentrically applied load on a short column as determined by the provisions of Sections 1109 and 1110. In long columns subjected to definite bending stresses, as determined in Section 1108, the ratio $\frac{h}{d}$ shall not exceed twenty.

1108. Bending Moment in Columns. (*a*) The bending moments in the columns of all reinforced concrete structures shall be determined on the basis of loading conditions and restraint and shall be provided for in the design. When the stiffness and strength of the columns are utilized to reduce moments in beams, girders, or slabs, as in the case of rigid frames, or in other forms of continuous construction wherein column moments are unavoidable, they shall be provided for in the design. In building frames, particular attention shall be given to the effect of unbalanced floor loads on both exterior and interior columns and of eccentric loading due to other causes. Wall columns shall be designed to resist moments produced by

1. Loads on all floors of the building
2. Loads on a single exterior bay at two adjacent floor levels, or
3. Loads on a single exterior bay at one floor level

Resistance to bending moments at any floor level shall be provided by distributing the moment between the columns immediately above and below the given floor in proportion to their relative stiffnesses and conditions of restraint.

1109. Determination of Combined Axial and Bending Stresses. (*a*) In a reinforced concrete column, designed by the methods of this chapter, which is (1) symmetrical about two perpendicular planes through its axis and (2) subject to an axial load, N, combined with bending in one or both of the planes of symmetry (but with the ratio of eccentricity to depth, e/t, no greater than 1.0 in either plane), the combined fiber stress in compression may be computed on the basis of recognized theory applying to uncracked sections, using formula (27).

$$f_c = \frac{N}{A_g}\left[\frac{1 + \frac{De}{t}}{1 + (n - 1)p_g}\right] \tag{27}$$

Equating this calculated stress, f_c, to the allowable stress, f_p, in formula (29), it follows that the column can be designed for an equivalent axial load, P, as given by formula (28).[6]

$$P = N\left[1 + \frac{CDe}{t}\right] \tag{28}$$

When bending exists on both axes of symmetry, the quantity $\frac{De}{t}$ is to be computed as the numerical sum of the $\frac{De}{t}$ quantities in the two directions.

(*b*) For columns in which the load, N, has an eccentricity, e, greater than the column depth, t, or for beams subject to small axial loads, the determination of the fiber stress f_c shall be made by use of recognized theory for cracked sections, based on the assumption that no tension exists in the concrete. For such cases the tensile steel stress shall also be investigated.

1110. Allowable Combined Axial and Bending Stress. (*a*) For spiral and tied columns, eccentrically loaded or otherwise subjected to combined axial compression and flexural stress, the maximum allowable compressive stress, f_p, is given by formula (29).

$$f_p = f_a\left[\frac{1 + \frac{De}{t}}{1 + \frac{CDe}{t}}\right] = f_a\left[\frac{t + De}{t + CDe}\right] \tag{29}$$

Wherein the notation is that of Sections 1103 and 1109, and, in addition f_a is the average allowable stress in the concrete of an axially loaded reinforced concrete column, and C is the ratio of f_a to the allowable fiber stress for members in flexure. Thus $f_a = \frac{0.225f'_c + f_s p_g}{1 + (n - 1)p}$ for spiral columns and 0.8 of this value for tied columns, and $C = \frac{f_a}{0.45f'_c}$.

1111. Wind Stresses. (*a*) When the allowable stress in columns is modified to provide for combined axial load and bending, and the stress due to wind loads is

[6] For approximate or trial computations, D may be taken as eight for a circular spiral column and five for a rectangular tied or spiral column.

also added, the total shall still come within the allowable values specified for wind loads in Section 603(c).

1112. Reinforced Concrete Walls. (a) The allowable working stresses in reinforced concrete bearing walls with minimum reinforcement as required by Section 1112(i), shall be $0.25f'_c$ for walls having a ratio of height to thickness of ten or less, and shall be reduced proportionally to $0.15f'_c$ for walls having a ratio of height to thickness of twenty-five. When the reinforcement in bearing walls is designed, placed and anchored in position as for tied columns, the allowable working stresses shall be on the basis of Section 1104, as for columns. In the case of concentrated loads, the length of the wall to be considered as effective for each shall not exceed the center to center distance between loads, nor shall it exceed the width of the bearing plus four times the wall thickness. The ratio p_g shall not exceed 0.04.

(b) Walls shall be designed for any lateral or other pressure to which they are subjected. Proper provision shall be made for eccentric loads and wind stresses. In such designs the allowable stresses shall be as given in Sections 305(a) and 603(c).

(c) Panel and enclosure walls of reinforced concrete shall have a thickness of not less than five inches and not less than one thirtieth the distance between the supporting or enclosing members.

(d) Bearing walls of reinforced concrete in building of fire-resistive construction shall be not less than six inches in thickness for the uppermost fifteen feet of their height; and for each successive twenty-five feet downward, or fraction thereof, the minimum thickness shall be increased one inch. In two story dwellings the walls may be six inches in thickness throughout.

(e) In buildings of non-fire resistive construction bearing walls of reinforced concrete shall not be less than one and one-third times the thickness required for buildings of fire-resistive construction, except that for dwellings of two stories or less in height the thickness of walls may be the same as specified for buildings of fire-resistive construction.

(f) Exterior basement walls, foundation walls, fire walls and party walls shall not be less than eight inches thick whether reinforced or not.

(g) Reinforced concrete bearing walls shall have a thickness of at least one twenty-fifth of the unsupported height or width, whichever is the shorter; provided, however, that approved buttresses, built-in columns, or piers designed to carry all the vertical loads, may be used in lieu of increased thickness.

(h) Reinforced concrete walls shall be anchored to the floors, columns, pilasters, buttresses and intersecting walls with reinforcement at least equivalent to three-eighths inch round bars twelve inches on centers, for each layer of wall reinforcement.

(i) Reinforced concrete walls shall be reinforced with an area of steel in each direction, both vertical and horizontal, at least equal to 0.0025 times the cross-sectional area of the wall, if of bars, and 0.0018 times the area if of electrically welded wire fabric.[7] The wire of the welded fabric shall be of not less than No. 10 W. & M. gage. Walls more than ten inches in thickness shall have the reinforcement for each direction placed in two layers parallel with the faces of the wall. One layer consisting of not less than one-half and not more than two-thirds the total required shall be placed not less than two inches nor more than one-third the thickness of the wall from the exterior surface. The other layer, comprising the balance of the required reinforcement, shall be placed not less than three-fourths inches and not more than one-third the thickness of the wall from the interior surface. Bars, if used, shall not be less than the equivalent of three-eighths inch round bars, nor shall they be spaced

[7] Expanded metal has been omitted until a specification can be formulated.

more than eighteen inches on centers. Welded wire [8] reinforcement for walls shall be in flat sheet form.

(*j*) In addition to the minimum as prescribed in 1112(*i*) there shall be not less than two five-eighths inch diameter bars around all window or door openings. Such bars shall extend at least twenty-four inches beyond the corner of the openings.

(*k*) Where reinforced concrete bearing walls consist of studs or ribs tied together by reinforced concrete members at each floor level, the studs may be considered as columns, but the restrictions as to minimum diameter or thickness of columns shall not apply.

FOOTINGS

1201. Scope. (*a*) The requirements prescribed in Sections 1202 to 1209 apply only to isolated footings.[9]

1202. Loads and Reactions. (*a*) Footings shall be proportioned to sustain the applied loads and induced reactions without exceeding the allowable stresses as prescribed in Sections 305 and 306, and as further provided in Sections 1205, 1206 and 1207.

(*b*) In cases where the footing is concentrically loaded and the member being supported does not transmit any moment to the footing, computations for moments and shears shall be based on an upward reaction assumed to be uniformly distributed per unit area or per pile and a downward applied load assumed to be uniformly distributed over the area of the footing covered by the column, pedestal, wall, or metallic column base.

(*c*) In cases where the footing is eccentrically loaded and/or the member being supported transmits a moment to the footing, proper allowance shall be made for any variation that may exist in the intensities of reaction and applied load consistent with the magnitude of the applied load and the amount of its actual or virtual eccentricity.

(*d*) In the case of footings on piles, computations for moments and shears may be based on the assumption that the reaction from any pile is concentrated at the center of the pile.

1203. Sloped or Stepped Footings. (*a*) In sloped or stepped footings, the angle of slope or depth and location of steps shall be such that the allowable stresses are not exceeded at any section.

(*b*) In sloped or stepped footings, the effective cross-section in compression shall be limited by the area above the neutral plane.

(*c*) Sloped or stepped footings shall be cast as a unit.

1204. Bending Moment. (*a*) The external moment on any section shall be determined by passing through the section a vertical plane which extends completely across the footing, and computing the moment of the forces acting over the entire area of the footing on one side of said plane.

(*b*) The greatest bending moment to be used in the design of an isolated footing shall be the moment computed in the manner prescribed in Section 1204(*a*) at sections located as follows:

1. At the face of the column, pedestal or wall, for footings supporting a concrete column, pedestal or wall.

[8] See footnote 7.

[9] The committee is not prepared at this time to make recommendations for combined footings—those supporting more than one column or wall.

2. Halfway between the middle and the edge of the wall, for footings under masonry walls.

3. Halfway between the face of the column or pedestal and the edge of the metallic base, for footings under metallic bases.

(*c*) The width resisting compression at any section shall be assumed as the entire width of the top of the footing at the section under consideration.

(*d*) In one-way reinforced footings, the total tensile reinforcement at any section shall provide a moment of resistance at least equal to the moment computed in the manner prescribed in Section 1204(*a*); and the reinforcement thus determined shall be distributed uniformly across the full width of the section.

(*e*) In two-way reinforced footings, the total tensile reinforcement at any section shall provide a moment of resistance at least equal to eighty-five per cent of the moment computed in the manner prescribed in Section 1204(*a*); and the total reinforcement thus determined shall be distributed across the corresponding resisting section in the manner prescribed for square footings in Section 1204(*f*), and for rectangular footings in Section 1204(*g*).

(*f*) In two-way square footings, the reinforcement extending in each direction shall be distributed uniformly across the full width of the footing.

(*g*) In two-way rectangular footings, the reinforcement in the long direction shall be distributed uniformly across the full width of the footing. In the case of the reinforcement in the short direction, that portion determined by formula (30) shall be uniformly distributed across a band-width (*B*) centered with respect to the center line of the column or pedestal and having a width equal to the length of the short side of the footing. The remainder of the reinforcement shall be uniformly distributed in the outer portions of the footing.

$$\frac{\textit{Reinforcement in band-width } (B)}{\textit{Total reinforcement in short direction}} = \frac{2}{(S+1)} \qquad (30)$$

In formula (30), S is the ratio of the long side to the short side of the footing.

1205. Shear and Bond. (*a*) The critical section for shear to be used as a measure of diagonal tension shall be assumed as a vertical section obtained by passing a series of vertical planes through the footing, each of which is parallel to a corresponding face of the column, pedestal, or wall and located a distance therefrom equal to the depth d for footings on soil, and one-half the depth d for footings on piles.

(*b*) Each face of the critical section as defined in Section 1205(*a*) shall be considered as resisting an external shear equal to the load on an area bounded by said face of the critical section for shear, two diagonal lines drawn from the column or pedestal corners and making 45-degree angles with the principal axes of the footing, and that portion of the corresponding edge or edges of the footing intercepted between the two diagonals.

(*c*) Critical sections for bond shall be assumed at the same planes as those prescribed for bending moment in Section 1204(*b*); also at all other vertical planes where changes of section or of reinforcement occur.

(*d*) Computations for shear to be used as a measure of bond shall be based on the same section and loading as prescribed for bending moment in Section 1204(*a*).

(*e*) The total tensile reinforcement at any section shall provide a bond resistance at least equal to the bond requirement as computed from the following percentages of the external shear at the section:

1. In one-way reinforced footings, 100 per cent.
2. In two-way reinforced footings, 85 per cent.

(*f*) In computing the external shear on any section through a footing supported on piles, the entire reaction from any pile whose center is located six inches or more outside the section shall be assumed as producing shear on the section; the reaction from any pile whose center is located six inches or more inside the section shall be assumed as producing no shear on the section. For intermediate positions of the pile center, the portion of the pile reaction to be assumed as producing shear on the section shall be based on straight-line interpolation between full value at six inches outside the section and zero value at six inches inside the section.

(*g*) For allowable shearing stresses, see Sections 305 and 808.

(*h*) For allowable bond stresses, see Sections 305 and 901 to 905.

1206. Transfer of Stress at Base of Column. (*a*) The stress in the longitudinal reinforcement of a column or pedestal shall be transferred to its supporting pedestal or footing either by extending the longitudinal bars into the supporting member, or by dowels.

(*b*) In case the transfer of stress in the reinforcement is accomplished by extension of the longitudinal bars, they shall extend into the supporting member the distance required to transfer to the concrete, by allowable bond stress, their full working value.

(*c*) In cases where dowels are used, their total sectional area shall be not less than the sectional area of the longitudinal reinforcement in the member from which the stress is being transferred. In no case shall the number of dowels per member be less than four and the diameter of the dowels shall not exceed the diameter of the column bars by more than one-eighth inch.

(*d*) Dowels shall extend up into the column or pedestal a distance at least equal to that required for lap of longitudinal column bars (see Section 1103) and down into the supporting pedestal or footing the distance required to transfer to the concrete, by allowable bond stress, the full working value of the dowel.

(*e*) The compressive stress in the concrete at the base of a column or pedestal shall be considered as being transferred by bearing to the top of the supporting pedestal or footing. The unit compressive stress on the loaded area shall not exceed the bearing stress allowable for the quality of concrete in the supporting member as limited by the ratio of the loaded area to the supporting area.

(*f*) For allowable bearing stresses see Table 305(*a*), Section 305.

(*g*) In sloped or stepped footings, the supporting area for bearing may be taken as the top horizontal surface of the footing, or assumed as the area of the lower base of the largest frustum of a pyramid or cone contained wholly within the footing and having for its upper base the area actually loaded, and having side slopes of one vertical to two horizontal.

1207. Pedestals and Footings (Plain Concrete). (*a*) The allowable compressive unit stress on the gross area of a concentrically loaded pedestal shall not exceed $0.25f'_c$. Where this stress is exceeded, reinforcement shall be provided and the member designed as a reinforced concrete column.

(*b*) The depth and width of a pedestal or footing of plain concrete shall be such that the tension in the concrete shall not exceed $.03f'_c$, and the average shearing stress shall not exceed $.02f'_c$ taken on sections as prescribed in Sections 1204 and 1205 for reinforced concrete footings.

1208. Footings Supporting Round Columns. (*a*) In computing the stresses in footings which support a round or octagonal concrete column or pedestal, the "face" of the column or pedestal shall be taken as the side of a square having an area equal to the area enclosed within the perimeter of the column or pedestal.

1209. Minimum Edge-Thickness. (*a*) In reinforced concrete footings, the thickness above the reinforcement at the edge shall be not less than six inches for footings on soil, nor less than twelve inches for footings on piles.

(*b*) In plain concrete footings, the thickness at the edge shall be not less than eight inches for footings on soil, nor less than fourteen inches above the tops of the piles for footings on piles.

FIRE PROTECTION

Excerpt from 1928 (A.C.I.) Joint Standard Building Code

506. (*b*) In fire-resistive construction, metal reinforcement shall be protected by not less than 1 in. of concrete in slabs and walls, and not less than $1\frac{1}{2}$ in. in beams, girders, and columns, provided coarse aggregate is used, which is free from disruptive action under high temperatures, as, for example, limestone or trap rock; when impracticable to obtain aggregate of this grade, the protective covering shall be $\frac{1}{2}$ in. thicker and shall be reinforced with metal mesh having openings not exceeding 3 in. placed 1 in. from the finished surface. In similar structures where the fire hazard is limited, the metal reinforcement shall not be placed nearer the exposed surface than $\frac{3}{4}$ in. in slabs and walls, or 1 in. in beams, girders, and columns.

FLAT SLAB

Excerpt from 1928 (A.C.I.) Joint Standard Building Code

Placing Steel. *Lengths and Bends to Satisfy Bond and Anchorage, if Panels Are Approximately the Same Size.*

1007. Point of Inflection. (*a*) In the middle strip the point of inflection for slabs without dropped panels shall be assumed at a line $0.33l$ distant from the center of the span and for slabs with dropped panels $0.3l$ distant from the center of the span.

(*b*) In the column strip, the point of inflection for slabs without dropped panels shall be at a line $0.33(l - c)$ distant from the center of the panel and $0.3(l - c)$ for slabs with dropped panels.

1008. Arrangement of Reinforcement at Column Heads—Two- and Four-Way Systems. (*a*) In both two- and four-way systems, provision shall be made for securing the reinforcement in place so as to resist properly not only the critical moments, but also the moments at intermediate sections. The full area of steel required for negative moment at the column head shall be continued in the same plane close to the upper surface of the slab to the edge of the dropped panel, but in no case less than a distance $0.2l$ from the center line of column. Lapped splices shall not be permitted at or near regions of maximum stress except as described in Section 505.

1009. Arrangement of Reinforcement—Two-Way System. (*a*) For column strips at least four-tenths of the area of steel required at the section for positive moment in the column strip shall be of such length and so placed as to reinforce the negative moment section at the two adjacent column heads. These bars, and any other bars for negative reinforcement shall extend into the adjacent panel to a point at least $0.05l$ beyond the point of inflection. Not less than one-third of the bars used for positive reinforcement in the column strip shall extend into the dropped panel at least twenty diameters of the bar, but not less than 12 in. or in case no dropped panel is used, shall extend to within $0.125l$ of the center line of the columns or the supports. The balance of the bars for positive reinforcement in the column strip shall extend at least $0.33l$ on either side of the center line of panel.

(*b*) For the middle strip at least one-half of the bars for positive moment shall be bent up and extend over the main bands at both sides of the panel to a point at least $0.25l$ beyond the center line of columns. The location of the bends shall be such that for a distance $0.15l$ for slabs with dropped panels (or $0.125l$ for slabs without dropped panels), on each side of the center line of columns, the full reinforcement required for negative moment will be provided in the top face of the slab. The full reinforcement for positive moment in the middle strip shall extend in the bottom face of the slab to a point at least $0.3l$ on either side of the panel center line, and at least 50 per cent of it shall extend to points $0.325l$ on either side of the panel center line for slabs with dropped panels, or $0.35l$ for slabs without dropped panels.

1010. Arrangement of Reinforcement—Four-Way System. (*a*) For direct bands, all provisions governing the placing of steel in column strips in two-way systems apply as well to the direct bands in four-way systems.

(*b*) For diagonal bands, at least four-tenths of the area of steel required at the section for positive moment shall be of such length and so placed as to reinforce the negative moment section at the two diagonally opposite column heads. These bars and any other bars for negative reinforcement shall extend into the adjoining panel to points at least $0.4l$ beyond a line drawn through the column center perpendicular to the direction of the band. The straight bars for positive moment in the diagonal bands shall not be shorter than the longer straight bars in the direct bands.

(*c*) For negative moment in the middle strip, the required steel shall extend not less than $0.25l$ on either side of the column center line.

TWO-WAY SLABS

The following articles from the "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete" (Joint Committee) have been reprinted with the kind permission of the American Society of Civil Engineers. These articles cover design recommendations for two-way slabs with supports on four sides.

TWO-WAY SLABS WITH SUPPORTS ON FOUR SIDES

809. General. (*a*) These recommendations are intended to apply to slabs (solid or ribbed), isolated or continuous, supported on all four sides by walls or beams, in either case built monolithically with the slabs. The recommended coefficients, as in the case of the design provisions for flat slabs, are based partly on analysis and partly on test data.[10] The analysis indicates that for square panels the moments

[10] In general, the coefficients and methods given in these recommendations are based upon the coefficients proposed by Dr. H. M. Westergaard (Formulas for the Design of Rectangular Floor Slabs and Supporting Girders, p. 26, *Proceedings of the American Concrete Institute for 1926*). Some modifications of these coefficients have been made and the series extended to include cases not covered by Dr. Westergaard. In making these modifications and extensions full consideration has been given to the results of available test data.

may be substantially less than those determined on the basis of independent prismatic beam elements. Similar decrease is considered to hold for other than square panels but at reducing percentages until a ratio of short to long span of 0.5 is reached. For this and all lesser ratios the entire distributed load (except that in the column strips) is assumed to be carried in the short direction of the panel. Available test data indicate that these assumptions are justified.

(*b*) Available data also indicate that, when two-way slabs are cast monolithically with supporting beams, the distribution and the numerical values for bending moments in slabs with one or more discontinuous edges do not differ widely from those of interior panels. However, these data are rather limited, but the moment coefficients recommended in Table 5 for the slabs with discontinuous edges are conservative and in general agreement with accepted theoretical considerations and general practice.

(*c*) In the special case of slabs discontinuous at 4 edges (isolated panels, case 5, Table 5), the coefficients may be assumed to apply also to slabs which are built into masonry walls, provided the weight of masonry above the slab is sufficient to restrain the slab properly at the edges. The average parapet wall is probably lacking in this respect.

810. Limitations and Notations. (*a*) The recommended moment coefficients in Table 5, Sec. 811, are intended to apply to panels fully loaded with a uniformly distributed load. For values of m intermediate between those shown in Table 5, interpolated values of the moment coefficients may be used. For values of m less than 0.5 the coefficients given for this ratio should be used.

(*b*) Panels are considered as being divided into middle strips and column strips as in flat slabs. [See Sec. 834(*a*)]. For panels in which the ratio m is less than 0.5 the middle strip in the short direction of the panel should have a width equal to the difference between the long and short spans, the remaining area representing the two column strips.

(*c*) *Notation.* Span lengths of panels should be taken as the center-to-center distance between supports or as the clear span plus twice the thickness of the slabs, whichever value is the smaller.

$$S = \text{short span as defined above}$$

$$m = \text{ratio } \frac{\text{short span}}{\text{long span}}$$

$$w = \text{load per unit area}$$

(*d*) *Principal Design Sections.* The critical sections for moment calculations are referred to as principal design sections and are located as follows:

For negative moment, along the edges of the panel at the faces of the supporting beams.
For positive moment, along the center lines of the panels.

811. Bending Moment Coefficients. (*a*) *Middle Strips.* In Table 5 are given the bending moment coefficients for the middle strips for both short and long spans for varying values of the ratio m of short to long span. These coefficients, when multiplied by wS^2 give the bending moment per unit width of slab. The basis of this table is a maximum negative moment of $.033wS^2$ per unit width in the middle strip for square interior panels. The coefficients for other than square panels and for panels

TABLE 5. BENDING MOMENT COEFFICIENTS FOR RECTANGULAR PANELS SUPPORTED ON FOUR SIDES AND BUILT MONOLITHICALLY WITH SUPPORTS

(Coefficients are for moments in middle strips.)

Moments	Short Span						Long Span All Values of m
	Values of m						
	1.0	0.9	0.8	0.7	0.6	0.5 and less	
Case 1—Interior Panels							
Negative Moment at							
Continuous edge	.033	.040	.048	.055	.063	.083	.033
Discontinuous edge	...	...	...	...	...	...	...
Positive Moment at Midspan	.025	.030	.036	.041	.047	.062	.025
Case 2—One Edge Discontinuous							
Negative Moment at							
Continuous edge	.041	.048	.055	.062	.069	.085	.041
Discontinuous edge	.021	.024	.027	.031	.035	.042	.021
Positive Moment at Midspan	.031	.036	.041	.047	.052	.064	.031
Case 3—Two Edges Discontinuous							
Negative Moment at							
Continuous edge	.049	.057	.064	.071	.078	.090	.049
Discontinuous edge	.025	.028	.032	.036	.039	.045	.025
Positive Moment at Midspan	.037	.043	.048	.054	.059	.068	.037
Case 4—Three Edges Discontinuous							
Negative Moment at							
Continuous edge	.058	.066	.074	.082	.090	.098	.058
Discontinuous edge	.029	.033	.037	.041	.045	.049	.029
Positive Moment at Midspan	.044	.050	.056	.062	.068	.074	.044
Case 5—Four Edges Discontinuous							
Negative Moment at							
Continuous edge	...	...	...	...	...	...	...
Discontinuous edge	.033	.038	.043	.047	.053	.055	.033
Positive Moment at Midspan	.050	.057	.064	.072	.080	.083	.050

These coefficients, when multiplied by wS^2, give the moment per foot of width. w = load per sq. ft.; S = short span as defined in Sec. 810(c).
Note that wS^2 is the multiplier for both short and long span moments.

with one or more edges discontinuous are based on the following modifications of this basic moment for the square interior panel:

1. Bending moments in the short span increase as the ratio m decreases.
2. Bending moments in the short span increase successively with the introduction of one or more discontinuous edges, the increase being independent of the position of the discontinuous edges.
3. Bending moments for the long span for all values of m are equal to the bending moments in a square panel having sides equal to the short span.
4. Negative moments at discontinuous edges are taken as equal to $\frac{1}{2}$ of the corresponding moment at the continuous edge.
5. Positive moments at the center are taken as $\frac{3}{4}$ of the negative moment at the continuous edge.

(b) *Column Strips.* For moments in the column strips, coefficients two-thirds of those given in Table 5 for the corresponding moments in the middle strip should be used. In determining the spacing of the reinforcement for the column strip, the moment at any section may be assumed to vary from a maximum at the edge of the middle strip to a minimum at the edge of the panel, but the average should be that computed from the coefficient as given herein.

(c) *Corner Reinforcement.* Experience and theoretical considerations have shown the need for reinforcement at exterior corners to prevent cracks in diagonal directions. The effective amount of such reinforcement per foot of width should be equal to that for the positive moment in the middle strip. This is required in both the top and bottom face of the slab. By the effective amount of the steel is meant the normal area multiplied by the sine of the angle which the bar makes with the critical section. In the top of the slab the critical section is perpendicular to the diagonal; in the bottom of the slab it is parallel to the diagonal.

812. Distribution of Unequal Negative Moments at Supports. (a) In applying the moment coefficients of Table 5 to adjacent panels of varying dimensions and unequal loading, the negative moments on either side of a supporting beam may differ materially. Under these conditions some modification of the moments should be made, based on the relative rigidity of the slabs and the resistance offered by the support. For this purpose the assumption that the supporting beams offer a restraint equivalent to the average of the stiffness factors of the adjacent slabs may be used in a manner similar to that given in Sec. 808(b) for beams framing into girders. On this basis two-thirds of the unbalanced negative moment should be distributed to the two spans in proportion to their respective stiffness factors.

(b) Where conditions are such as to require modification of the support moments, as given in (a) above, the corresponding midspan moments may be obtained by the procedure ordinarily followed for continuous beams. For this purpose, the unadjusted negative moments obtained from Table 5, considered as equivalent to fixed end moments, may be multiplied by $1\frac{1}{2}$ to obtain the simple span moments. The midspan moment then would be equal to the average adjusted end moments less $1\frac{1}{2}$ times the unadjusted end moments. The coefficients in Table 5 for positive moments at midspan are sufficiently conservative to cover ordinary cases. However, where large adjustment of the support moments is required, the midspan moments should be investigated.

813. Shear in Slabs. The shearing stresses in the slab should be computed on the assumption that the load is distributed to the supporting beams in accordance with Sec. 815.

814. Minimum Slab Thickness. The slab thickness should not be less than 4 inches, nor less than the value computed by the following formula:

$$t = \left[S + \frac{S}{m} - \frac{N}{10}\right]\frac{1}{72}\sqrt[3]{\frac{2500}{f'_c}}$$

Where t = Slab thickness in inches.

S (in inches) and m as in Sec. 810(c).

N = Total length in inches of slab periphery which is continuous with adjacent slabs.

815. Loads and Bending Moments in Supporting Beams. (a) *Distribution of Load.* The loads on the supporting beams for a two-way rectangular panel may be assumed as the uniformly distributed load within the tributary areas of the panel bounded by the intersection of 45° lines from the corners with the median line of the panel parallel to the long side.

(b) *Total Load and Shear.* On the basis of the load distribution in (a) above, the total loads on the short and long span beams due to one loaded panel are given by the following formulas, respectively.

$$W_S = \frac{wS^2}{4}$$

$$W_L = \frac{wS^2}{4}\left[\frac{2 - m}{m}\right]$$

The end shears may be obtained from the above loads by the usual modifications of the reactions for any difference in end moments.

(c) *Bending Moments.* The bending moments may be obtained for the load distribution assumed by the methods of mechanics appropriate to the conditions of support, or they may be determined approximately by transforming the load on the beams to equivalent uniform load per lineal foot of beam as follows:

For the short span, $= \dfrac{wS}{3}$

For the long span, $= \dfrac{wS}{3}\left[\dfrac{3 - m^2}{2}\right]$

BUILDING CODE, N. E. BUILDING OFFICIALS CONFERENCE

Reduction of Live Loads. (Section 2310)

Tributary Floor Area or Flat 2-Way Slab Area sq. ft.	Reduction Allowed, %			
	In Office Buildings and Manufacturing Buildings	In Public Garages	In Gymnasiums, Wholesale Stores, Storage Buildings and Assembly Halls	In Other Types
100	5	. .	0	10
200	10	. .	0	15
300	15	25	0	25

The following reductions shall be permitted in all buildings except storage buildings, wholesale stores, public garages and office and manufacturing buildings for all columns, girders, trusses, walls, piers, and foundations:

Carrying one floor	Same reductions as tabulated above
Carrying two floors	25% reduction
Carrying three floors	40% reduction
Carrying four floors	50% reduction
Carrying five floors	55% reduction
Carrying six floors or more	60% reduction

For office buildings and manufacturing buildings

Carrying one floor	Same reductions as tabulated above
Carrying two floors	10% reduction
Carrying three floors	20% reduction
Carrying four floors	30% reduction
Carrying five floors	40% reduction
Carrying six floors or more	50% reduction

For warehouses, storage buildings, and wholesale stores

Carrying one floor	No reduction
Carrying two floors	5% reduction
Carrying three floors	10% reduction
Carrying four floors	15% reduction
Carrying five floors	20% reduction

For public garages

Carrying one floor	Same reductions as tabulated above
Carrying two floors or more	25% reduction

No reduction shall be allowed in the roof load . . . on any portion of any structure.

These reductions shall not be made if the member carries more than one floor and has its live load reduced according to the table above.

TABLE 1. AREA OF STEEL PER FOOT OF WIDTH

Diameter of Bar

	Round Rods							Square Rods			
Spacing in.	$\frac{1}{4}$ in.	$\frac{3}{8}$ in.	$\frac{1}{2}$ in.	$\frac{5}{8}$ in.	$\frac{3}{4}$ in.	$\frac{7}{8}$ in.	1 in.	$\frac{1}{2}$ in.	1 in.	$1\frac{1}{8}$ in.	$1\frac{1}{4}$ in.
2	0.29	0.66	1.18	1.84	2.65	3.61	4.71	1.50	6.00	7.59	9.37
$2\frac{1}{2}$	0.23	0.53	0.94	1.47	2.12	2.89	3.77	1.20	4.80	6.08	7.50
3	0.20	0.44	0.79	1.23	1.77	2.41	3.14	1.00	4.00	5.06	6.24
$3\frac{1}{2}$	0.17	0.38	0.67	1.05	1.51	2.06	2.69	0.86	3.43	4.34	5.36
4	0.15	0.33	0.59	0.92	1.33	1.80	2.36	0.75	3.00	3.80	4.68
$4\frac{1}{2}$	0.13	0.29	0.52	0.82	1.18	1.60	2.09	0.67	2.67	3.37	4.16
5	0.12	0.26	0.47	0.74	1.06	1.44	1.88	0.60	2.40	3.04	3.75
$5\frac{1}{2}$	0.11	0.24	0.43	0.67	0.96	1.31	1.71	0.55	2.18	2.76	3.41
6	0.10	0.22	0.39	0.61	0.88	1.20	1.57	0.50	2.00	2.53	3.12
$6\frac{1}{2}$	0.09	0.20	0.36	0.57	0.82	1.11	1.45	0.46	1.85	2.33	2.88
7	0.08	0.19	0.34	0.53	0.76	1.03	1.35	0.43	1.72	2.17	2.68
$7\frac{1}{2}$	0.08	0.18	0.31	0.49	0.71	0.96	1.26	0.40	1.60	2.02	2.50
8	0.07	0.17	0.29	0.46	0.66	0.90	1.18	0.38	1.50	1.90	2.34
$8\frac{1}{2}$	0.07	0.16	0.28	0.43	0.62	0.85	1.11	0.35	1.41	1.78	2.20
9	0.07	0.15	0.26	0.41	0.59	0.80	1.05	0.33	1.33	1.69	2.08
$9\frac{1}{2}$	0.06	0.14	0.25	0.39	0.56	0.76	0.99	0.32	1.26	1.60	1.97
10	0.06	0.13	0.24	0.37	0.53	0.72	0.94	0.30	1.20	1.52	1.87
11	0.05	0.12	0.21	0.33	0.48	0.66	0.86	0.27	1.09	1.38	1.70
12	0.05	0.11	0.19	0.31	0.44	0.60	0.78	0.25	1.00	1.27	1.56

Cross-Sectional Area of Bars

Number of Bars

	1	2	3	4	5	6	7	8
Round in.								
$\frac{1}{4}$	0.05	0.10	0.15	0.20	0.25	0.29	0.34	0.39
$\frac{3}{8}$	0.11	0.22	0.33	0.44	0.55	0.66	0.77	0.88
$\frac{1}{2}$	0.20	0.39	0.59	0.78	0.98	1.18	1.37	1.57
$\frac{5}{8}$	0.31	0.61	0.92	1.23	1.53	1.84	2.15	2.45
$\frac{3}{4}$	0.44	0.88	1.33	1.77	2.21	2.65	3.09	3.54
$\frac{7}{8}$	0.60	1.20	1.80	2.41	3.01	3.61	4.21	4.81
1	0.78	1.57	2.36	3.14	3.93	4.71	5.50	6.28
Square in.								
$\frac{1}{2}$	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00
$1\frac{1}{8}$	1.27	2.53	3.80	5.06	6.33	7.59	8.86	10.13
$1\frac{1}{4}$	1.56	3.12	4.69	6.25	7.81	9.38	10.95	12.50

TABLE 1. *Continued*

Perimeter of Bars

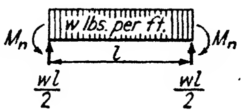
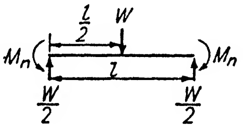
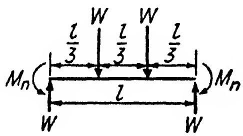
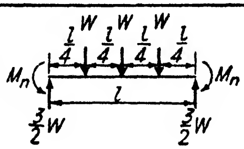
Number of Bars

	1	2	3	4	5	6	7	8
Round in.								
$\frac{1}{4}$	0.79	1.57	2.36	3.14	3.93	4.71	5.50	6.28
$\frac{3}{8}$	1.18	2.36	3.53	4.71	5.89	7.07	8.25	9.42
$\frac{1}{2}$	1.57	3.14	4.72	6.28	7.85	9.42	11.00	12.57
$\frac{5}{8}$	1.96	3.93	5.89	7.85	9.82	11.78	13.74	15.71
$\frac{3}{4}$	2.36	4.71	7.06	9.42	11.78	14.13	16.49	18.85
$\frac{7}{8}$	2.75	5.50	8.25	11.00	13.74	16.49	19.24	21.99
1	3.14	6.28	9.42	12.57	15.71	18.86	22.00	25.13
Square in.								
$\frac{1}{2}$	2	4	6	8	10	12	14	16
1	4	8	12	16	20	24	28	32
$1\frac{1}{8}$	4.5	9	13.5	18	22.5	27	31.5	36
$1\frac{1}{4}$	5	10	15	20	25	30	35	40

Weight of Bars

Round in.								
$\frac{1}{4}$	0.17	0.33	0.50	0.67	0.83	1.00	1.17	1.34
$\frac{3}{8}$	0.38	0.75	1.13	1.50	1.88	2.26	2.63	3.01
$\frac{1}{2}$	0.67	1.34	2.01	2.67	3.34	4.01	4.68	5.35
$\frac{5}{8}$	1.04	2.09	3.13	4.17	5.22	6.26	7.30	8.34
$\frac{3}{4}$	1.50	3.00	4.51	6.01	7.51	9.01	10.51	12.02
$\frac{7}{8}$	2.04	4.09	6.13	8.18	10.22	12.26	14.31	16.35
1	2.67	5.34	8.01	10.68	13.35	16.02	18.69	21.36
Square in.								
$\frac{1}{2}$	0.85	1.70	2.55	3.40	4.25	5.10	5.95	6.80
1	3.40	6.80	10.20	13.60	17.00	20.40	23.80	27.20
$1\frac{1}{8}$	4.30	8.61	12.91	17.21	21.52	25.82	30.12	34.42
$1\frac{1}{4}$	5.31	10.62	15.94	21.25	26.56	31.87	37.18	42.50

Table 2. Deflection Coefficients for Beams

Loading	M_n	C_1
	0	$\frac{5}{384} = 0.0130$
	$\frac{wl^2}{24}$	$\frac{3}{384} = 0.0078$
	$\frac{3}{56}wl^2$	$\frac{17}{2688} = 0.0063$
	$\frac{wl^2}{16}$	$\frac{1}{192} = 0.0052$
	$\frac{wl^2}{12}$	$\frac{1}{384} = 0.0026$
	0	$\frac{1}{48} = 0.0208$
	$\frac{Wl}{12}$	$\frac{1}{96} = 0.0104$
	$\frac{3}{28}Wl$	$\frac{5}{672} = 0.0074$
	$\frac{5}{36}Wl$	$\frac{1}{288} = 0.0035$
	0	$\frac{23}{648} = 0.0355$
	$\frac{Wl}{12}$	$\frac{65}{2592} = 0.0251$
	$\frac{Wl}{9}$	$\frac{7}{324} = 0.0216$
	$\frac{7}{48}Wl$	$\frac{179}{10{,}368} = 0.0173$
	$\frac{5}{33}Wl$	$\frac{118}{7128} = 0.0165$
	0	$\frac{19}{384} = 0.0495$
	$\frac{Wl}{7}$	$\frac{85}{2688} = 0.0316$
	$\frac{3}{16}Wl$	$\frac{5}{192} = 0.0260$
	$\frac{7}{30}Wl$	$\frac{13}{640} = 0.0203$

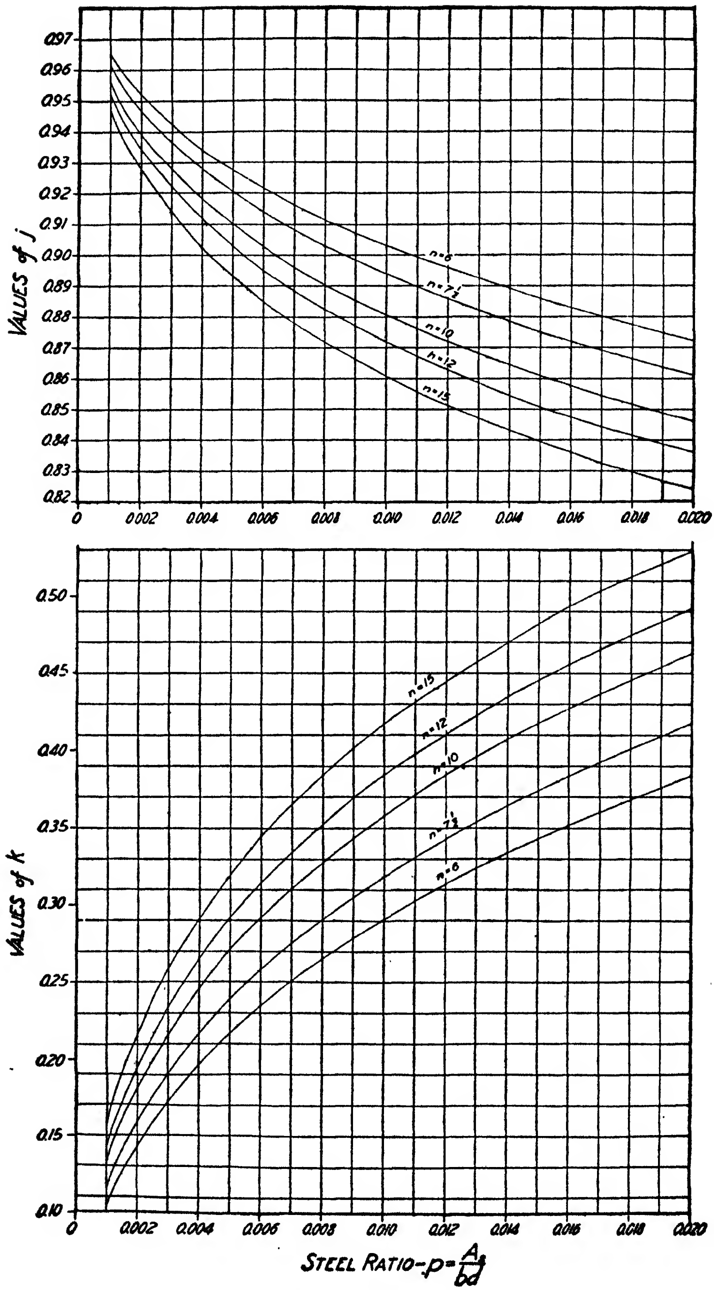


DIAGRAM 1. Rectangular beams. Values of j and k.

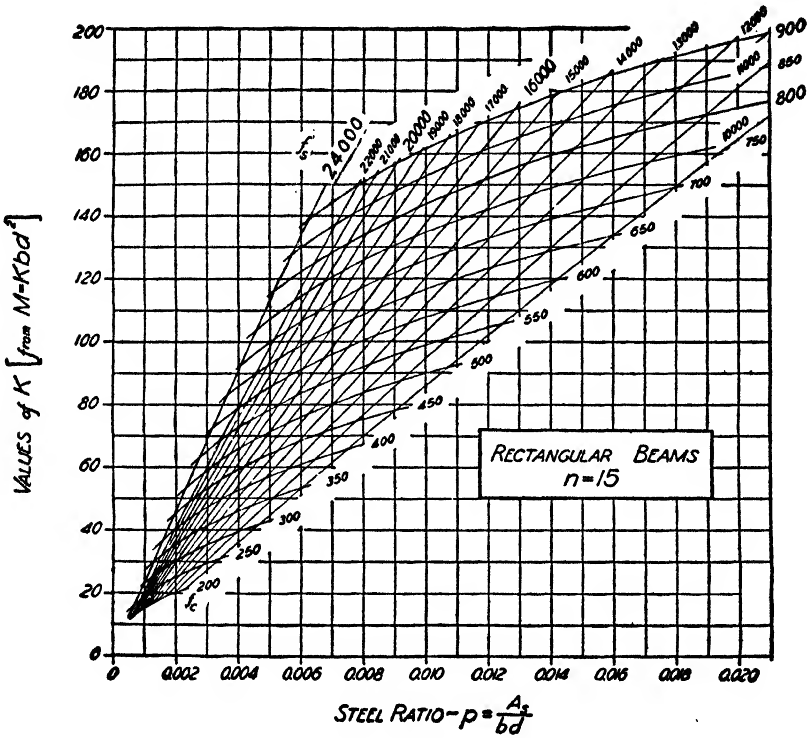


DIAGRAM 2. Rectangular beams. Fiber stress solutions.

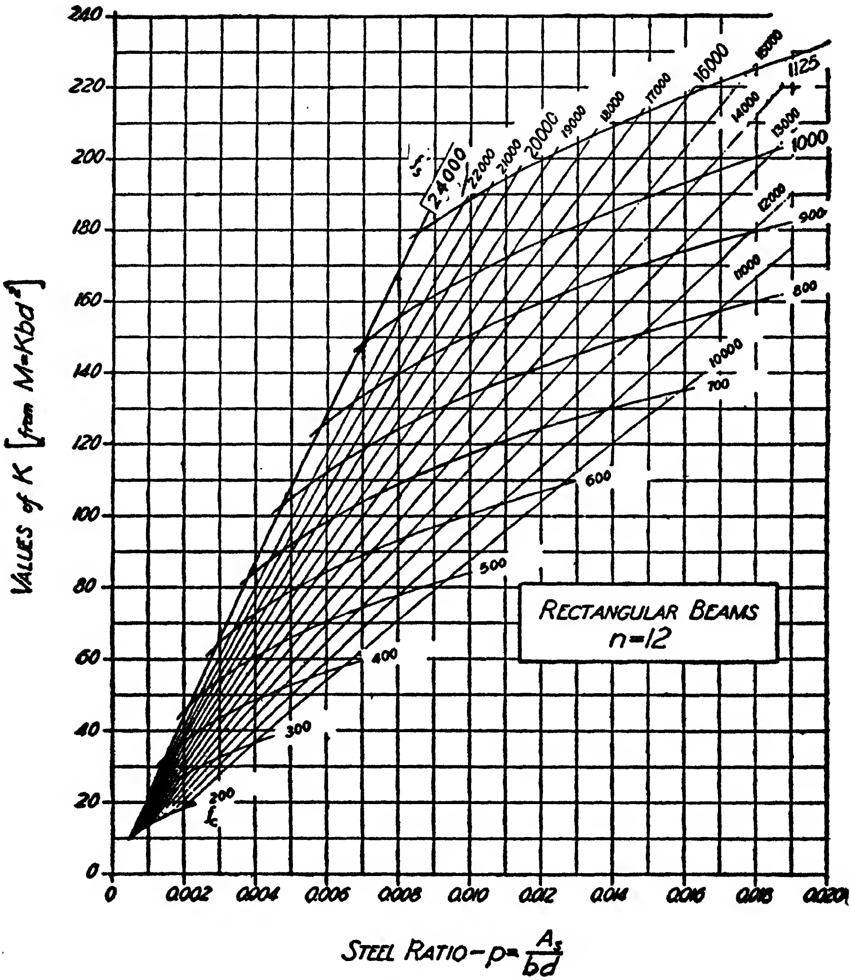


Diagram 3. Rectangular beams. Fiber stress solutions.

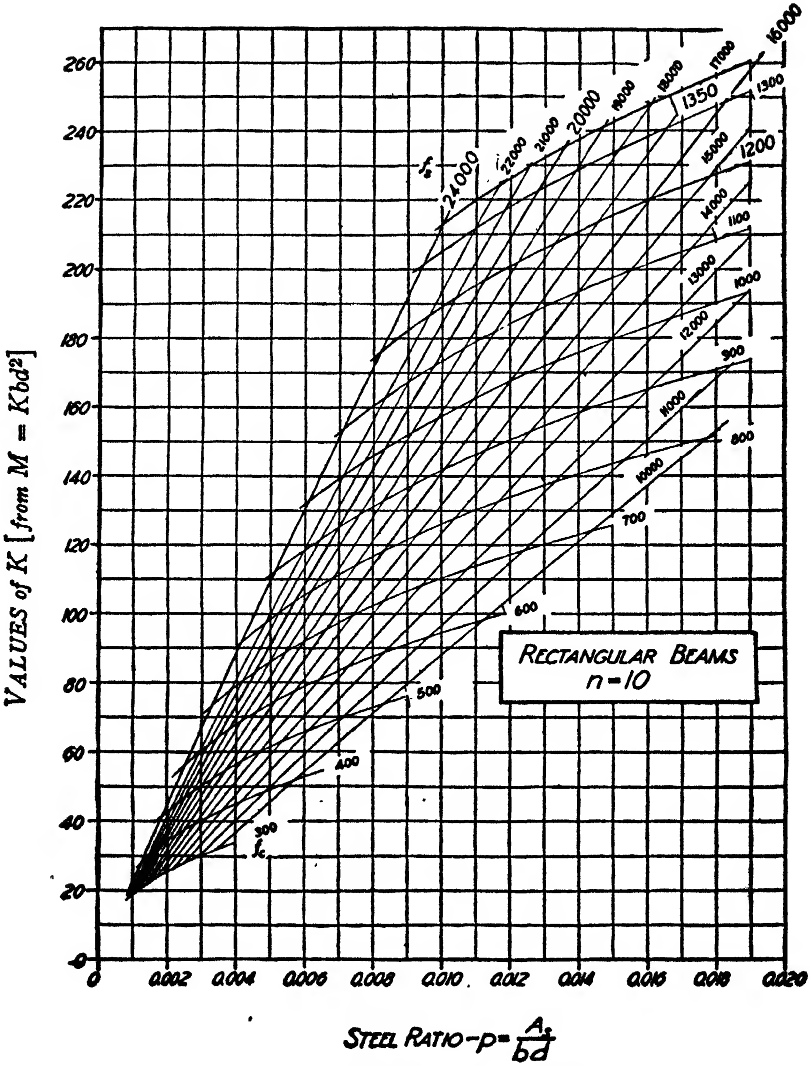


DIAGRAM 4. Rectangular beams. Fiber stress solutions.

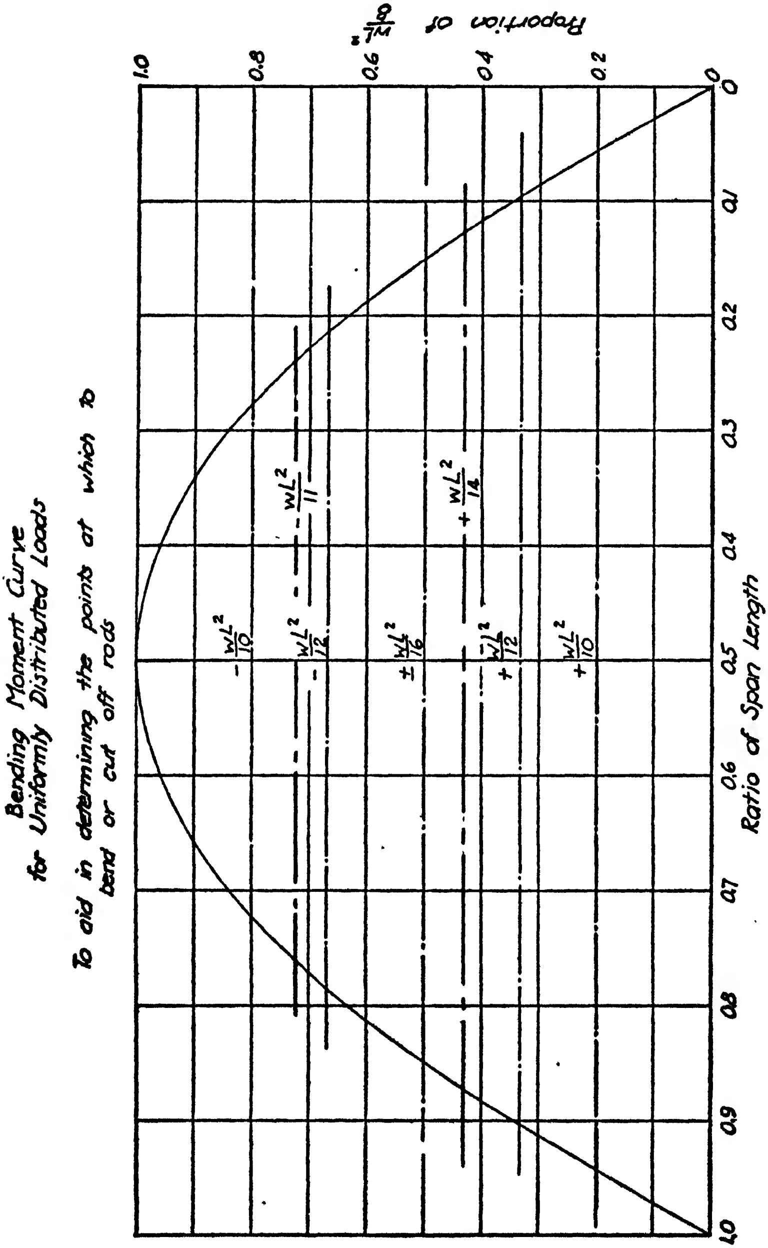


DIAGRAM 5. Bending moment for uniformly distributed loads.

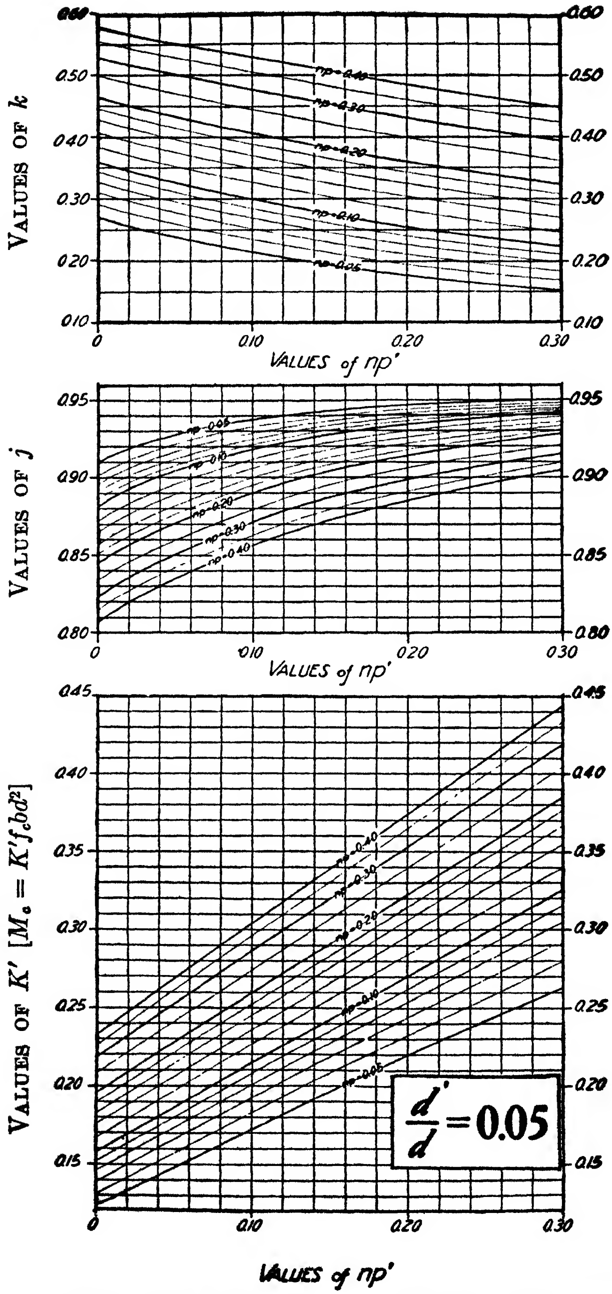


DIAGRAM 6. Rectangular beams with compression steel. Fiber stress solutions.

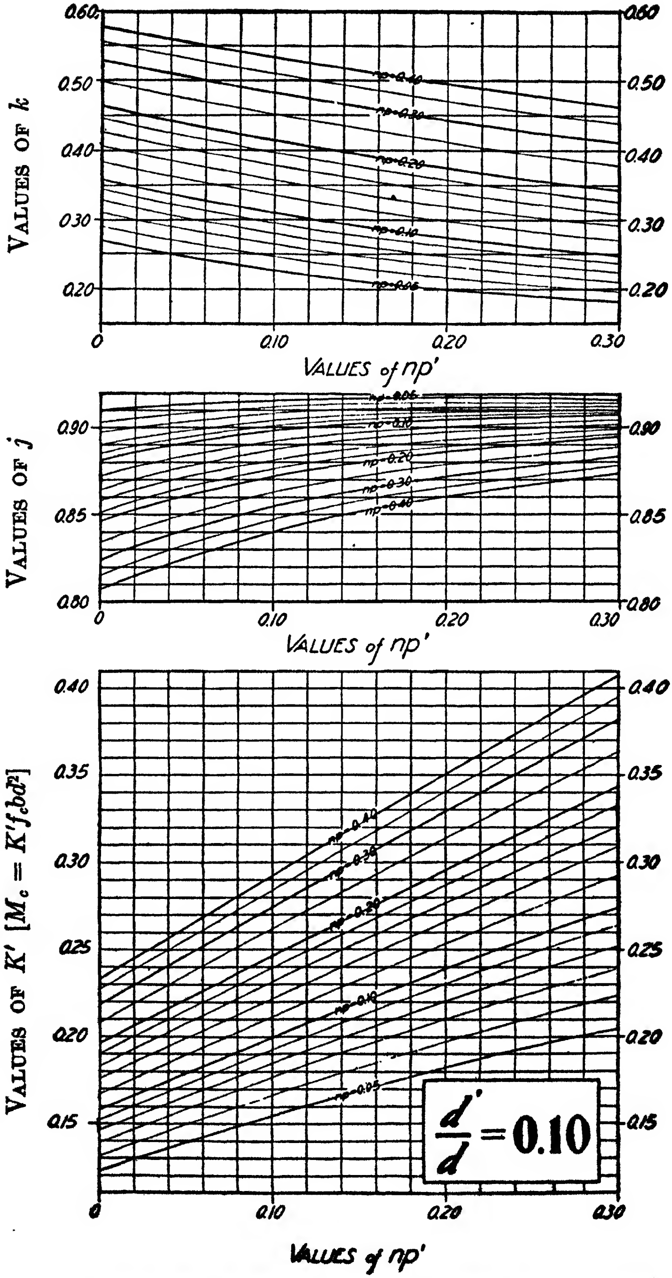


DIAGRAM 7. Rectangular beams with compression steel. Fiber stress solutions.

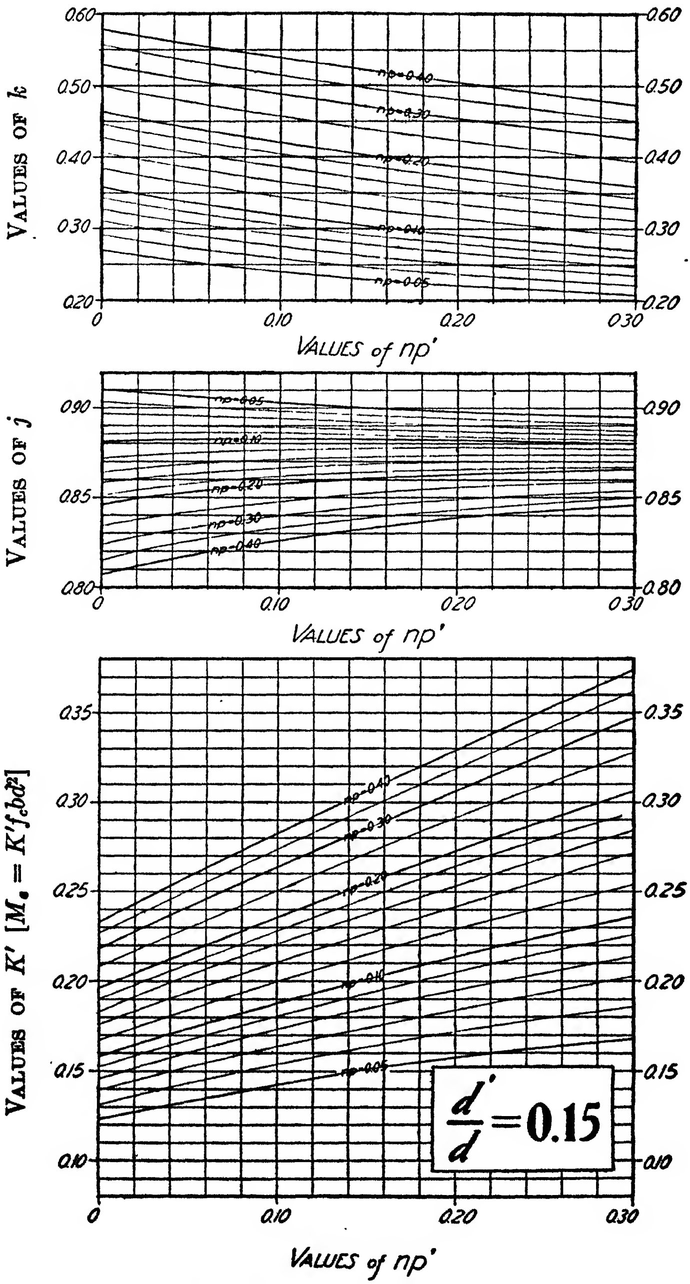


Diagram 8. Rectangular beams with compression steel. Fiber stress solutions.

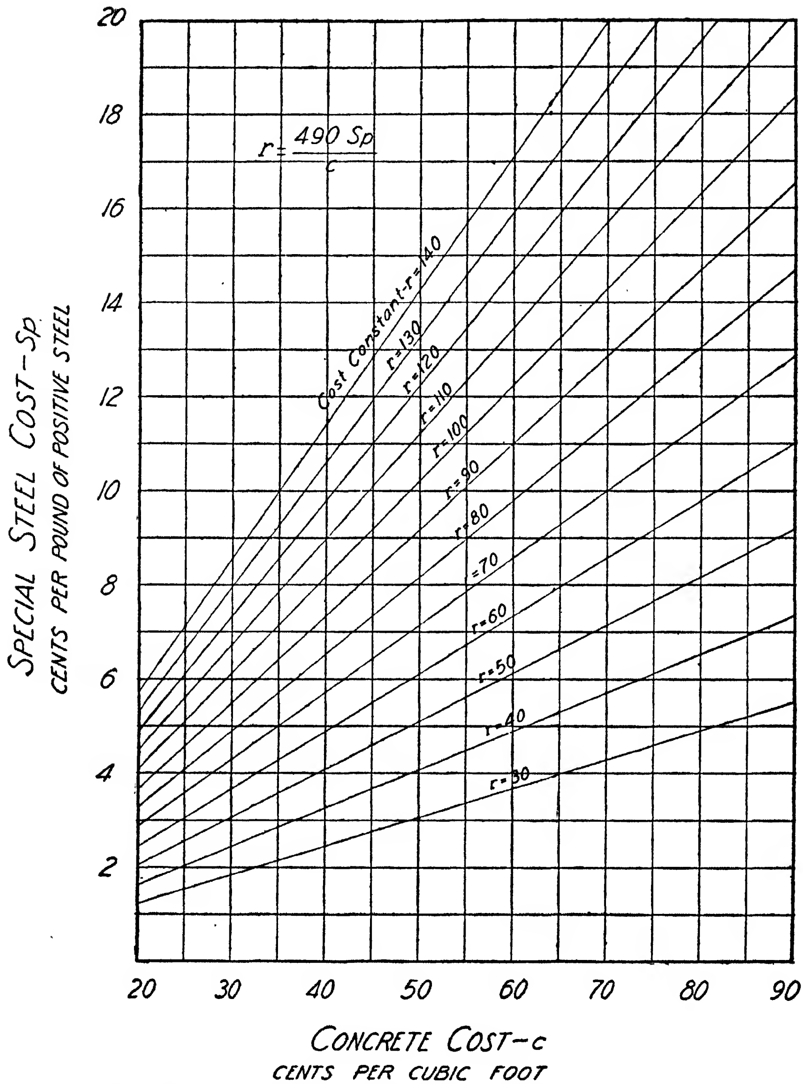


DIAGRAM 9. Tee beams. Cost constant for economical depth.

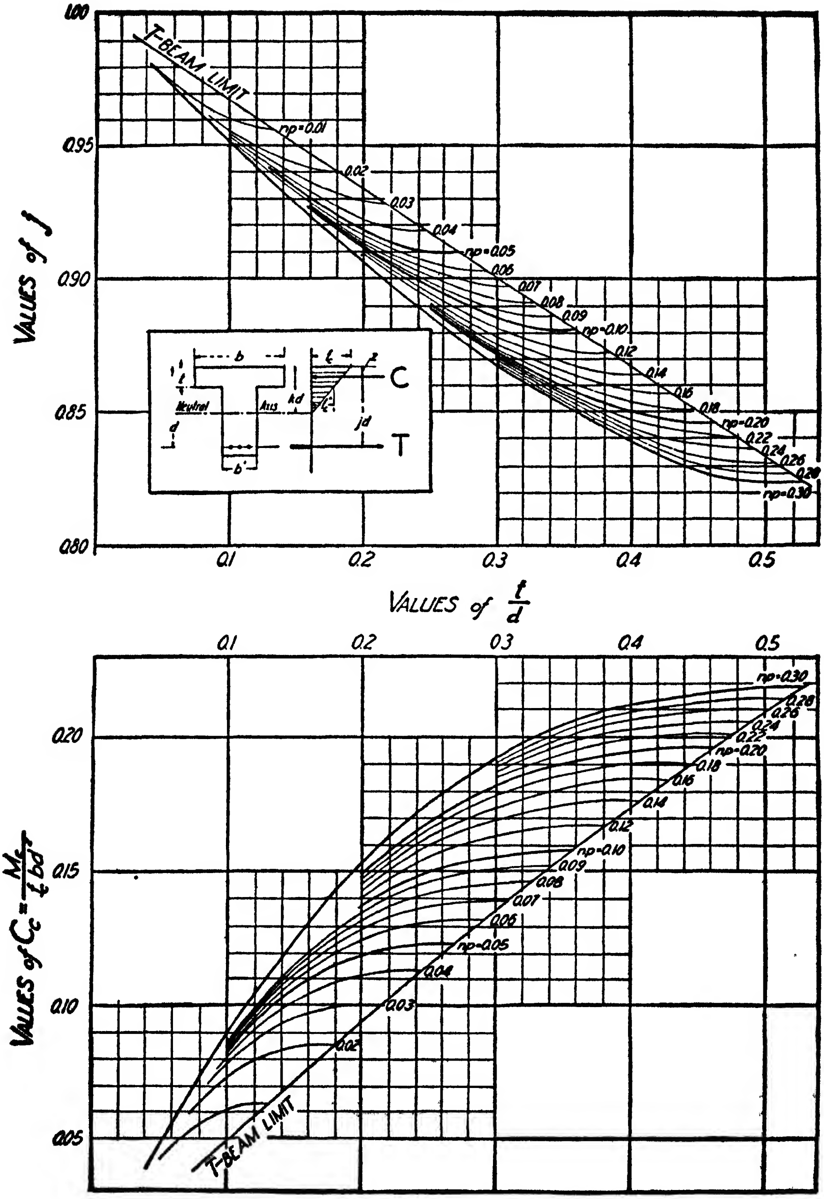
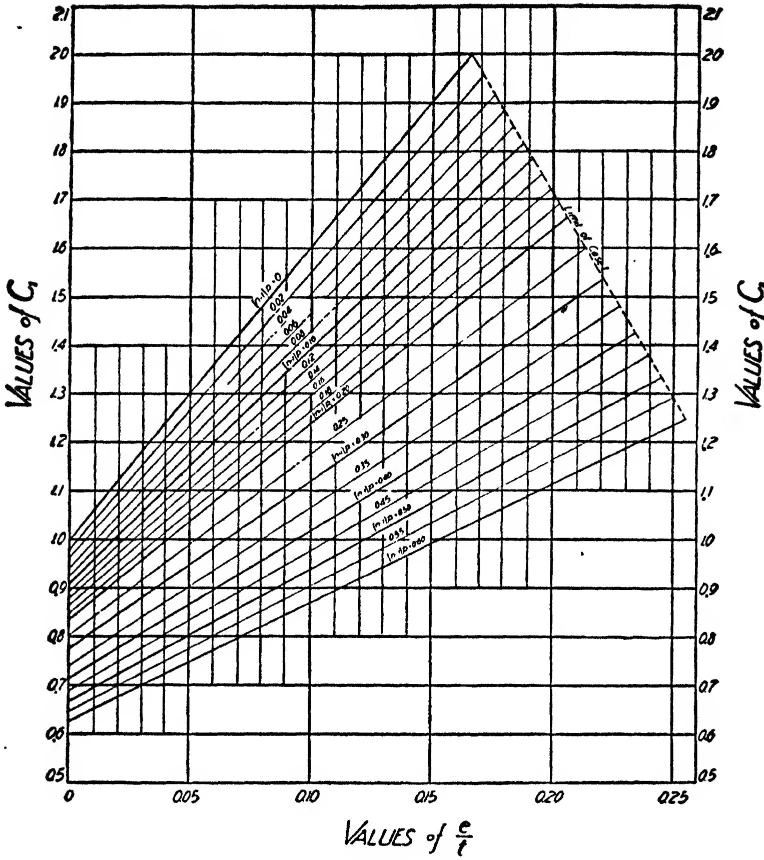


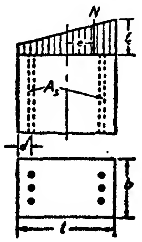
DIAGRAM 10. Tee beams. Fiber stress solutions.



$$k = \frac{1 + 12(n-1)p\,(g)^2}{12\left(\frac{e}{t}\right)\left[1 + (n-1)p\right]} + \frac{1}{2}$$

$$C_1 = \frac{2k}{(2k-1)\left[1 + (n-1)p\right]}$$

$$f = \frac{N}{bt} \cdot C_1$$

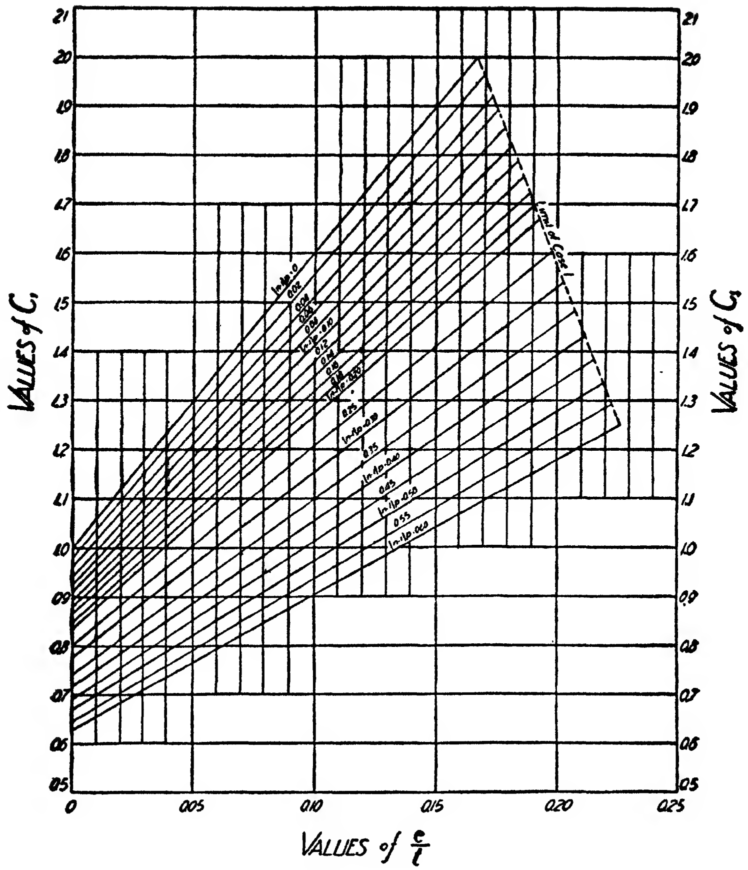


p = Total Steel Ratio = $\frac{A_s}{bt}$

$n = \frac{E_s}{E_c}$

CASE I
$d/t = 0.05$

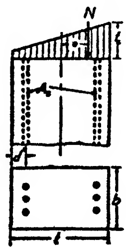
Diagram 11. Eccentrically loaded columns. Rectangular section. Case I.



$$k = \frac{1 + 12(n-1)p\left(\frac{d'}{t}\right)^2}{12\left(\frac{e}{t}\right)\left[1 + (n-1)p\right]} + \frac{1}{2}$$

$$C_1 = \frac{2k}{(2k-1)\left[1 + (n-1)p\right]}$$

$$f_c = \frac{N}{bt} \cdot C_1$$

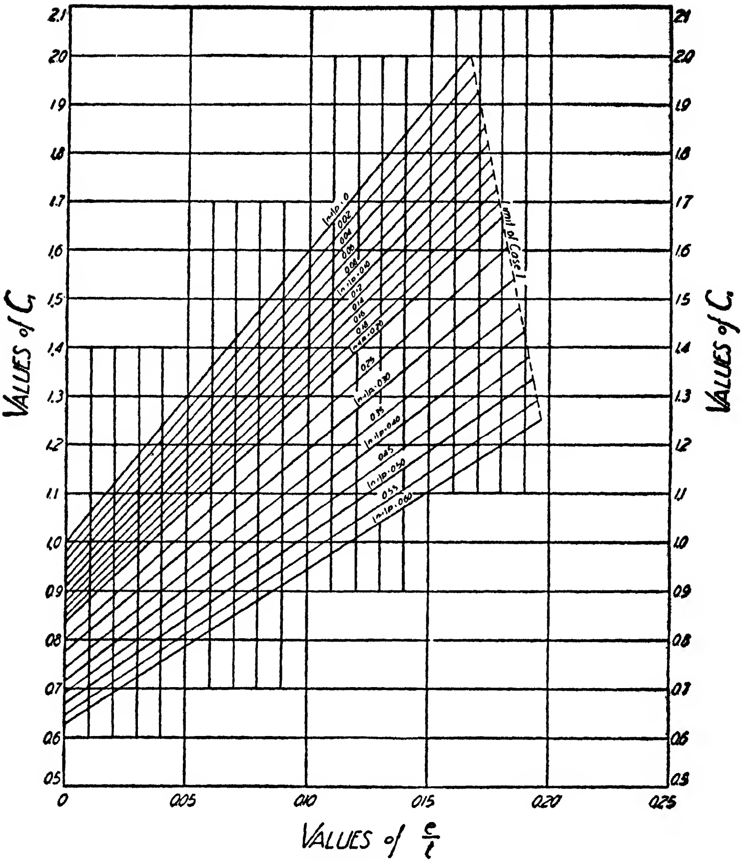


p = Total Steel Ratio = $\frac{A_s}{bt}$

$n = \frac{E_s}{E_c}$

CASE I
$d'/t = 0.10$

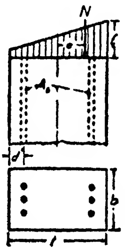
DIAGRAM 12. Eccentrically loaded columns. Rectangular section. Case I.



$$k = \frac{1+12(n-1)p\left(\frac{1}{2}-\frac{d'}{t}\right)^2}{12\left(\frac{e}{t}\right)[1+(n-1)p]} + \frac{1}{2}$$

$$C_1 = \frac{2k}{(2k-1)[1+(n-1)p]}$$

$$f_c = \frac{N}{bt}\,C_1$$

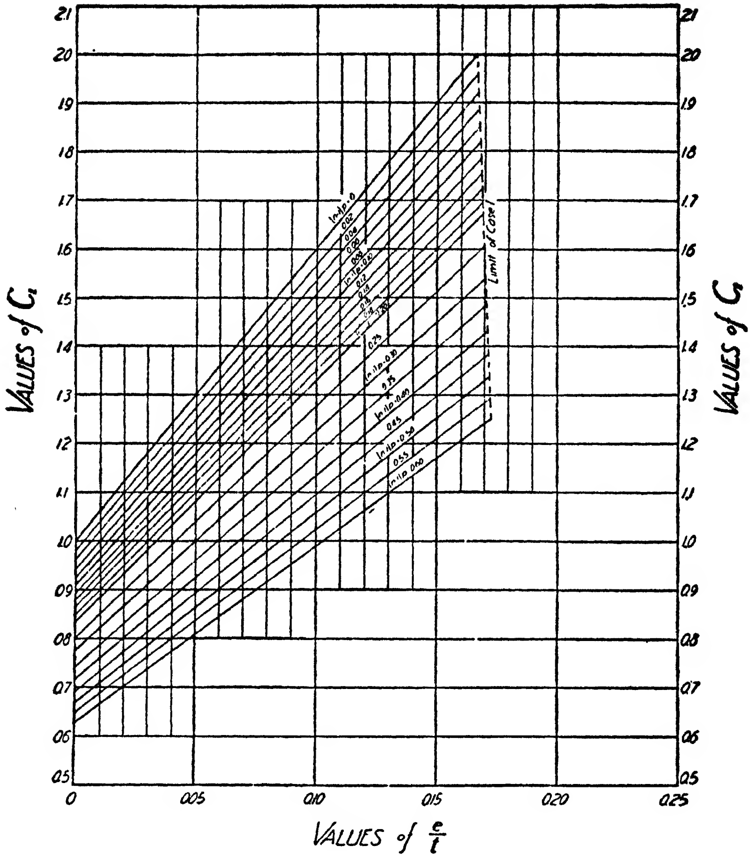


p = Total Steel Ratio = $\frac{A_s}{bt}$

$n = \frac{E_s}{E_c}$

CASE I
$d'/t = 0.15$

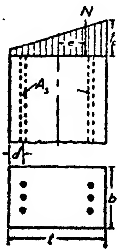
DIAGRAM 13. Eccentrically loaded columns. Rectangular section. Case I.



$$k = \frac{1 + 12(n-1)p(g)^2}{12\left(\frac{e}{t}\right)\left[1 + (n-1)p\right]} + \frac{1}{2}$$

$$C_1 = \frac{2k}{(2k-1)\left[1 + (n-1)p\right]}$$

$$f_c = \frac{N}{bt} C_1$$

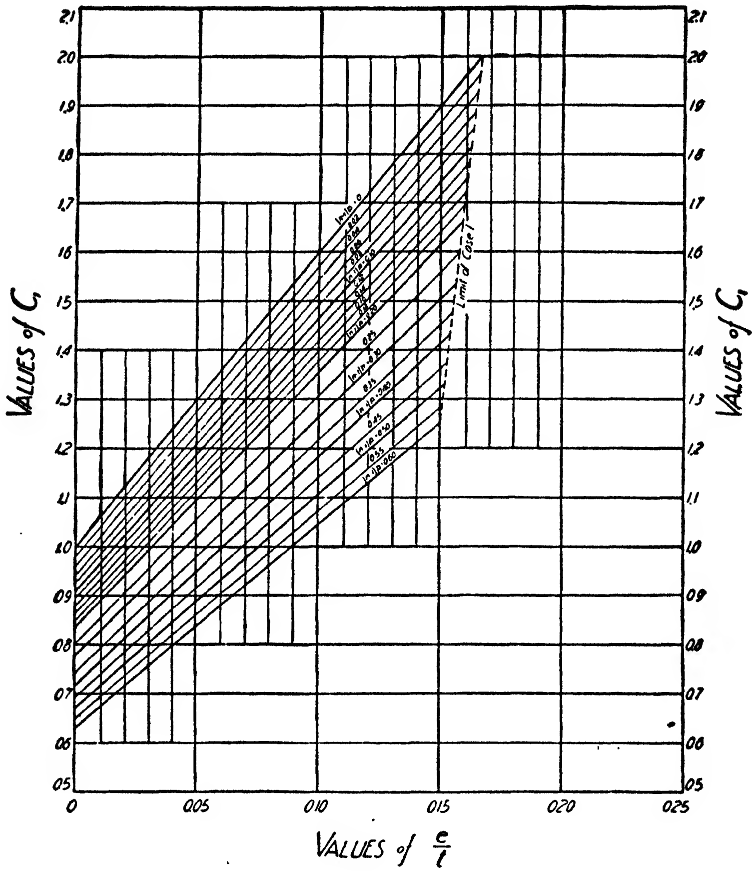


p = Total Steel Ratio = $\frac{A_s}{bt}$

$n = \frac{E_s}{E_c}$

CASE I
$d'/t = 0.20$

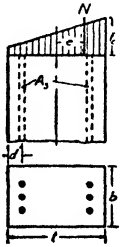
DIAGRAM 14. Eccentrically loaded columns. Rectangular section. Case I.



$$k = \frac{1+12(n-1)p\left(\frac{g}{t}\right)^2}{12\left(\frac{e}{t}\right)[1+(n-1)p]} + \frac{1}{2}$$

$$C_1 = \frac{2k}{(2k-1)[1+(n-1)p]}$$

$$f_c = \frac{N}{bt} C_1$$



p = Total Steel Ratio = $\frac{A_s}{bt}$

$n = \frac{E_s}{E_c}$

CASE I
$d'/t = 0.25$

DIAGRAM 15. Eccentrically loaded columns. Rectangular section. Case I.

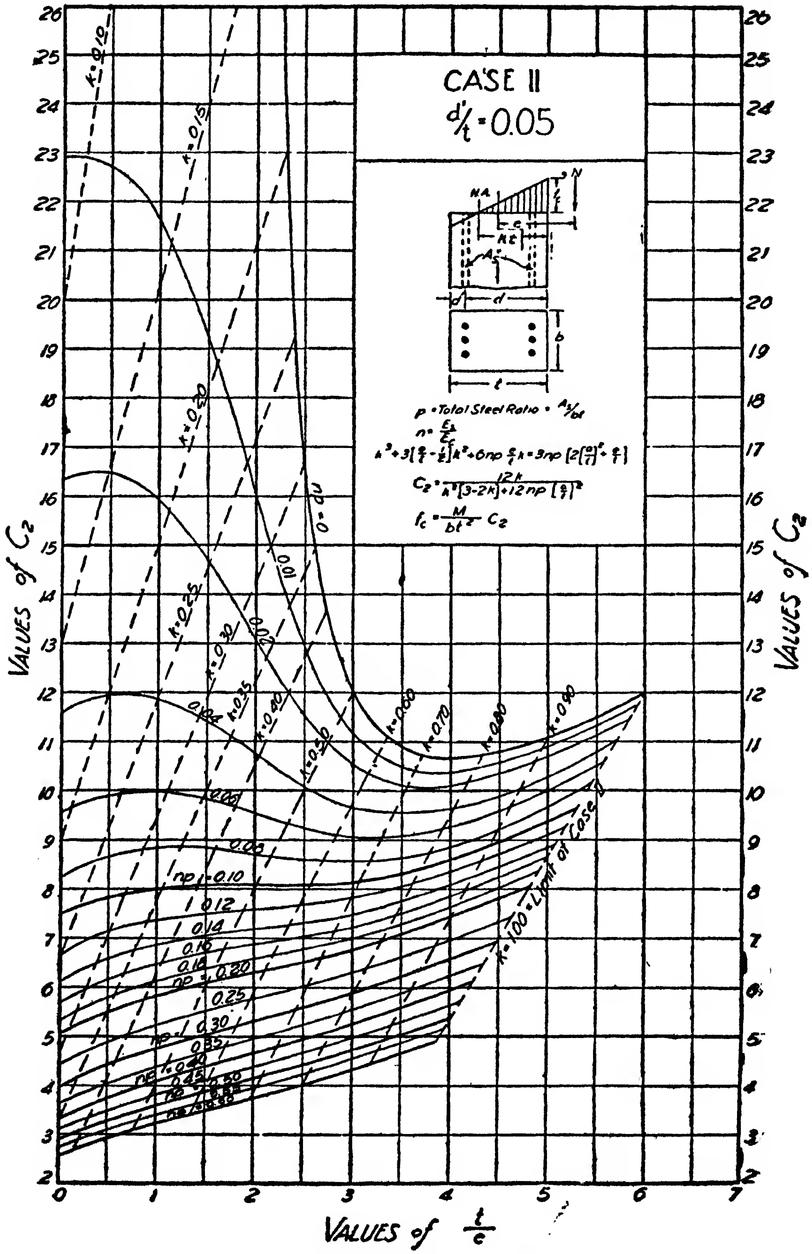


Diagram 16. Eccentrically loaded columns. Rectangular section. Case II. (After Turneaure and Maurer.)

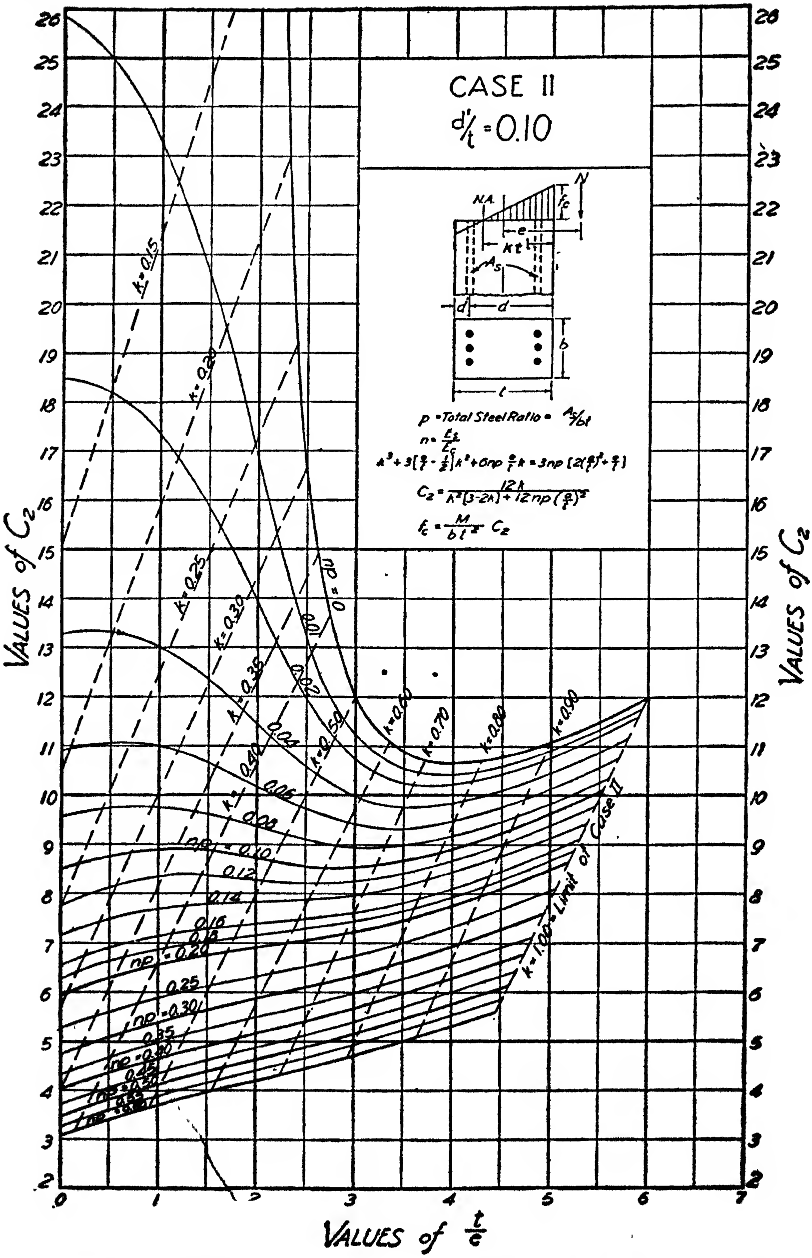


DIAGRAM 17. Eccentrically loaded columns. Rectangular section. Case II. (After Turneaure and Maurer.)

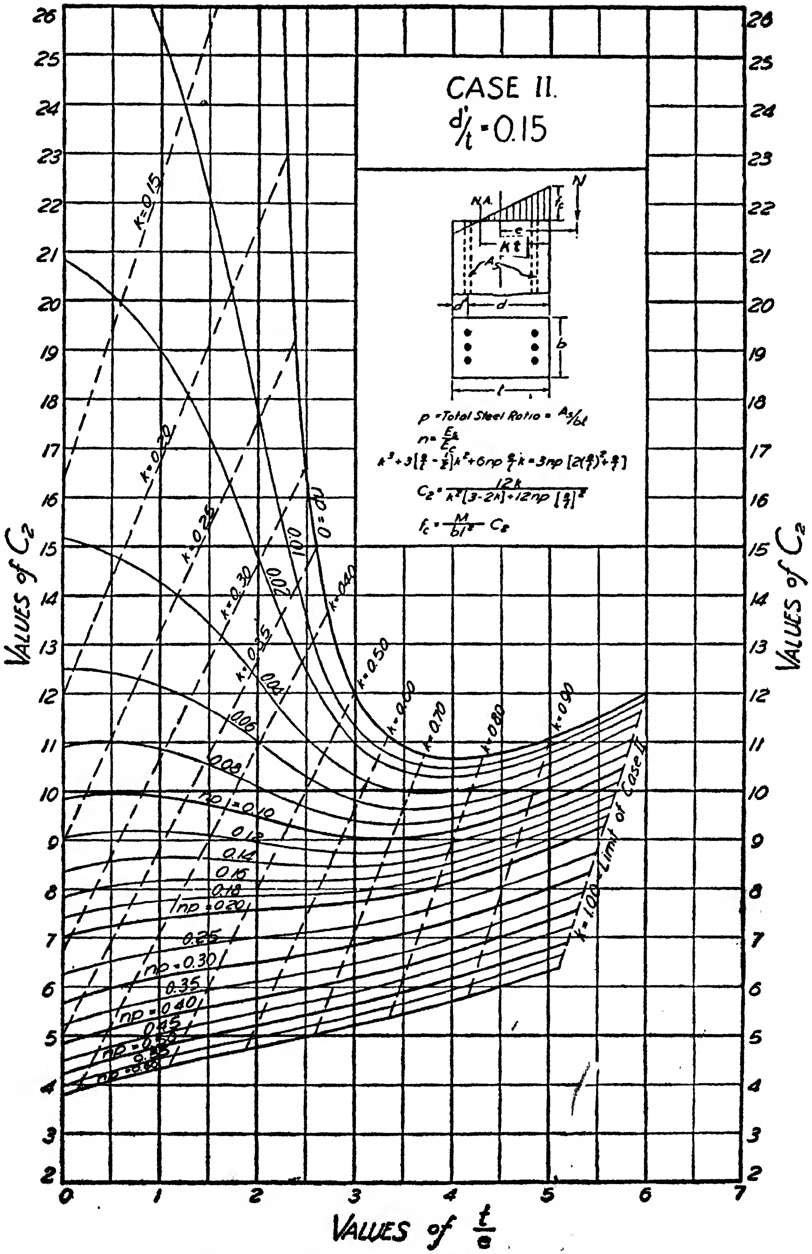


Diagram 18. Eccentrically loaded columns. Rectangular section. Case II. (After Turneaure and Maurer.)

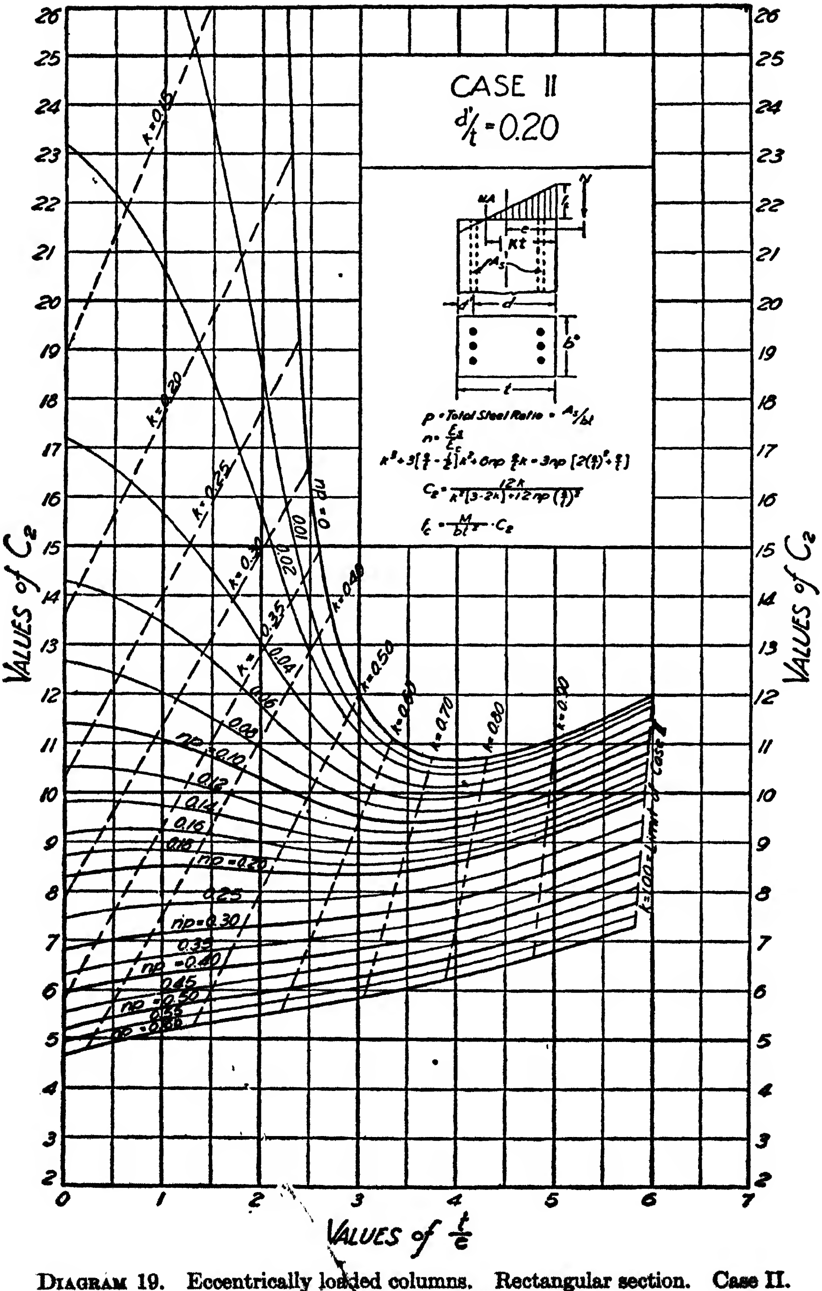


DIAGRAM 19. Eccentrically loaded columns. Rectangular section. Case II. (After Turneaure and Maurer.)

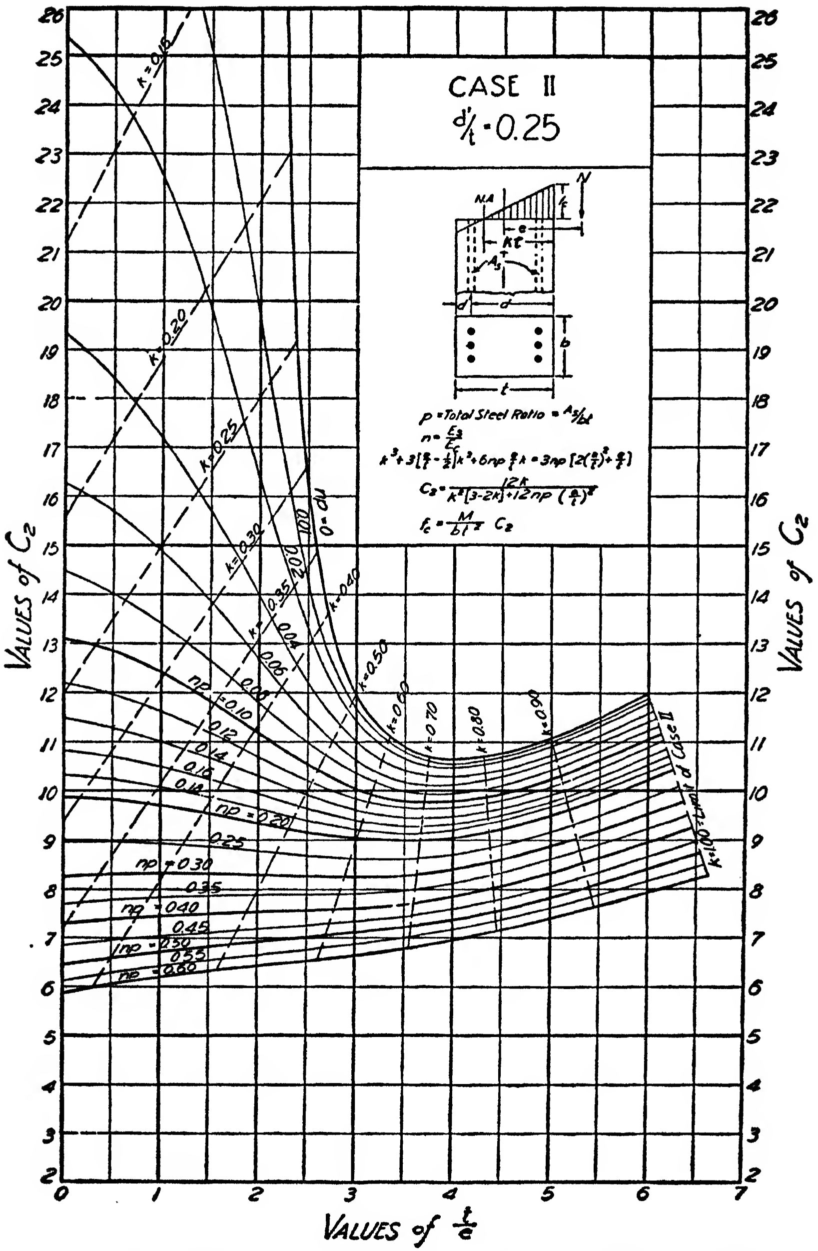


Diagram 20. Eccentrically loaded columns. Rectangular section. Case II. (After Turneaure and Maurer.)

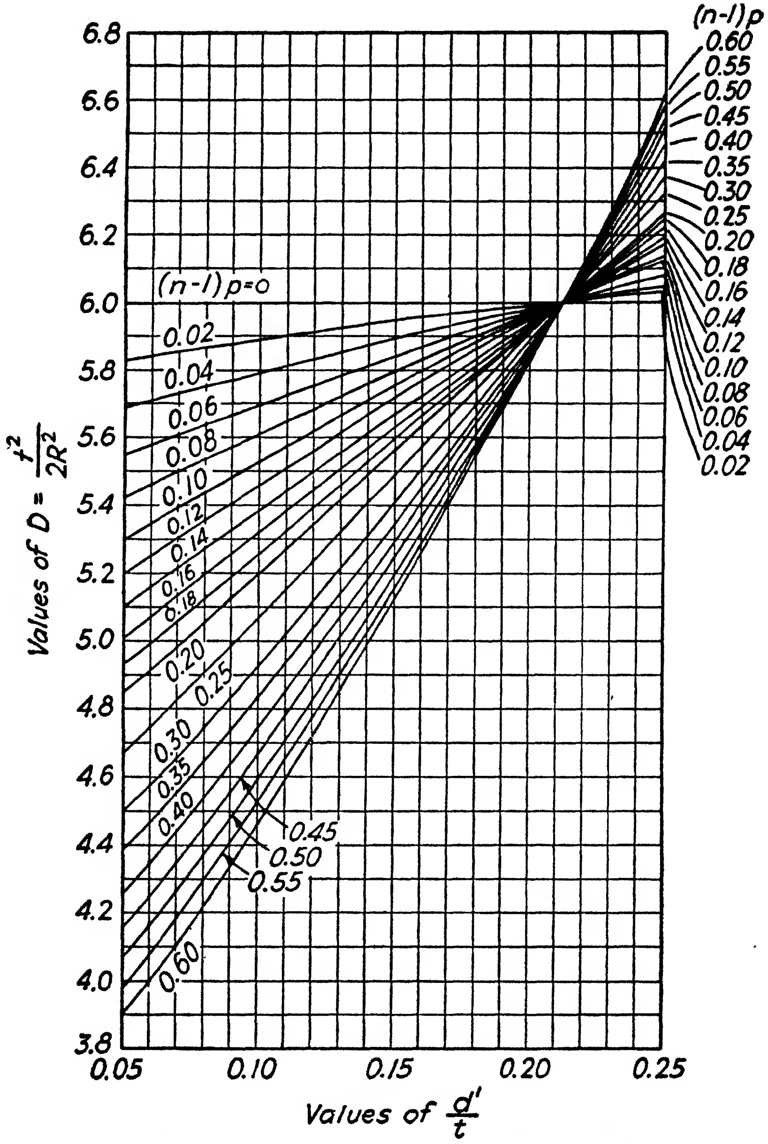


DIAGRAM 21. Valves of D. Columns with rectangular section.

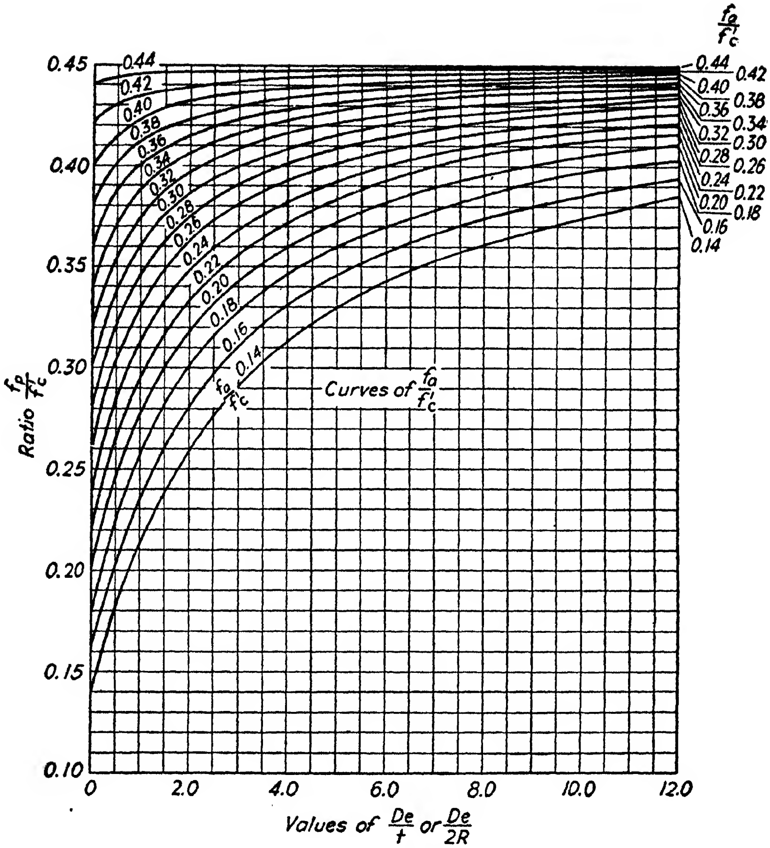


DIAGRAM 22. Allowable column stresses.

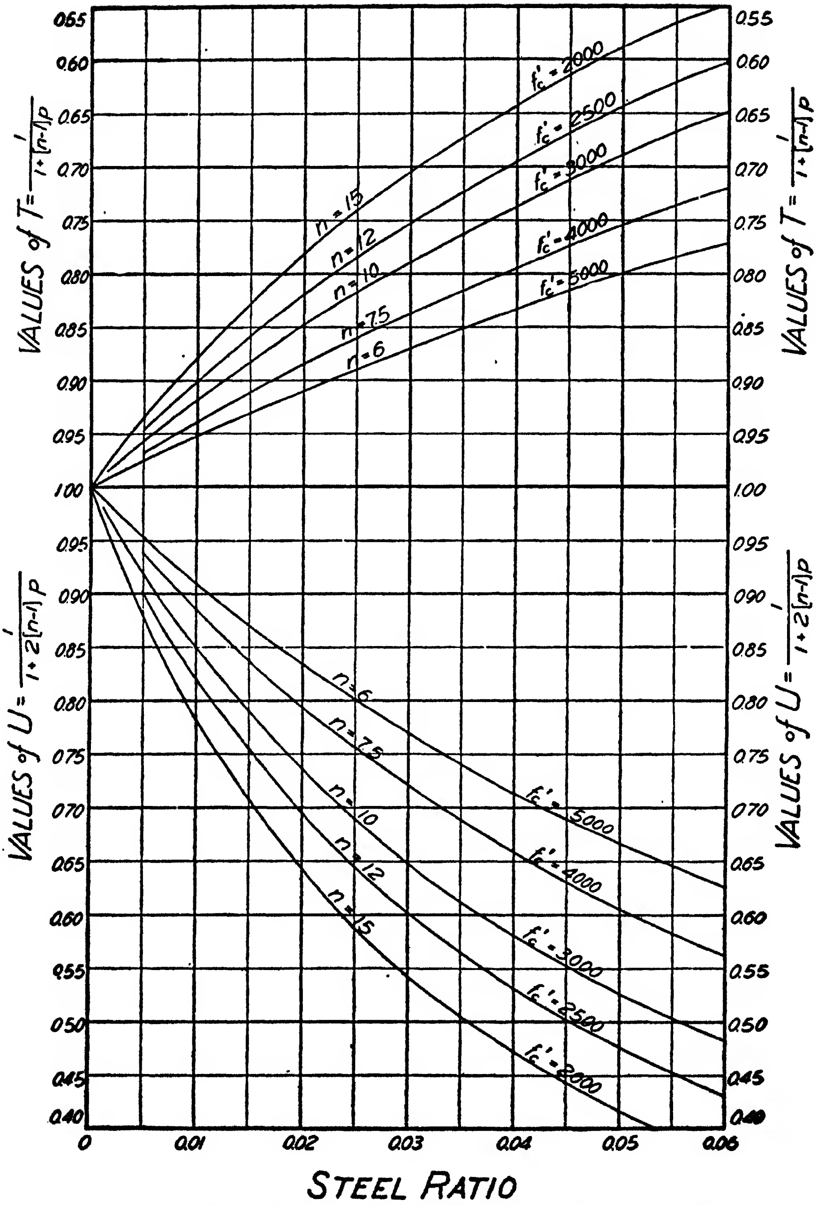


DIAGRAM 23. Eccentrically loaded columns. Circular section. Case I.

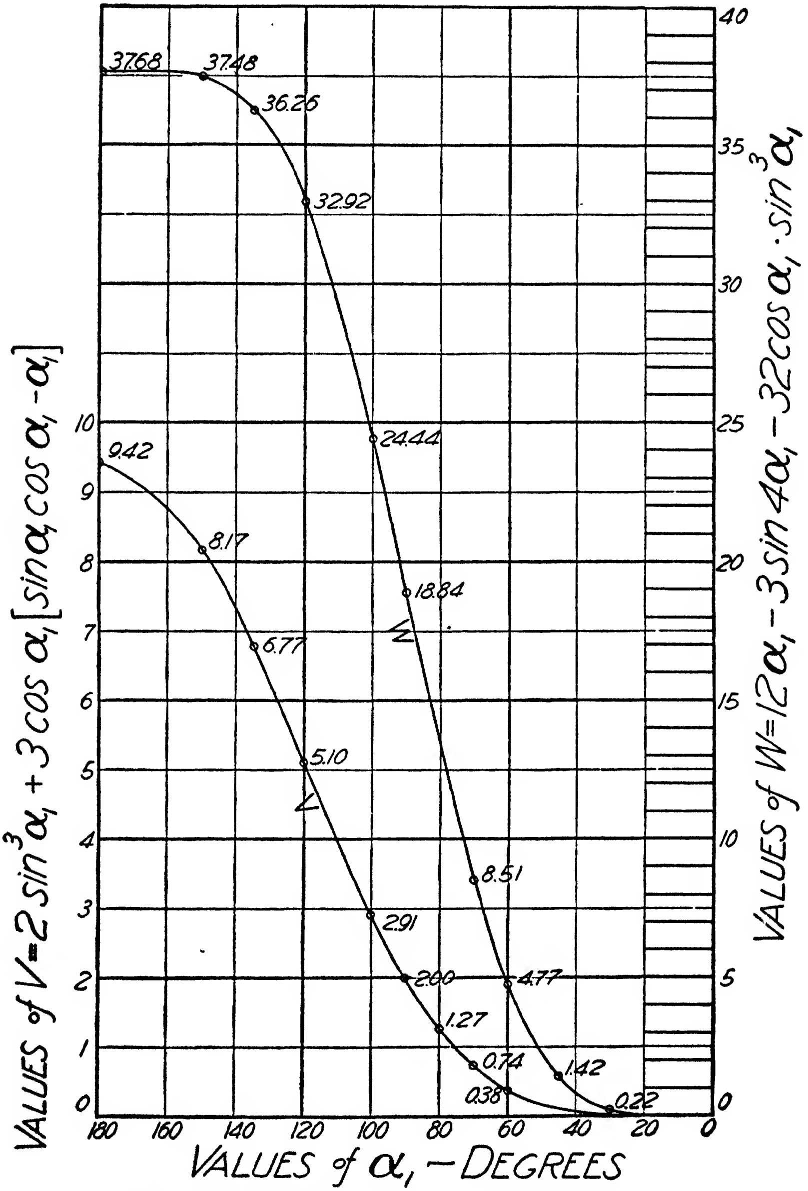


DIAGRAM 24. Eccentrically loaded columns. Circular section. Case II.

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